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COUNTING ENDS ON COMPLETE SMOOTH METRIC MEASURE SPACES

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Abstract. Let $(M, g, e^{-f}dv)$ be a complete smooth metric measure space with Bakry-Émery Ricci curvature nonnegative outside a compact set. We prove that the number of ends of such a measure space is finite if f has at most sublinear growth outside the compact set. In particular, we give an explicit upper bound for the number.

1. Introduction and main results

A complete smooth metric measure space, denoted by $(M^n, g, e^{-f}dv)$, is an n-dimensional complete smooth Riemannian manifold (M^n, g) together with a weighted volume form $e^{-f}dv$, where f is a smooth function on M and dv is the volume element of Riemannian metric g. The Bakry-Émery Ricci curvature [1] (also called ∞-Bakry-Émery Ricci curvature) associated with this space is defined by

$$\mathrm{Ric}_f := \mathrm{Ric} + \mathrm{Hess}(f),$$

where Ric is the Ricci curvature of the manifold and Hess is the Hessian with respect to the metric g. The Bakry-Émery Ricci curvature extends the usual Ricci curvature, and often shares similar properties with the Ricci curvature; see for example [4], [5], [13], [14], [18], [19] and [20]. This tensor is also related to the gradient Ricci soliton, which is a special solution of the Ricci flow. Here a gradient Ricci soliton means a complete manifold (M, g) and a smooth function f in M satisfy

$$\mathrm{Ric}_f = \lambda g$$

for some constant λ. The soliton is called expanding, steady or shrinking if $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$, respectively. Ricci solitons possess many interesting geometrical and topological results. See for example, [3] and [17] for nice surveys on this subject and the references therein.

On a complete smooth metric measure space $(M, g, e^{-f}dv)$, one can define the f-Laplacian

$$\Delta_f := \Delta - \nabla f \cdot \nabla,$$

which is linked with the Bakry-Émery Ricci curvature by the Bochner identity

$$\frac{1}{2}\Delta_f|\nabla u|^2 = |\mathrm{Hess}(u)|^2 + \langle \nabla u, \nabla \Delta_f u\rangle + \mathrm{Ric}_f(\nabla u, \nabla u).$$

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Since $\mathrm{tr}(\mathrm{Hess}(u)) \neq \Delta_f u$, one does not immediately obtain geometrical comparison results from the Bochner identity like the classical comparison theorems for Ricci curvature. But under appropriate assumptions on f, Wei and Wylie [18] successfully generalized classical comparison results to the cases of smooth metric measure spaces. Another way to deal with the above Bochner identity is to replace Ric_f by the m-Bakry-Émery Ricci curvature

$$\mathrm{Ric}_f^m := \mathrm{Ric}_f - \frac{1}{m} df \otimes df$$

for some constant $m > 0$. Using this tensor, one can get

$$\frac{1}{2}\Delta_f |\nabla u|^2 \geq \frac{(\Delta_f u)^2}{m+n} + \langle \nabla \Delta_f u, \nabla u\rangle + \mathrm{Ric}_f^m(\nabla u, \nabla u),$$

which is regarded as the Bochner formula for the Ricci curvature of $(n+m)$-dimensional manifolds. Then one can proceed as in the classical case to get various results for the m-Bakry-Émery Ricci curvature; see for example [10] and [21] and the references therein.

As we all know, the end of a complete smooth metric measure space is an important geometrical quantity, and it has received much attention recently. Lichnerowicz [11], Wei and Wylie [18] proved that if $\mathrm{Ric}_f \geq 0$ for some bounded f and M contains a line, then $M = N^{n-1} \times \mathbb{R}$ and f is constant along the line. In [5], Fang, Li, and Zhang showed that only an upper bound on f is needed in their splitting result. As a consequence, if M has at least two ends, their splitting theorem implies that the manifold N must be compact. Wei and Wylie [18] also showed that any complete smooth metric measure space with $\mathrm{Ric}_f > 0$ for some bounded f has only one end.

Munteanu and Wang [15] showed that any complete smooth metric measure space with $\mathrm{Ric}_f \geq 0$ has at most one f-nonparabolic end. Moreover, if a complete smooth metric measure space $(M^n, g, e^{-f}dv)$ has its Bakry-Émery Ricci curvature bounded below and the spectrum of f-Laplacian achieves its optimal positive upper bound, then Munteanu and Wang [15], [16] showed that M has only one end or is isometric to $N^{n-1} \times \mathbb{R}$ for some compact manifold N^{n-1}. In particular, their result implies that any nontrivial gradient steady Ricci soliton must have only one end.

In view of their results, we are interested in the number of ends on complete smooth metric measure spaces under more general curvature assumptions. Our result gives an explicit upper bound for the number of all ends (include f-nonparabolic and f-parabolic) on smooth metric measure spaces with nonnegative Bakry-Émery Ricci curvature outside a compact set.

Theorem 1.1. *Let $(M, g, e^{-f}dv)$ be an n-dimensional complete smooth metric measure space. Fix a point $o \in M$ and $R > 0$. Suppose* $\mathrm{Ric}_f \geq -(n-1)K$ *for some constant $K \geq 0$ in the geodesic ball $B_o(R)$ of radius R, and* $\mathrm{Ric}_f \geq 0$ *outside the geodesic ball $B_o(R)$. If f grows sublinearly outside the geodesic ball $B_o(R)$, then*

$$N_R(M) \leq \frac{2(n+4A)}{n-1+4A}(\sqrt{K}R)^{-(n+4A)} \exp\left(\frac{17}{2}(n-1+4A)\sqrt{K}R\right), \tag{1.1}$$

where $N_R(M)$ denotes the number of ends of M with respect to $B_o(R)$, and $A := A(R) = \sup_{x \in B_o(25/2\,R)} |f(x)|$.

Remark 1.2. We can apply a similar argument to obtain an upper bound for the number of ends under the only conditions of Ric_f^m (without any assumption on f). That is, if $\mathrm{Ric}_f^m \geq 0$ outside the geodesic ball $B_o(R)$, and $\mathrm{Ric}_f^m \geq -(m-1)K$ for some constant $K \geq 0$ on $B_o(R)$, then

$$N_R(M) \leq \frac{2(n+m)}{n+m-1}(\sqrt{K}R)^{-(n+m)} \exp\left(\frac{17}{2}(n+m-1)\sqrt{K}R\right).$$

When f is constant, Cai [2] and Li-Tam [9] independently proved that any complete manifold with nonnegative Ricci curvature outside a compact set has at most finitely many ends (see also [12]).

The proof of Theorem 1.1 adapts the argument of Cai [2], which relies on the f-volume comparison proved by Wei-Wylie [18] and local properties of Busemann functions on smooth metric measure spaces [5]. Since these results all depend on some assumption of f, the growth condition of f in our theorem seems to be necessary. It is interesting to know whether the growth condition of f is sharp.

The paper is organized as follows. In Section 2, we recall comparison theorems for the Bakry-Émery Ricci curvature, and state some definitions of geometrical invariants on complete smooth metric measure spaces. We also explain some basic facts about Busemann functions on complete smooth metric measure spaces. In Section 3, we apply Cai's arguments to prove our main theorem.

2. Preliminaries

In this section, we recall some previous results, which will be prepared to prove our theorem. We first give the relative f-volume comparison theorem of Wei and Wylie [18] (see also [19]).

Lemma 2.1. *Let $(M, g, e^{-f}dv)$ be an n-dimensional complete noncompact smooth metric measure space. If $\mathrm{Ric}_f \geq -(n-1)K$ for some constant $K \geq 0$, then*

$$\frac{V_f(B_x(R_1,R_2))}{V_f(B_x(r_1,r_2))} \leq \frac{V_K^{n+4A}(B_x(R_1,R_2))}{V_K^{n+4A}(B_x(r_1,r_2))} \tag{2.1}$$

for any $x \in M$ and $0 < r_1 < r_2$, $0 < R_1 < R_2$, $r_1 \leq R_1$, $r_2 \leq R_2$, where $B_x(R_1,R_2) = B_x(R_2)\backslash B_x(R_1)$, and $A = A(x,R_2) = \sup_{y\in B_x(R_2)}|f(y)|$. Here $V_K^{n+4A}(B_x(r))$ denotes the volume of the ball in the model space M_K^{n+4A}, i.e., the simply connected space form with constant sectional curvature $-K$ and dimension $n+4A$.

From Lemma 2.1, we immediately get

$$\frac{V_f(B_x(r_2))}{V_f(B_x(r_1))} \leq \frac{\int_0^{r_2}(\sinh^{n-1+4A}\sqrt{K}\,t)dt}{\int_0^{r_1}(\sinh^{n-1+4A}\sqrt{K}\,t)dt} \tag{2.2}$$

for any $x \in M$ and $0 < r_1 < r_2$, where $A = A(x,r_2) = \sup_{y\in B_x(r_2)}|f(y)|$.

We also recall some definitions of basic geometric quantities on complete smooth metric measure spaces.

Definition 2.2. On a complete smooth metric measure space $(M, g, e^{-f}dv)$, a smooth function $G_f(x,y)$ defined on $(M\times M)\backslash\{(x,x)\}$ is called f-Green's function if it satisfies

$$G_f(x,y) = G_f(y,x) \quad \text{and} \quad \Delta_{f,y}G(x,y) = -\delta_{f,x}(y)$$

for all $x \neq y$, where $\delta_{f,x}(y)$ is defined by

$$\int_M \psi(y)\delta_{f,x}(y)e^{-f}dv = \psi(x)$$

for every compactly supported function $\psi \subset M$.

Every complete smooth metric measure space admits an f-Green's function. Indeed we can follow Li-Tam [9] to give a constructive argument for the existence of $G_f(x,y)$. However, some metric measure spaces admit f-Green's functions which are positive and others may not. This special property distinguishes the function theory of smooth measure metric measure spaces into two classes.

Definition 2.3. A complete smooth metric measure space $(M, g, e^{-f}dv)$ is said to be f-nonparabolic if it admits a positive f-Green's function. Otherwise, it is said to be f-parabolic.

By the criterion of Li-Tam [8], [9], we easily see that a complete measure manifold is f-nonparabolic if and only if there exists a positive f-superharmonic function whose infimum is achieved at infinity.

Definition 2.4. Let M be a complete manifold. A geodesic $\gamma : (-\infty, +\infty) \to M$ is called a line if

$$d(\gamma(s), \gamma(t)) = |s - t|$$

for all s and t. Furthermore, a geodesic $\gamma : [0, +\infty) \to M$ is called a ray if

$$d(\gamma(0), \gamma(t)) = t$$

for all $t > 0$.

It is easy to see that if M is complete noncompact, it must contain a ray.

Definition 2.5. Let γ_1 and γ_2 be two rays emanating from the base point $o \in M$. They are called cofinal if for any $R > 0$ and any $t > R$, then $\gamma_1(t)$ and $\gamma_2(t)$ lie in the same component of $M - B_o(R)$. An equivalence class of cofinal rays is called an *end* of M. In this paper we will denote by $[\gamma]$ the equivalence class of the ray γ.

In general, when we say that E is an end of the manifold M we mean that it is an end with respect to some compact subset $\Omega \subset M$. Notice that the above definition does not depend on the base point o and the particular complete metric on M. Thus the number of ends is a topological invariant of M.

Definition 2.6. An end E on $(M, g, e^{-f}dv)$ is said to be f-nonparabolic if it admits a positive f-Green's function with Neumann boundary condition on ∂E. Otherwise, it is said to be f-parabolic.

From Li-Tam [9], we know that a complete smooth metric measure space is f-nonparabolic if and only if it has an f-nonparabolic end. Of course, it is possible for an f-nonparabolic space to have many f-parabolic ends.

In the following we recall some basic facts about Busemann functions on complete smooth metric measure spaces.

For each ray $\gamma \subset M$, the *Busemann function* associated to γ is defined by

$$b_\gamma(x) := \lim_{t\to\infty}(t - d(x, \gamma(t))).$$

This function is Lipschitz continuous with Lipschitz constant 1 and it is differential almost everywhere. At the points where b_γ is not smooth we interpret the f-Laplacian in the sense of barriers.

Definition 2.7. A lower barrier for a continuous function h at the point $p \in M$ is a C^2 function h_p, defined in a neighborhood U of p, such that

$$h_p(p) = h \quad \text{and} \quad h_p(x) \leq h(x), \quad x \in U.$$

We say that a continuous function h on M satisfies $\Delta_f h \geq a$ at p in the barrier sense, if for any $\epsilon > 0$, there exists a lower barrier function $h_{p,\epsilon}$ of h at p such that

$$\Delta_f h_{p,\epsilon} \geq a - \epsilon.$$

A continuous function h that satisfies $\Delta_f h \leq a$ in the barrier sense is similarly defined as above.

For any given point $p \in M$, let $\alpha(t)$ be a minimal geodesic from p to the ray $\gamma(t)$. As $t \to \infty$, $\alpha(t)$ has a convergent subsequence which converges to a ray at p. Such a ray is called an asymptotic ray to $\gamma(t)$ at p.

Let γ be a line in M. Then we have rays $\gamma^+ : [0, \infty) \to M$ by

$$\gamma^+(t) = \gamma(t)$$

and $\gamma^- : [0, \infty) \to M$ by

$$\gamma^-(t) = \gamma(-t).$$

Similar to the above procedure, we can define b^+_γ (or b^-_γ, respectively) to be the associated Busemann function of γ^+ (or γ^-, respectively).

If Bakry-Émery Ricci curvature is nonnegative and f is at most sublinear growth of distance function, Fang, Li, and Zhang [5] showed that Busemann functions $b^\pm$ associated to rays $\gamma^\pm$, are smooth f-harmonic functions, and satisfy $b^+ + b^- = 0$ and $\operatorname{Hess} b^\pm = 0$. In fact, applying their arguments locally, we have the following lemma.

Lemma 2.8. *Let N be the δ-tubular neighborhood of a line γ on $(M, g, e^{-f}dv)$. Suppose that from every point p in N, there are asymptotic rays to $\gamma^\pm$ such that* $\mathrm{Ric}_f \geq 0$ *and f is of sublinear growth of distance function on both asymptotic rays. Then through every point in N, there exists a line α such that*

$$b^+_\gamma(\alpha^+(t)) = t, \quad b^-_\gamma(\alpha^-(t)) = t.$$

Proof of Lemma 2.8. The proof is a local Fang-Li-Zhang's discussion [5]. For any point $p \in N$, by [5] (or [18]) we firstly can prove that the two asymptotic rays to $\gamma^\pm$ are uniquely determined at p and form a line, say γ_p. We can also show that Busemann functions $b^\pm_\gamma$ at p with $b^+_\gamma + b^-_\gamma = 0$, which are smooth f-harmonic functions (in the barrier sense), satisfy

$$||\nabla b^\pm_\gamma|| = 1 \quad \text{and} \quad \operatorname{Hess} b^\pm_\gamma = 0$$

on γ_p when $\mathrm{Ric}_f \geq 0$ and f is of sublinear growth. Notice that the restriction of $b^\pm_\gamma$ to γ_p is a linear function with derivative 1. Therefore the lemma follows by reparametrizing γ_p. □

3. Proof of Theorem 1.1

Following the argument of Lemma 3.3 in [2], applying Lemma 2.8 we get

Lemma 3.1. *Under the same assumptions of Theorem* 1.1*, M cannot admit a line γ, which satisfies the following property:*

$$d(\gamma(t), B_o(R)) \geq |t| + 2R \ \ for \ \ all \ \ t. \tag{3.1}$$

Using Lemma 3.1, we furthermore have

Proposition 3.2. *Under the same assumptions of Theorem* 1.1*, if* $[\gamma_1]$ *and* $[\gamma_2]$ *are two different ends of* M^n*, then for any* $t_1, t_2 \geq 3R$*, we have*

$$d(\gamma_1(t_1), \gamma_2(t_2)) > t_1 + t_2 - 6R.$$

Proof. Assume that the conclusion of the proposition is not true. That is, there exists $\tau_1, \tau_2 \geq 3R$ such that

$$d(\gamma_1(\tau_1), \gamma_2(\tau_2)) \leq \tau_1 + \tau_2 - 6R.$$

Then we want to get a contradiction. Since $[\gamma_1]$ and $[\gamma_2]$ are two different ends, there exists a large number $A > \tau_1 + \tau_2$ such that $\gamma_1(t)$ and $\gamma_2(t)$ are in different unbounded components of $M - B_o(A)$ for all $t > A$. Let σ_t, $t > A$ be a minimal geodesic joining $\gamma_1(t)$ and $\gamma_2(t)$. Then σ_t must pass through $B_o(A)$. Moreover, we *claim* that the middle point m_t of σ_t is in the ball $B_o(2A)$. In fact, let p be a point in $\sigma_t \cap B_o(A)$ and without loss of generality we may assume that $d(p, \gamma_1(t)) \leq d(p, \gamma_2(t))$. Then

$$\begin{aligned} d(o, m_t) &\leq d(o,p) + d(p, m_t) \\ &\leq A + \frac{1}{2}\rho_t - d(p, \gamma_1(t)) \\ &\leq A + \frac{1}{2}\rho_t - (t - A), \end{aligned}$$

where ρ_t is the length of σ_t. On the other hand,

$$\begin{aligned} \rho_t &= d(\gamma_1(t), \gamma_2(t)) \\ &\leq d(\gamma_1(t), \gamma_1(\tau_1)) + d(\gamma_1(\tau_1), \gamma_2(\tau_2)) + d(\gamma_2(\tau_2), \gamma_2(t)) \\ &\leq (t - \tau_1) + \tau_1 + \tau_2 - 6R + (t - \tau_2) \\ &= 2t - 6R. \end{aligned}$$

Hence,

$$\begin{aligned} d(o, m_t) &\leq A + \frac{1}{2}(2t - 6R) - (t - A) \\ &= 2A - 3R. \end{aligned}$$

This implies that the point m_t is in the ball $B_o(2A)$.

Now reparametrizing σ_t by translating the origin, we still denote it by σ_t such that

$$\sigma_t(-\frac{1}{2}\rho_t) = \gamma_1(t), \quad \sigma_t(0) = m_t, \quad \sigma_t(\frac{1}{2}\rho_t) = \gamma_2(t).$$

Then we *claim* that $\sigma_t(s)$ satisfies inequality (3.1) for $-\frac{1}{2}\rho_t \leq s \leq \frac{1}{2}\rho_t$. In fact, for any $s \geq 0$, we have

$$\begin{aligned} d(\sigma_t(s), B_o(R)) &\geq d\left(\sigma_t(\frac{1}{2}\rho_t), B_o(R)\right) - \left(\frac{1}{2}\rho_t - s\right) \\ &\geq (t - R) - \frac{1}{2}(2t - 6R) + s \\ &= s + 2R, \end{aligned}$$

where we used the above fact: $\rho_t \leq 2t - 6R$. Since $\sigma_t(0) \in B_o(2A)$ for all $t \geq A$, when $t \to \infty$ (hence $\rho_t \to \infty$), a subsequence of σ_t converges to a line $\gamma(s)$ satisfying inequality (3.1) for all s. This is a contradiction by Lemma 3.1. $\square$

In particular, Proposition 3.2 implies

Corollary 3.3. *Under the same assumptions of Theorem* 1.1*, if* $[\gamma_1]$ *and* $[\gamma_2]$ *are two different ends of* M^n*, then*

$$d(\gamma_1(4R), \gamma_2(4R)) > 2R.$$

We now apply Corollary 3.3 to prove Theorem 1.1, by adapting the argument of Cai [2].

Proof of Theorem 1.1. Fix a point $o \in M$. Let $\gamma_1, \gamma_2, \ldots, \gamma_k$ be k rays with k different ends starting from the base point o. In the following we only need to bound the number k from above.

We consider the sphere $S_o(4R)$ of radius $4R$, and let $\{p_j\}$ be a maximal set of points on $S_o(4R)$ such that the balls $B_{p_j}(R/2)$ are disjoint from each other. Clearly, the balls $B_{p_j}(R)$ cover $S_o(4R)$. Since the set $\{\gamma_i(4R), i = 1, 2, \ldots, k\}$ is contained in $S_o(4R)$, each $\gamma_i(4R)$ is contained in some $B_{p_j}(R)$. But by the Corollary 3.3, each ball $B_{p_j}(R)$ contains at most one $\gamma_i(4R)$, and hence the number of balls is not less than k. Therefore, to estimate upper bound of the number k, it suffices to bound the number of balls $B_{p_j}(R/2)$.

Notice that

$$B_{p_j}\left(\frac{R}{2}\right) \subset B_o\left(\frac{9}{2}R\right) \subset B_{p_j}\left(\frac{17}{2}R\right).$$

By the f-volume comparison theorem, i.e., (2.2), we have

$$V_f\left(B_{p_j}(\frac{17}{2}R)\right) \le \frac{\int_0^{17/2R}(\sinh^{n-1+4A}\sqrt{K}t)dt}{\int_0^{R/2}(\sinh^{n-1+4A}\sqrt{K}t)dt} V_f\left(B_{p_j}(\frac{R}{2})\right),$$

where $A = \sup_{x \in B_{p_j}(17/2R)} |f(x)|$. Therefore, the number of balls $B_{p_j}(R/2)$ is no more than

$$\frac{\int_0^{17/2R}(\sinh^{n-1+4B}\sqrt{K}t)dt}{\int_0^{R/2}(\sinh^{n-1+4B}\sqrt{K}t)dt}.$$

Here we choose $B := \sup_{x \in B_o(25/2R)} |f(x)|$, due to the fact: $B_{p_j}(17/2R) \subset B_o(25/2R)$.

Since

$$\frac{\int_0^{17/2R}(\sinh^{n-1+4B}\sqrt{K}t)dt}{\int_0^{R/2}(\sinh^{n-1+4B}\sqrt{K}t)dt} \le \frac{2(n+4B)}{n-1+4B} \cdot \frac{e^{\frac{17(n-1+4B)}{2}\sqrt{K}R}}{(\sqrt{K}R)^{n+4B}},$$

the theorem follows. $\square$

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