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THE FREE WREATH PRODUCT OF A DISCRETE GROUP BY A QUANTUM AUTOMORPHISM GROUP

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ABSTRACT. Let $\mathbb{G}$ be the quantum automorphism group of a finite dimensional C*-algebra (B, ψ) and Γ a discrete group. We want to compute the fusion rules of $\widehat{\Gamma} \wr_* \mathbb{G}$. First of all, we will revise the representation theory of $\mathbb{G}$ and, in particular, we will describe the spaces of intertwiners by using noncrossing partitions. It will allow us to find the fusion rules of the free wreath product in the general case of a state ψ. We will also prove the simplicity of the reduced C*-algebra, when ψ is a trace, as well as the Haagerup property of $L^\infty(\widehat{\Gamma} \wr_* \mathbb{G})$, when Γ is moreover finite.

INTRODUCTION

The core idea of the quantum automorphism group is to provide a quantum version of the standard notion of an automorphism group. As Wang proves in [Wan98], given a finite dimensional C*-algebra B endowed with a state ψ, the category of compact quantum groups acting on B and ψ-invariant admits a universal object which is called a quantum automorphism group and is denoted $\mathbb{G}^{aut}(B, \psi)$. In the particular case of $B = \mathbb{C}^n$ endowed with the uniform probability measure ψ, it is easy to prove that $\mathbb{G}^{aut}(\mathbb{C}^n, \psi)$ is the quantum symmetric group S_n^+. The representation theory of $\mathbb{G}^{aut}(B, \psi)$ has been taken into account for the first time by Banica in [Ban99, Ban02]: in such a paper, he showed that, if ψ is a δ-form and $\dim(B) \geq 4$, the spaces of intertwiners can be described by using Temperley-Lieb diagrams; moreover, the irreducible representations and the fusion rules are the same as of $SO(3)$. Later on, in [BS09] Banica and Speicher proved that, in the case of S_n^+ it was possible to describe the intertwiners in a simpler way by using noncrossing partitions. Bichon in [Bic04] introduced the definition of the free wreath product of a compact quantum group by a quantum permutation group and proved that it was possible to endow this product with a compact quantum group structure. Later on, in [BV09] Banica and Vergnioux showed that the quantum reflection group H_n^{s+} is isomorphic to $\widehat{\mathbb{Z}}_s \wr_* S_n^+$ and calculated its fusion rules in the case $n \geq 4$. In [Lem14], Lemeux generalized this result to the free wreath product $\widehat{\Gamma} \wr_* S_n^+$, where Γ is a discrete group and $n \geq 4$. In the last few years, the fusion rules of compact quantum groups have been used as a starting point for the proof of some of their properties such as simplicity, fullness, factoriality and Haagerup property.

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In the present work, we aim to provide a further generalization of such results, by considering the free wreath product of the dual of a discrete group by a quantum automorphism group. The first step is revising the representation theory of $\mathbb{G}^{aut}(B,\psi)$, $\dim(B)\geq 4$ by showing that, for a general δ-form ψ, it is possible to describe the intertwiners by means of noncrossing partitions. By relying on such a graphical interpretation, we find the fusion rules and the irreducible representations of $H^+_{(B,\psi)}(\widehat{\Gamma}) := \widehat{\Gamma}\wr_* \mathbb{G}^{aut}(B,\psi)$, $\dim(B)\geq 4$. In this case the intertwiners are interpreted by using noncrossing partitions decorated with the elements of the discrete group Γ. In the last section we prove that, when ψ is a δ-trace, $C_r(H^+_{(B,\psi)})$ is simple with unique trace if $\dim(B)\geq 8$, while $L^\infty(\widehat{\Gamma}\wr_* \mathbb{G})$ has the Haagerup property if Γ is finite. All these results are also presented in a more general form, by dropping the δ condition for ψ and by proving that in this case the free wreath product can be decomposed as a free product of quantum groups in which the condition still holds.

1. The quantum automorphism group $\mathbb{G}^{aut}(B,\psi)$

In this section we recall some known results concerning the quantum automorphism group and the category of noncrossing partitions, in order to give a new description of its intertwining spaces, by using noncrossing partitions instead of Temperley-Lieb diagrams.

A Woronowicz compact quantum group (see [Wor87, Wor98]) is a pair $(C(\mathbb{G}),\Delta)$ where $C(\mathbb{G})$ is a unital C*-algebra and $\Delta : C(\mathbb{G}) \longrightarrow C(\mathbb{G})\otimes C(\mathbb{G})$ a $*$-homo morphism together with a family of unitary matrices $(u^\alpha)_{\alpha\in I}$, $u^\alpha \in M_{d_\alpha}(C(\mathbb{G}))$, such that:

- the $*$-subalgebra generated by the entries u^α_{ij} of the u^α is dense in $C(\mathbb{G})$,
- for all $\alpha\in I$ and $1\leq i,j\leq d_\alpha$, we have $\Delta(u^\alpha_{ij}) = \sum_{k=1}^{d_\alpha} u^\alpha_{ik}\otimes u^\alpha_{kj}$,
- for all $\alpha \in I$ the transposed matrix $(u^\alpha)^t$ is invertible.

A finite dimensional representation of the compact quantum group $(C(\mathbb{G}),\Delta)$ is a matrix $v\in M_n(C(\mathbb{G}))$ such that $\Delta(v_{ij}) = \sum_{k=1}^n v_{ik}\otimes v_{kj}$. Given two representations $v\in M_{n_v}(C(\mathbb{G}))$ and $w\in M_{n_w}(C(\mathbb{G}))$, the space of intertwiners is $Hom(v,w) = \{T\in M_{n_w,n_v}(\mathbb{C}) |\ (T\otimes 1)v = w(T\otimes 1)\}$.

We recall that every finite dimensional C*-algebra B is isomorphic to a multimatrix C*-algebra. In the following we will consider the decomposition $B = \bigoplus_{\alpha=1}^c M_{n_\alpha}(\mathbb{C})$. Let $\mathscr{B} = \{(e^\alpha_{ij})_{i,j=1,\dots,n_\alpha}, \alpha = 1,\dots,c\}$ be the canonical basis of matrix units. Suppose B is endowed with the usual multiplication $m : B\otimes B \longrightarrow B$, $m(e^\alpha_{ij}\otimes e^\beta_{kl}) = \delta_{jk}\delta_{\alpha\beta}e^\alpha_{il}$, and unity $\eta : \mathbb{C}\longrightarrow B$, $\eta(1) = \sum_{\alpha=1}^c\sum_{i=1}^{n_\alpha} e^\alpha_{ii}$. Every finite dimensional C*-algebra B can be endowed with a Hilbert space structure by considering the scalar product $\langle x,y\rangle = \psi(y^*x)$ which is induced by a faithful state ψ. By normalizing the basis $\mathscr{B}$ with respect to the norm associated to this scalar product, we get the orthonormal basis $\mathscr{B}' = \{b^\alpha_{ij} | b^\alpha_{ij} = \psi(e^\alpha_{jj})^{-\frac{1}{2}} e^\alpha_{ij},\ i,j = 1,\dots,n_\alpha,\ \alpha = 1,\dots,c\}$.

We are now ready to give one of the main definitions of the paper. The quantum automorphism group of a finite dimensional C*-algebra (B,ψ) is the universal object in the category of the compact quantum groups acting on B and leaving the state ψ invariant (see [Ban99, Wan98]). More precisely it can be defined as follows.

Definition 1.1. Let B be an n-dimensional C*-algebra with multiplication $m : B\otimes B \longrightarrow B$ and unity $\eta : \mathbb{C} \longrightarrow B$. Let ψ be a state on B. Consider the following universal unital C*-algebra:

$$C(\mathbb{G}^{aut}(B,\psi)) = \langle (u_{ij})_{i,j=1}^n | u=(u_{ij}) \text{ is unitary}, m\in Hom(u^{\otimes 2},u), \eta \in Hom(1,u)\rangle.$$

Then $C(\mathbb{G}^{aut}(B,\psi))$ endowed with the usual comultiplication $\Delta(u_{ij}) = \sum_{k=1}^n u_{ik}\otimes u_{kj}$ is the quantum automorphism group of the C*-algebra (B,ψ).

Definition 1.2. Let B be an n-dimensional C*-algebra as in Definition 1.1 and $\delta > 0$. A faithful state $\psi : B \longrightarrow \mathbb{C}$ is a δ-form, if the multiplication map of B and its adjoint with respect to the inner product induced by ψ satisfy $mm^* = \delta \cdot \mathrm{id}_B$. If such a ψ is also a trace, then it is called a tracial δ-form or a δ-trace.

Remark 1.3. For every faithful state $\psi : M_n(\mathbb{C}) \longrightarrow \mathbb{C}$, there exists $Q \in M_n(\mathbb{C})$, $Q > 0$, $\mathrm{Tr}(Q) = 1$ such that $\psi = \mathrm{Tr}(Q\cdot)$. In particular, we notice that every such ψ is a δ-form, with $\delta = \mathrm{Tr}(Q^{-1})$. Broadly speaking, if $B = \bigoplus_{\alpha=1}^c M_{n_\alpha}(\mathbb{C})$, then every faithful state ψ over B is of the form $\psi = \bigoplus_{\alpha=1}^c \mathrm{Tr}(Q_\alpha\cdot)$ for a suitable family $Q_\alpha \in M_{n_\alpha}(\mathbb{C})$, $Q_\alpha > 0$, $\sum_\alpha \mathrm{Tr}(Q_\alpha) = 1$. In particular, if $\mathrm{Tr}(Q_\alpha^{-1}) = \delta$ for all α, then ψ is a δ-form.

Every positive complex matrix is diagonalizable. It follows that the matrices Q_α are always similar to diagonal matrices with positive real eigenvalues. The following proposition shows that we can always assume Q_α to be diagonal.

Proposition 1.4. *Let $B = \bigoplus_{\alpha=1}^c M_{n_\alpha}(\mathbb{C})$ be a finite dimensional* C*-*algebra and $\phi = \bigoplus_{\alpha=1}^c \mathrm{Tr}(Q_\alpha\cdot)$, $\psi = \bigoplus_{\alpha=1}^c \mathrm{Tr}(P_\alpha\cdot)$ two states on B. Suppose that the families of matrices Q_α and P_α are similar, i.e., there exists a family of unitary matrices U_α such that $P_\alpha = U_\alpha^* Q_\alpha U_\alpha$. Then, the quantum automorphism groups $\mathbb{G}^{aut}(B,\phi)$ and $\mathbb{G}^{aut}(B,\psi)$ are isomorphic.*

Proof. The proof consists of showing that, for a good choice of the basis of B, the two groups are generated by the same relations. Let $U = \bigoplus_{\alpha=1}^c U_\alpha$ and consider the $*$-homomorphism $\pi : (B,\psi) \longrightarrow (B,\phi)$ given by $b \mapsto UbU^*$. First of all, we observe that $\phi = \psi \circ \pi$; it follows that, for all $b, c \in B$, $\langle b,c\rangle_\psi = \psi(c^*b) = \phi(\pi(c^*b)) = \phi(\pi(c)^*\pi(b)) = \langle \pi(b),\pi(c)\rangle_\phi$, where the scalar products are induced by the two states on B. In particular, any orthonormal basis of B with respect to ψ is sent by π to an orthonormal basis with respect to ϕ. Let $(b_i)_i$ be an orthonormal basis of (B,ψ) and consider the quantum group $\mathbb{G}^{aut}(B,\psi)$ as introduced in Definition 1.1. $C(\mathbb{G}^{aut}(B,\psi))$ is generated by the coefficients of the matrix u^ψ subject to some relations. The core of such a proof is that, by considering $\pi(b_i)$ as the basis of B, the C*-algebra $C(\mathbb{G}^{aut}(B,\phi))$ is generated by a matrix u^ϕ with the same relations as $C(\mathbb{G}^{aut}(B,\psi))$. If we use these well chosen presentations, it follows that the isomorphism between $C(\mathbb{G}^{aut}(B,\psi))$ and $C(\mathbb{G}^{aut}(B,\phi))$ is simply given by $u_{ij}^\psi \mapsto u_{ij}^\phi$. □

Because of Proposition 1.4 in the following we will always be able to consider B as a multimatrix C*-algebra endowed with a δ-form ψ such that the associated matrices Q_α are diagonal. We denote $Q_{i,\alpha}$ as the eigenvalue in position (i,i) of Q_α and observe that $\psi(e_{ij}^\alpha) = \mathrm{Tr}(Q_\alpha e_{ij}^\alpha) = \delta_{ij} Q_{i,\alpha}$.

The representation theory of $\mathbb{G}^{aut}(B,\psi)$ is well known from [Ban02], but, in order to generalize it to the free wreath product, we need a description of the intertwining spaces in terms of noncrossing partitions, as it was done for $\mathbb{G}^{aut}(\mathbb{C}^n,\mathrm{tr})$

(see [BS09]). For this aim, we need to recall some standard definitions about noncrossing partitions.

Definition 1.5. Let $k, l \in \mathbb{N}$. We use the notation $NC(k,l)$ to denote the set of noncrossing partitions between k upper points and l lower points. These noncrossing partitions can be represented by diagrams obtained by connecting k points lying on an upper imaginary line and l points lying on a lower line in all the possible ways by using strings that pass only in the part of the plane which lies between the two rows of points and that do not cross each other. A block in a noncrossing partition is a set of points connected to each other. A point without connections can be considered as a trivial block. The total number of blocks of $p \in NC(k,l)$ is denoted $b(p)$.

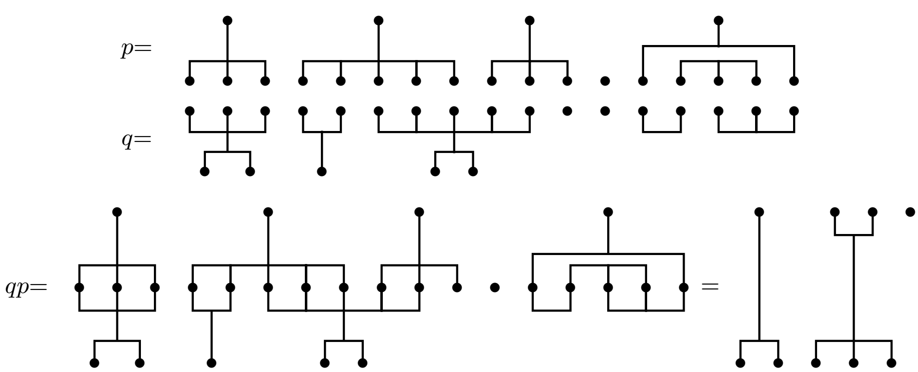
Definition 1.6. Let $p \in NC(k,l), q \in NC(v,w)$. We define the following diagram operations:

(1) the tensor product $p \otimes q$ is the diagram in $NC(k+v, l+w)$ obtained by horizontal concatenation of the diagrams p and q,
(2) if $l = v$ it is possible to define the composition qp as the diagram in $NC(k,w)$ obtained by identifying the lower points of p with the upper points of q and by removing all the blocks which have possibly appeared and which contain neither one of the upper points of p nor one of the lower points of q; such operation, when it is defined, is associative,
(3) the adjoint p^* is the diagram in $NC(l,k)$ obtained by reflecting the diagram p with respect to an horizontal line which lies between the rows of the upper and of the lower points.

Notation 1.7. The blocks possibly removed when multiplying two noncrossing partitions $p \in NC(k,l)$, $q \in NC(l,w)$, will be called central blocks and their number denoted by $cb(p,q)$. Furthermore, the vertical concatenation can produce some (closed) cycles which will not appear in the final noncrossing partition either. Intuitively, they are the rectangles which are obtained when two or more central points are connected both in the upper and in the lower noncrossing partition (see the example below). In a more formal way, the number of cycles, denoted $cy(p,q)$, is defined as

$$cy(p,q) = l + b(qp) + cb(p,q) - b(p) - b(q).$$

Example 1.8. In order to clarify the multiplication operation and the concepts of block, central block as well as cycle, we can think of $p \in NC(4,17)$ and $q \in NC(17,5)$ in the following example:



We suddenly have $b(p) = 6$, $b(q) = 7$, $b(qp) = 3$ and $cb(p, q) = 1$. Then, the number of cycles is $cy(p, q) = 17 + 3 + 1 - 6 - 7 = 8$.

In the following proposition, we introduce a simple but useful relation concerning the number of cycles obtained by multiplying three noncrossing partitions.

Proposition 1.9. *Let $p \in NC(k, l)$, $r \in NC(l, m)$ and $s \in NC(m, v)$. Then the following relation holds:*

$$cy(p, sr) = cy(p, r) + cy(rp, s) - cy(r, s). \tag{1.1}$$

Proof. By making use of the definition of a cycle and of the associativity of the composition, the relation (1.1) reduces to $cb(p, sr) = cb(p, r) + cb(rp, s) - cb(r, s)$. In order to complete the proof, it is then enough to observe that $cb(p, sr) = cb(p, r)$, because the number of central blocks obtained by concatenating p and sr does not depend on the noncrossing partition s; similarly $cb(rp, s) = cb(r, s)$. □

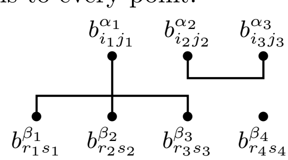
The next goal is to assign a linear map to every noncrossing partition. In order to define such a map, we will make use of the following notation which generalizes the classical one.

Notation 1.10. Consider a diagram $p \in NC(k, l)$ and associate to every point a matrix unit of the canonical basis of B. Let $(b^{\alpha_1}_{i_1j_1}, \dots, b^{\alpha_k}_{i_kj_k})$ be the ordered set of elements associated to the upper points and $(b^{\beta_1}_{r_1s_1}, \dots, b^{\beta_l}_{r_ls_l})$ the elements associated to the lower points. Let ij and α be the multi-index notation for the indices of these matrices, in particular $ij = ((i_1, j_1), \dots, (i_k, j_k))$ and $\alpha = (\alpha_1, \dots, \alpha_k)$; in a similar way, we define rs and β.

Denote by b_v, $v = 1, \dots, m$ the different blocks of p, and let $b_v^{\uparrow}$ ($b_v^{\downarrow}$) be the ordered product of the matrix units associated to the upper (lower) points of the block b_v. Such a product is conventionally the identity matrix, if there are no upper (lower) points in the block. Define

$$\delta_p^{\alpha,\beta}(ij, rs) := \prod_{v=1}^{m} \psi((b_v^{\downarrow})^* b_v^{\uparrow}). \tag{1.2}$$

Example 1.11. Consider the following noncrossing partition p in which we associated an element of the basis to every point:



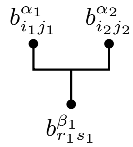
In this case the coefficient just introduced is

$$\delta_p^{\alpha,\beta}(ij, rs) = \psi((b^{\beta_1}_{r_1s_1} b^{\beta_2}_{r_2s_2} b^{\beta_3}_{r_3s_3})^* b^{\alpha_1}_{i_1j_1}) \psi(b^{\alpha_2}_{i_2j_2} b^{\alpha_3}_{i_3j_3}) \psi((b^{\beta_4}_{r_4s_4})^*).$$

Definition 1.12. We associate to every element $p \in NC(k, l)$ the linear map $T_p : B^{\otimes k} \longrightarrow B^{\otimes l}$ which is defined by:

$$T_p(b^{\alpha_1}_{i_1j_1} \otimes \cdots \otimes b^{\alpha_k}_{i_kj_k}) = \sum_{r,s,\beta} \delta_p^{\alpha,\beta}(ij, rs) b^{\beta_1}_{r_1s_1} \otimes \cdots \otimes b^{\beta_l}_{r_ls_l}.$$

Example 1.13. The diagram p which is associated to the multiplication map m (writing explicitly $b^{\alpha_1}_{i_1,j_1}, b^{\alpha_2}_{i_2,j_2}$ on the upper points and $b^{\beta_1}_{r_1,s_1}$ on the lower point) is:



Here, by applying the definition

$$\begin{aligned}
&\delta_p^{\alpha,\beta}((i_1,j_1,i_2,j_2),(r_1,s_1))\\
&= \psi((\psi(e^{\beta_1}_{s_1s_1})^{-\frac{1}{2}} e^{\beta_1}_{r_1s_1})^* \psi(e^{\alpha_1}_{j_1j_1})^{-\frac{1}{2}} e^{\alpha_1}_{i_1j_1} \psi(e^{\alpha_2}_{j_2j_2})^{-\frac{1}{2}} e^{\alpha_2}_{i_2j_2})\\
&= \psi(e^{\beta_1}_{s_1s_1})^{-\frac{1}{2}} \psi(e^{\alpha_1}_{j_1j_1})^{-\frac{1}{2}} \psi(e^{\alpha_2}_{j_2j_2})^{-\frac{1}{2}} \psi(e^{\beta_1}_{s_1r_1} e^{\alpha_1}_{i_1j_1} e^{\alpha_2}_{i_2j_2})\\
&= \delta_{\beta_1,\alpha_1}\delta_{\alpha_1,\alpha_2}\delta_{r_1,i_1}\delta_{j_1,i_2}\delta_{s_1,j_2}\psi(e^{\alpha_1}_{j_1j_1})^{-\frac{1}{2}}
\end{aligned}$$

so the associated map $T_p : B^{\otimes 2} \longrightarrow B$ is given by

$$T_p(b^{\alpha_1}_{i_1j_1} \otimes b^{\alpha_2}_{i_2j_2}) = \delta_{\alpha_1,\alpha_2}\delta_{j_1,i_2}\psi(e^{\alpha_1}_{j_1j_1})^{-\frac{1}{2}} b^{\alpha_1}_{i_1j_2}$$

which is the multiplication m.

We have then the following compatibility result (see [BS09, Proposition 1.9] for the case of $\mathbb{C}^n$ with the canonical trace).

Proposition 1.14. *Let $p \in NC(l,k), q \in NC(v,w)$. We have:*

(1) $T_{p\otimes q} = T_p \otimes T_q$,
(2) $T_p^* = T_{p^*}$,
(3) *if $k = v$, then* $T_{qp} = \delta^{-cy(p,q)} T_q T_p$.

Proof. Even if the core of the proof is essentially the same as [BS09], it is necessary to pay much more attention to the computations.

The relation 1 is clear because $\delta_p^{\alpha,\beta}(ij,rs)\delta_q^{\alpha',\beta'}(IJ,RS) = \delta_{p\otimes q}^{\alpha\alpha',\beta\beta'}(ijIJ,rsRS)$.

The relation 2 follows from $\delta_p^{\alpha,\beta}(ij,rs) = \delta_{p^*}^{\beta,\alpha}(rs,ij)$, which is true because $\psi((b^\alpha_{ij})^*) = \psi(b^\alpha_{ij})$ (we point out that the matrix associated to ψ has real eigenvalues).

The relation 3 is more difficult to prove true. In particular, we need to show that

$$\sum_\beta \sum_{r,s=1}^{n_\beta} \delta_p^{\alpha,\beta}(ij,rs)\delta_q^{\beta,\gamma}(rs,RS) = \delta^{cy(p,q)}\delta_{qp}^{\alpha,\gamma}(ij,RS). \tag{1.3}$$

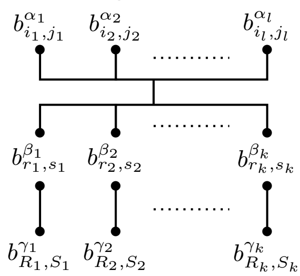
We remark that every noncrossing partition is obtained by using compositions, tensor products and adjoints of the basic morphisms m, η and id_B. In order to prove the composition formula between the maps associated to two noncrossing partitions p and q, we can think of decomposing q in the composition of a sequence of noncrossing partitions corresponding to elementary maps of type $\mathrm{id}_B^{\otimes u} \otimes f \otimes \mathrm{id}_B^{\otimes v}$ where $u, v \in \mathbb{N}$ and $f = \mathrm{id}_B, m, m^*, \eta, \eta^*$.

If we suppose that relation 3 holds for the composition of a general noncrossing partition p with this kind of map and let $q = q_s \cdots q_2 q_1$ be such a decomposition of q, then we have

$$\begin{aligned}
T_{qp} &= T_{q_s\cdots q_2 q_1 p}\\
&= \delta^{-cy(q_{s-1}\cdots q_1 p, q_s)} T_{q_s} T_{q_{s-1}\cdots q_1 p}\\
&= \delta^{-cy(q_{s-1}\cdots q_1 p, q_s) - cy(q_{s-2}\cdots q_1 p, q_{s-1}) - \cdots - cy(p,q_1)} T_{q_s} T_{q_{s-1}} \cdots T_{q_1} T_p\\
&= \delta^{-cy(q_{s-1}\cdots q_1 p, q_s) - cy(q_{s-2}\cdots q_1 p, q_{s-1}) - \cdots - cy(p,q_1) + cy(q_1,q_2)} T_{q_s} \cdots T_{q_2 q_1} T_p\\
&= \delta^{-cy(q_{s-1}\cdots q_1 p, q_s) - \cdots - cy(p,q_1) + cy(q_1,q_2) + \cdots + cy(q_{s-1}\cdots q_1, q_s)} T_{q_s\cdots q_2 q_1} T_p\\
&= \delta^{-cy(p, q_s\cdots q_1)} T_{q_s\cdots q_1} T_p\\
&= \delta^{-cy(p,q)} T_q T_p
\end{aligned}$$

where the second to last equality follows by applying s times Proposition 1.9.

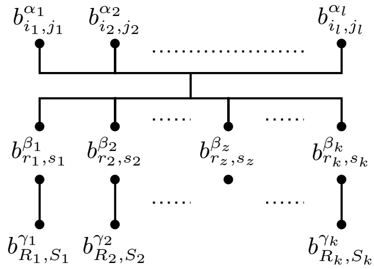
Now we only need to show the relation (1.3) when composing the map associated to p with an elementary map. Furthermore, we can consider that the noncrossing partition p has only one block (or possibly two when $f = m$); indeed it is possible to reconstruct the multi-block case by using a tensor product argument or by generalizing the following proof in an obvious way. Let's now start the computations in the different cases. Let $p \in NC(l,k)$ be a one block noncrossing partition. The first case we take into account is the composition of T_p with $T_q = \mathrm{id}_B^{\otimes k}$. The corresponding diagram is the following:



In this case relation (1.3) is satisfied; indeed,

$$\begin{aligned}
&\sum_\beta \sum_{r,s=1}^{n_\beta} \delta_p^{\alpha,\beta}(ij,rs)\delta_q^{\beta,\gamma}(rs,RS)\\
&= \sum_\beta \sum_{r,s=1}^{n_\beta} \psi((b^{\beta_1}_{r_1,s_1}\cdots b^{\beta_k}_{r_k,s_k})^*(b^{\alpha_1}_{i_1,j_1}\cdots b^{\alpha_l}_{i_l,j_l}))\prod_{t=1}^k \psi((b^{\gamma_t}_{R_t,S_t})^* b^{\beta_t}_{r_t,s_t})\\
&= \sum_\beta \sum_{r,s=1}^{n_\beta} \psi((b^{\beta_1}_{r_1,s_1}\cdots b^{\beta_k}_{r_k,s_k})^*(b^{\alpha_1}_{i_1,j_1}\cdots b^{\alpha_l}_{i_l,j_l}))\prod_{t=1}^k \delta_{\gamma_t\beta_t}\delta_{R_t r_t}\delta_{S_t s_t}\\
&= \psi((b^{\gamma_1}_{R_1,S_1}\cdots b^{\gamma_k}_{R_k,S_k})^*(b^{\alpha_1}_{i_1,j_1}\cdots b^{\alpha_l}_{i_l,j_l}))\\
&= \delta_{qp}^{\alpha,\gamma}(ij,RS).
\end{aligned}$$

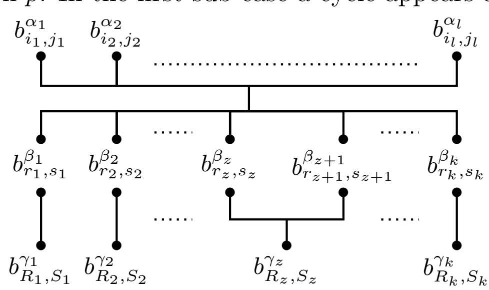
A second possible case is the composition of T_p with $T_q = \mathrm{id}_B^{\otimes u} \otimes \eta^* \otimes \mathrm{id}_B^{\otimes v}$. Here there are two different situations which deserve to be considered. If $T_p = \eta$, a simple computation shows that $\eta^*\eta = \mathrm{id}_{\mathbb{C}}$ (because ψ is normalized) and the relation (1.3) is satisfied. For all the other possible T_p, the general diagram is the following.



With respect to the previous case in the lower noncrossing partition q, one of the identity maps was replaced by η^* (in this case, with a little abuse of notation, we removed $b^{\gamma_z}_{R_z,S_z}$ but we did not reassign the index z). The relation (1.3) is verified because

$$\begin{aligned}
&\sum_{\beta}\sum_{r,s=1}^{n_\beta} \delta_p^{\alpha,\beta}(ij,rs)\delta_q^{\beta,\gamma}(rs,RS)\\
=\;& \sum_{\beta}\sum_{r,s=1}^{n_\beta} \psi((b^{\beta_1}_{r_1,s_1}\cdots b^{\beta_k}_{r_k,s_k})^*(b^{\alpha_1}_{i_1,j_1}\cdots b^{\alpha_l}_{i_l,j_l}))\psi(b^{\beta_z}_{r_z,s_z})\prod_{t=1,t\neq z}^{k}\psi((b^{\gamma_t}_{R_t,S_t})^* b^{\beta_t}_{r_t,s_t})\\
=\;& \sum_{\beta}\sum_{r,s=1}^{n_\beta} \psi((b^{\beta_1}_{r_1,s_1}\cdots b^{\beta_k}_{r_k,s_k})^*(b^{\alpha_1}_{i_1,j_1}\cdots b^{\alpha_l}_{i_l,j_l}))\delta_{r_z s_z} Q^{\frac{1}{2}}_{s_z,\beta_z}\prod_{t=1,t\neq z}^{k}\delta_{\gamma_t\beta_t}\delta_{R_t r_t}\delta_{S_t s_t}\\
=\;& \sum_{\beta}\sum_{r_z=1}^{n_\beta} \psi((b^{\gamma_1}_{R_1,S_1}\cdots e^{\beta_z}_{r_z,s_z}\cdots b^{\gamma_k}_{R_k,S_k})^*(b^{\alpha_1}_{i_1,j_1}\cdots b^{\alpha_l}_{i_l,j_l}))\\
=\;& \psi((b^{\gamma_1}_{R_1,S_1}\cdots b^{\gamma_{z-1}}_{R_{z-1},S_{z-1}} b^{\gamma_{z+1}}_{R_{z+1},S_{z+1}}\cdots b^{\gamma_k}_{R_k,S_k})^*(b^{\alpha_1}_{i_1,j_1}\cdots b^{\alpha_l}_{i_l,j_l}))\\
=\;& \delta^{\alpha,\gamma}_{qp}(ij,RS)
\end{aligned}$$

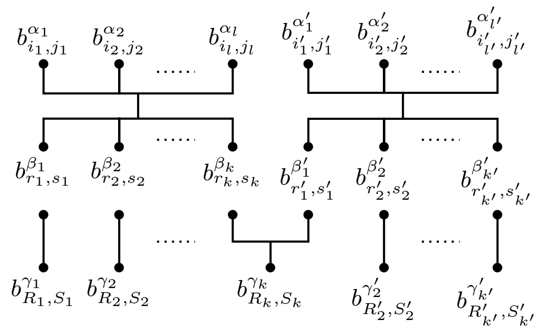
The third case we analyze is the composition of T_p with $T_q = \mathrm{id}_B^{\otimes u}\otimes m\otimes \mathrm{id}_B^{\otimes v}$. There are two different situations which deserve to be considered: the two upper points of m can be connected either to one block or to two different blocks of the noncrossing partition p. In the first sub-case a cycle appears and the diagram is:



As in the previous case, one of the points of the lower line was removed and its index (in this case $z+1$) was not reassigned. The relation (1.3) is still verified and the factor δ implied by the cycle appears. Indeed, we have

$$\begin{aligned}
&\sum_{\beta}\sum_{r,s=1}^{n_\beta}\delta_p^{\alpha,\beta}(ij,rs)\delta_q^{\beta,\gamma}(rs,RS)\\
=\;&\sum_{\beta}\sum_{r,s=1}^{n_\beta}\psi((b_{r_1,s_1}^{\beta_1}\cdots b_{r_k,s_k}^{\beta_k})^*(b_{i_1,j_1}^{\alpha_1}\cdots b_{i_l,j_l}^{\alpha_l}))\psi((b_{R_z,S_z}^{\gamma_z})^*b_{r_z,s_z}^{\beta_z}b_{r_{z+1},s_{z+1}}^{\beta_{z+1}})\\
&\qquad\qquad\qquad\qquad\times\prod_{t=1,t\neq z,z+1}^{k}\psi((b_{R_t,S_t}^{\gamma_t})^*b_{r_t,s_t}^{\beta_t})\\
=\;&\sum_{\beta}\sum_{r,s=1}^{n_\beta}\psi((b_{r_1,s_1}^{\beta_1}\cdots b_{r_k,s_k}^{\beta_k})^*(b_{i_1,j_1}^{\alpha_1}\cdots b_{i_l,j_l}^{\alpha_l}))\delta_{\beta_z\gamma_z}\delta_{\beta_{z+1}\gamma_z}\delta_{r_zR_z}\delta_{s_zr_{z+1}}\delta_{s_{z+1}S_z}\\
&\qquad\qquad\qquad\qquad\times Q_{s_z,\beta_z}^{-\frac{1}{2}}\prod_{t=1,t\neq z,z+1}^{k}(\delta_{\gamma_t\beta_t}\delta_{R_tr_t}\delta_{S_ts_t})\\
=\;&\sum_{s_z=1}^{n_{\gamma_z}}Q_{s_z,\gamma_z}^{-1}\psi((b_{R_1,S_1}^{\gamma_1}\cdots e_{R_z,s_z}^{\gamma_z}b_{s_z,S_z}^{\gamma_z}b_{R_{z+2},S_{z+2}}^{\gamma_{z+2}}\cdots b_{R_k,S_k}^{\gamma_k})^*(b_{i_1,j_1}^{\alpha_1}\cdots b_{i_l,j_l}^{\alpha_l}))\\
=\;&\delta\cdot\psi((b_{R_1,S_1}^{\gamma_1}\cdots b_{R_z,S_z}^{\gamma_z}b_{R_{z+2},S_{z+2}}^{\gamma_{z+2}}\cdots b_{R_k,S_k}^{\gamma_k})^*(b_{i_1,j_1}^{\alpha_1}\cdots b_{i_l,j_l}^{\alpha_l}))\\
=\;&\delta\cdot\delta_{qp}^{\alpha,\gamma}(ij,RS).
\end{aligned}$$

The diagram of the sub-case with two blocks (with $p\in NC(l+l',k+k')$) follows.



Relation (1.3) is still verified, indeed

$$\begin{aligned}
&\sum_{\beta,\beta'}\sum_{r,s=1}^{n_\beta}\sum_{r',s'=1}^{n_{\beta'}}\delta_p^{\alpha\alpha',\beta\beta'}(ii'jj',rr'ss')\delta_q^{\beta\beta',\gamma\gamma'}(rr'ss',RR'SS')\\
&=\sum_{\beta,\beta'}\sum_{r,s=1}^{n_\beta}\sum_{r',s'=1}^{n_{\beta'}}\psi((b_{r_1,s_1}^{\beta_1}\cdots b_{r_k,s_k}^{\beta_k})^*(b_{i_1,j_1}^{\alpha_1}\cdots b_{i_l,j_l}^{\alpha_l}))\\
&\qquad\qquad\times\psi((b_{r'_1,s'_1}^{\beta'_1}\cdots b_{r'_{k'},s'_{k'}}^{\beta'_{k'}})^*(b_{i'_1,j'_1}^{\alpha'_1}\cdots b_{i'_{l'},j'_{l'}}^{\alpha_l}))\\
&\qquad\qquad\times\prod_{t=1}^{k-1}\psi((b_{R_t,S_t}^{\gamma_t})^*b_{r_t,s_t}^{\beta_t})\psi((b_{R_k,S_k}^{\gamma_k})^*b_{r_k,s_k}^{\beta_k}b_{r'_1,s'_1}^{\beta'_1})\prod_{t=2}^{k'}\psi((b_{R'_t,S'_t}^{\gamma'_t})^*b_{r'_t,s'_t}^{\beta'_t})
\end{aligned}$$

$$= \sum_{\beta,\beta'} \sum_{r,s=1}^{n_\beta} \sum_{r',s'=1}^{n_{\beta'}} \psi((b^{\beta_1}_{r_1,s_1} \cdots b^{\beta_k}_{r_k,s_k})^*(b^{\alpha_1}_{i_1,j_1} \cdots b^{\alpha_l}_{i_l,j_l}))$$
$$\times \psi((b^{\beta'_1}_{r'_1,s'_1} \cdots b^{\beta'_{k'}}_{r'_{k'},s'_{k'}})^*(b^{\alpha'_1}_{i'_1,j'_1} \cdots b^{\alpha_l}_{i'_{l'},j'_{l'}}))$$
$$\times \prod_{t=1}^{k-1} (\delta_{\gamma_t\beta_t}\delta_{R_t r_t}\delta_{S_t s_t})\delta_{\beta_k\gamma_k}\delta_{\beta'_1\gamma_k}\delta_{r_k R_k}\delta_{s_k r'_1}\delta_{s'_1 S_k} Q^{-\frac{1}{2}}_{s_k,\beta_k}$$
$$\times \prod_{t=2}^{k'} (\delta_{\gamma'_t\beta'_t}\delta_{R'_t r'_t}\delta_{S'_t s'_t})$$
$$= \sum_{s_k=1}^{n_{\gamma_k}} Q^{-\frac{1}{2}}_{s_k,\gamma_k} \psi((b^{\gamma_1}_{R_1,S_1} \cdots b^{\gamma_k}_{R_k,s_k})^*(b^{\alpha_1}_{i_1,j_1} \cdots b^{\alpha_l}_{i_l,j_l}))$$
$$\times \psi((b^{\gamma_k}_{s_k,S_k} b^{\gamma'_2}_{R'_2,S'_2} \cdots b^{\gamma'_{k'}}_{R'_{k'},S_{k'}})^*(b^{\alpha'_1}_{i'_1,j'_1} \cdots b^{\alpha'_{l'}}_{i'_{l'},j'_{l'}}))$$
$$= Q^{-\frac{1}{2}}_{j_l,\gamma_k} \psi((b^{\gamma_1}_{R_1,S_1} \cdots b^{\gamma_k}_{R_k,j_l})^*(b^{\alpha_1}_{i_1,j_1} \cdots b^{\alpha_l}_{i_l,j_l}))$$
$$\times \psi((b^{\gamma_k}_{j_l,S_k} b^{\gamma'_2}_{R'_2,S'_2} \cdots b^{\gamma'_{k'}}_{R'_{k'},S'_{k'}})^*(b^{\alpha'_1}_{i'_1,j'_1} \cdots b^{\alpha'_{l'}}_{i'_{l'},j'_{l'}}))$$
$$= Q^{-1}_{j_l,\gamma_k} \psi((b^{\gamma_1}_{R_1,S_1} \cdots e^{\gamma_k}_{R_k,j_l})^*(b^{\alpha_1}_{i_1,j_1} \cdots b^{\alpha_l}_{i_l,j_l}))$$
$$\times \psi((b^{\gamma_k}_{j_l,S_k} b^{\gamma'_2}_{R'_2,S'_2} \cdots b^{\gamma'_{k'}}_{R'_{k'},S'_{k'}})^*(b^{\alpha'_1}_{i'_1,j'_1} \cdots b^{\alpha'_{l'}}_{i'_{l'},j'_{l'}}))$$
$$= \psi((b^{\gamma_1}_{R_1,S_1} \cdots b^{\gamma_k}_{R_k,S_k} b^{\gamma'_2}_{R'_2,S'_2} \cdots b^{\gamma'_{k'}}_{R'_{k'},S'_{k'}})^*(b^{\alpha_1}_{i_1,j_1} \cdots b^{\alpha_l}_{i_l,j_l} b^{\alpha'_1}_{i'_1,j'_1} \cdots b^{\alpha'_{l'}}_{i'_{l'},j'_{l'}}))$$
$$= \delta^{\alpha\alpha',\gamma\gamma'}_{qp}(ii'jj', RR'SS')$$

where the second to last equality is obtained by simplifying $Q^{-1}_{j_l,\gamma_k}$ with the value given by the first ψ and by inserting all the δ conditions and coefficients of its argument in the argument of the second ψ. This is essentially due to the fact that the index j_l is in both arguments.

With similar computations, it is finally possible to prove that the formula still holds in the remaining cases of η (trivial case) and m^*. □

Remark 1.15. On the one hand, in the classical case of $S_n^+ = \mathbb{G}^{aut}(\mathbb{C}^n, \mathrm{tr})$ (see Proposition 1.9 in [BS09]), the composition formula we have just proved contains the scalar coefficient $n^{-cb(p,q)}$, which in this case disappears because ψ is normalized (while $\mathrm{tr}(1) = n$). On the other hand, we need to introduce the correction factor $\delta^{-cy(p,q)}$ which is necessary because ψ is a δ-form (while tr is a 1-form).

In conclusion, by generalizing a result from [BS09], we can describe a basis of the space of intertwiners of $\mathbb{G}^{aut}(B,\psi)$ by making use of noncrossing partitions (instead of Temperley-Lieb diagrams as in [Ban02]).

Theorem 1.16. *Let B be an n-dimensional* C*-*algebra, $n \geq 4$ and consider the compact quantum automorphism group $\mathbb{G}^{aut}(B,\psi)$ with fundamental representation u. Then, for all $k, l \in \mathbb{N}$,*

$$Hom(u^{\otimes k}, u^{\otimes l}) = span\{T_p | p \in NC(k,l)\}.$$

Furthermore, the maps associated to distinct noncrossing partitions in $NC(k,l)$ are linearly independent.

Proof. The first inclusion ($\subseteq$) comes from the fact that in $Hom(u^{\otimes k}, u^{\otimes l})$ all the morphisms are generated by using linear combinations of compositions and tensor products of the maps $m, m^*, \eta, \eta^*, \mathrm{id}$. All these 'basic' maps correspond to noncrossing partitions so all the morphisms are linear combinations of suitable T_p.

The other inclusion ($\supseteq$) follows by observing that all noncrossing partitions can be obtained from the basic ones (diagrams of multiplication, unity and identity) by using the operations of Definition 1.6 (this is true because the theorem was already proved for $(B,\psi) = (\mathbb{C}^n, \mathrm{tr})$). The independence of the maps follows from a dimension count, as observed in [BS09], because the dimensions of the intertwining spaces computed in [Ban99] are still true in this case (see also [Ban02]). $\square$

We recall now a proposition attributed to Brannan in [DCFY14] in order to generalize the representation theory to the case of a state ψ:

Proposition 1.17. *Let (B,ψ) be a finite dimensional* C*-*-algebra equipped with a state. Let $B = \bigoplus_{i=1}^k B_i$ be the coarsest direct sum decomposition into* C*-*-algebras such that, for each i, the normalization ψ_i of $\psi_{|B_i}$ is a δ_i-form for a suitable δ_i. Then, $\mathbb{G}^{aut}(B,\psi)$ is isomorphic to the free product $\hat{*}_{i=1}^k \mathbb{G}^{aut}(B_i,\psi_i)$.*

The representation theory of a free product has been described by Wang in [Wan95] and it is completely determined by the representation theory of the factors.

2. Intertwiners and fusion rules of $\widehat{\Gamma} \wr_* \mathbb{G}^{aut}(B,\psi)$

In this section, we introduce the free wreath product of a discrete group by a quantum automorphism group and we consider its representation theory.

Definition 2.1. Let Γ be a discrete group and consider the quantum automorphism group $\mathbb{G}^{aut}(B,\psi)$ where ψ is a faithful state on B. The free wreath product $C(\widehat{\Gamma} \wr_* \mathbb{G}^{aut}(B,\psi))$ is the universal unital C*-algebra with generators $a(g) \in \mathcal{L}(B) \otimes C(\widehat{\Gamma} \wr_* \mathbb{G}^{aut}(B,\psi))$, $g \in \Gamma$, and relations such that:

- $a(g)$ is unitary for every $g \in \Gamma$,
- $m \in Hom(a(g) \otimes a(h), a(gh))$ for every $g, h \in \Gamma$,
- $\eta \in Hom(1, a(e))$.

Such a universal C*-algebra can be endowed with a compact quantum group structure, but as far as this construction is concerned, we need to go deeper into the generators $a(g)$.

Notation 2.2. Consider the matrices $a = (a_{ij,\alpha}^{kl,\beta})$ and $b = (b_{ij,\alpha}^{kl,\beta})$ with coefficients in a C*-algebra where $1 \le \alpha,\ \beta \le c$, $1 \le i, j \le n_\alpha$, $1 \le k, l \le n_\beta$. The operations of multiplication and adjoint are defined respectively by:

$$(ab)_{ij,\alpha}^{kl,\beta} = \sum_{\gamma=1}^{c} \sum_{r,s=1}^{n_\gamma} a_{rs,\gamma}^{kl,\beta} b_{ij,\alpha}^{rs,\gamma}, \qquad a^* = ((a_{kl,\beta}^{ij,\alpha})^*)_{ij,\alpha}^{kl,\beta}.$$

Remark 2.3. The generators of $C(\widehat{\Gamma} \wr_* \mathbb{G}^{aut}(B,\psi))$ can be seen as matrices of type $a(g) = (a_{ij,\alpha}^{kl,\beta}(g))$, $1 \le \alpha, \beta \le c$, $1 \le i, j \le n_\alpha$, $1 \le k, l \le n_\beta$, $g \in \Gamma$. By using the conventions introduced in Notation 2.2, we can change the three conditions of

Definition 2.1 into the following relations:

$$\sum_{l=1}^{n_\gamma} Q_{l,\gamma}^{-\frac{1}{2}} a_{ik,\alpha}^{rl,\gamma}(g) a_{pj,\beta}^{ls,\gamma}(h) = \delta_{\alpha\beta}\delta_{kp} Q_{k,\alpha}^{-\frac{1}{2}} a_{ij,\alpha}^{rs,\gamma}(gh),$$

$$\sum_{k=1}^{n_\alpha} Q_{k,\alpha}^{-\frac{1}{2}} a_{ik,\alpha}^{rp,\beta}(g) a_{kj,\alpha}^{qs,\gamma}(h) = \delta_{\beta\gamma}\delta_{pq} Q_{p,\beta}^{-\frac{1}{2}} a_{ij,\alpha}^{rs,\beta}(gh),$$

$$\sum_{\alpha=1}^{c}\sum_{j=1}^{n_\alpha} Q_{j,\alpha}^{\frac{1}{2}} a_{jj,\alpha}^{kl,\beta}(e) = \delta_{kl} Q_{l,\beta}^{\frac{1}{2}}, \qquad \sum_{\beta=1}^{c}\sum_{k=1}^{n_\beta} Q_{k,\beta}^{\frac{1}{2}} a_{ij,\alpha}^{kk,\beta}(e) = \delta_{ij} Q_{i,\alpha}^{\frac{1}{2}},$$

$$(a_{ij,\alpha}^{kl,\beta}(g))^* = (\frac{Q_{l,\beta}}{Q_{j,\alpha}})^{\frac{1}{2}} (\frac{Q_{k,\beta}}{Q_{i,\alpha}})^{-\frac{1}{2}} a_{ji,\alpha}^{lk,\beta}(g^{-1}). \tag{2.1}$$

Proposition 2.4. *There exists a unique comultiplication map on $C(\widehat{\Gamma} \wr_* \mathbb{G}^{aut}(B,\psi))$ such that:*

$$\Delta : C(\widehat{\Gamma} \wr_* \mathbb{G}^{aut}(B,\psi)) \longrightarrow C(\widehat{\Gamma} \wr_* \mathbb{G}^{aut}(B,\psi)) \otimes C(\widehat{\Gamma} \wr_* \mathbb{G}^{aut}(B,\psi)),$$

$$\Delta(a_{ij,\alpha}^{kl,\beta}(g)) = \sum_{\gamma=1}^{c}\sum_{r,s=1}^{n_\gamma} a_{rs,\gamma}^{kl,\beta}(g) \otimes a_{ij,\alpha}^{rs,\gamma}(g).$$

Then $(C(\widehat{\Gamma} \wr_* \mathbb{G}^{aut}(B,\psi)), \Delta)$ is a compact quantum group, denoted $H^+_{(B,\psi)}(\widehat{\Gamma})$.

Proof. This proof consists of verifying that the defining properties of a compact quantum group are satisfied. We observe that the matrices $a(g)$ are unitary and, by construction, their entries generate a dense $*$-subalgebra of $C(\widehat{\Gamma} \wr_* \mathbb{G}^{aut}(B,\psi))$. The existence and uniqueness of the suitable comultiplication Δ follow by an easy computation. We only need to prove that the transposed matrices $(a(g))^t$ are invertible. For every $(a(g))^t$, the inverse is given by $b(g) = ((\frac{Q_{l,\beta}}{Q_{j,\alpha}})^{-\frac{1}{2}} (\frac{Q_{k,\beta}}{Q_{i,\alpha}})^{\frac{1}{2}} a_{ji,\alpha}^{lk,\beta}(g^{-1}))_{ij,\alpha}^{kl,\beta}$; indeed, we have that

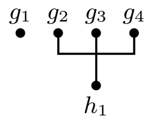
$$\begin{aligned}(b(g)(a(g))^t)_{rs,\gamma}^{kl,\beta} &= \sum_{\alpha=1}^{c}\sum_{i,j=1}^{n_\alpha} (\frac{Q_{l,\beta}}{Q_{j,\alpha}})^{-\frac{1}{2}} (\frac{Q_{k,\beta}}{Q_{i,\alpha}})^{\frac{1}{2}} a_{ji,\alpha}^{lk,\beta}(g^{-1}) a(g)_{ij,\alpha}^{rs,\gamma} \\ &= \delta_{\beta\gamma}\delta_{kr} \sum_{\alpha=1}^{c}\sum_{j=1}^{n_\alpha} (\frac{Q_{l,\beta}}{Q_{j,\alpha}})^{-\frac{1}{2}} a_{jj,\alpha}^{ls,\beta}(e) \\ &= \delta_{\beta,\gamma}\delta_{kr}\delta_{ls}\end{aligned}$$

so $b(g)(a(g))^t = \mathrm{Id}$. In the same way it is possible to prove that $(a(g))^t b(g) = \mathrm{Id}$ so $a(g)$ is invertible and $H^+_{(B,\psi)}(\widehat{\Gamma})$ is a compact quantum group. □

Remark 2.5. The result given in Proposition 1.4 is clearly still valid for the free wreath product $H^+_{(B,\psi)}(\widehat{\Gamma})$ so the matrices Q_α associated to the state ψ on B will always be assumed to be diagonal.

We can now focus on the representation theory of $H^+_{(B,\psi)}(\Gamma)$ when ψ is a δ-form.

Notation 2.6. We denote by $NC_{\widehat{\Gamma}}(g_1,\dots,g_k;h_1,\dots,h_l)$ the set of diagrams in $NC(k,l)$ where the k upper points are decorated by some $g_i \in \Gamma$ and the l lower points by elements $h_j \in \Gamma$ such that, in every block, the product of the upper elements is equal to the product of the lower elements (with the convention that, if the block connects only upper or only lower points, the product must be the unit of Γ). For example



is in $NC_{\widehat{\Gamma}}(g_1, g_2, g_3, g_4; h_1)$ if $g_1 = e$, $g_2g_3g_4 = h_1$.

The operations between noncrossing partitions introduced in Definition 1.6 as well as the results of Proposition 1.14 naturally extend to decorated diagrams.

Proposition 2.7. *Let*

$$p \in NC_{\widehat{\Gamma}}(g_1, \dots, g_k; h_1, \dots, h_l), q \in NC_{\widehat{\Gamma}}(g'_1, \dots, g'_v; h'_1, \dots, h'_w).$$

We have:

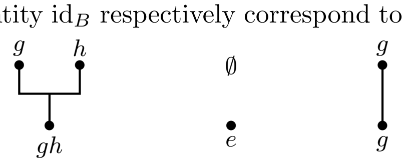
(1) $T_{p\otimes q} = T_p \otimes T_q$ *where*
$$p \otimes q \in NC_{\widehat{\Gamma}}(g_1, \dots, g_k, g'_1, \dots, g'_v; h_1, \dots, h_l, h'_1, \dots, h'_w)$$
is obtained by horizontal concatenation,

(2) $T_p^* = T_{p^*}$ *where* $p^* \in NC_{\widehat{\Gamma}}(h_1, \dots, h_l; g_1, \dots, g_k)$ *is obtained by reflecting* p *with respect to an horizontal line which lies between the rows of the upper and of the lower points,*

(3) *if* $(h_1, \dots, h_l) = (g'_1, \dots, g'_v)$ *then* $T_{qp} = \delta^{-cy(p,q)} T_q T_p$ *where*
$$qp \in NC_{\widehat{\Gamma}}(g_1, \dots, g_k; h'_1, \dots, h'_w)$$
is obtained by vertical concatenation.

Proof. The proof is essentially the same as of Proposition 1.14. We have only to observe that the operations between noncrossing partitions are well defined with respect to the decoration of the diagrams, i.e., the operations of tensor product, adjoint and composition always produce diagrams with an admissible decoration. □

Example 2.8. The fundamental maps m, η and id_B can be represented by using decorated noncrossing partitions. In particular, for all $g, h \in \Gamma$ the multiplication m, the unity η and the identity id_B respectively correspond to the following diagrams:



Theorem 2.9. *Let* Γ *be a discrete group and* (B, ψ) *be a finite dimensional* C*-*algebra with a* δ*-form* ψ *and* $\dim(B) \geq 4$. *The spaces of intertwiners of* $\widehat{\Gamma} \wr_* \mathbb{G}^{aut}(B,\psi)$ *are generated by the linear maps associated to some decorated noncrossing partitions. More precisely, for* $g_i, h_j \in \Gamma$, *we have*

$$Hom(a(g_1) \otimes \cdots \otimes a(g_k), a(h_1) \otimes \cdots \otimes a(h_l)) \\ = span\{T_p | p \in NC_{\widehat{\Gamma}}(g_1, \dots, g_k; h_1, \dots, h_l)\}$$

Proof. The first inclusion ($\subseteq$) is clear because the maps in the generator set of the morphisms (multiplication, unity, identity and their adjoints) are associated to the usual noncrossing partitions with all the possible decorations; then they are in the span. First, for the other inclusion, we need to recall that all noncrossing partitions can be built by using tensor products and compositions of multiplication, unity, identity and their adjoints. It is now enough to observe that we are allowed to give every admissible decoration because the 'product rule' (introduced in Notation 2.6) remains valid throughout all the tensoring and composition operations. □

In the same way as in [Lem14, Cor 2.21], it is then possible to prove that

Proposition 2.10. *The basic representations $a(g)$, $g \in \Gamma$ of $H^+_{(B,\psi)}(\widehat{\Gamma})$ are irreducible and pairwise nonequivalent if $g \neq e$; the remaining representation is $a(e) = 1 \oplus \omega(e)$, where $\omega(e)$ is irreducible and nonequivalent to any $a(g), g \neq e$.*

Definition 2.11. Let Γ be a discrete group and $M = \langle \Gamma \rangle$ the monoid of the words over Γ. It is endowed with the following operations:

- involution: $(g_1, \dots, g_k)^- = (g_k^{-1}, \dots, g_1^{-1})$,
- concatenation: $(g_1, \dots, g_k), (h_1, \dots, h_l) = (g_1, \dots, g_k, h_1, \dots, h_l)$,
- fusion: $(g_1, \dots, g_k).(h_1, \dots, h_l) = (g_1, \dots, g_k h_1, \dots, h_l)$.

We can now state the main theorem, generalizing Theorem 2.25 in [Lem14] to the case of $H^+_{(B,\psi)}(\widehat{\Gamma})$. The proof is the same as of [Lem14], because it relies only on the description of the intertwining spaces by using noncrossing partitions.

Theorem 2.12. *The irreducible representations of $H^+_{(B,\psi)}(\widehat{\Gamma})$ are indexed by the words of M and denoted $\omega(x), x \in M$ with involution $\bar{\omega}(x) = \omega(\bar{x})$. In particular, for $g \in \Gamma$, we have $\omega(g) = a(g) \ominus \delta_{g,e} 1$.*

The fusion rules are:

$$\omega(x) \otimes \omega(y) = \sum_{\substack{x=u,t \\ y=\bar{t},v}} \omega(u,v) \oplus \sum_{\substack{x=u,t \\ y=\bar{t},v \\ u\neq\emptyset, v\neq\emptyset}} \omega(u.v).$$

In order to extend these results to the case of a state ψ, we use this proposition.

Proposition 2.13. *Let $B = \bigoplus_{\alpha=1}^c M_{n_\alpha}(\mathbb{C})$ be a finite dimensional C^*-algebra with a state $\psi = \bigoplus_{\alpha=1}^c \mathrm{Tr}(Q_\alpha \cdot)$ on it. The state ψ restricted to every summand $M_{n_\alpha}(\mathbb{C})$ (and normalized) is a δ-form with $\delta = \mathrm{Tr}(Q_\alpha^{-1})$. Consider the decomposition $B = \bigoplus_{i=1}^d B_i$ where every B_i is the direct sum of all the $M_{n_\alpha}(\mathbb{C})$ such that $\mathrm{Tr}(Q_\alpha^{-1})$ is a constant value denoted δ_i. Let ψ_i be the state on B_i which is obtained by normalizing $\psi_{|_{B_i}}$. Then*

$$\widehat{\Gamma} \wr_* \mathbb{G}^{aut}(B,\psi) \cong \hat{*}_{i=1}^k (\widehat{\Gamma} \wr_* \mathbb{G}^{aut}(B_i,\psi_i))$$

is a $*$-isomorphism which intertwines the comultiplications.

Proof. For this proof we will use the relations introduced in Remark 2.3 between the generators of the free wreath product $C(\widehat{\Gamma} \wr_* \mathbb{G}^{aut}(B,\psi))$. First, we notice that the map $mm^* \in Hom(a(g), a(g))$ is such that $mm^*_{|_{B_i}} = \delta_i \mathrm{id}_{B_i}$. The existence of this particular morphism implies that $a^{kl,\beta}_{ij,\alpha}(g) = 0$ for all i, j, k, l whenever $M_{n_\alpha}(\mathbb{C})$ and $M_{n_\beta}(\mathbb{C})$ are not in the same summand B_i. This allows us to simplify the defining relations above and we deduce that, for every i, there is a natural morphism $C(\widehat{\Gamma} \wr_* \mathbb{G}^{aut}(B_i,\psi_i)) \longrightarrow C(\widehat{\Gamma} \wr_* \mathbb{G}^{aut}(B,\psi))$. It follows that, because of the universality of the free product construction, there is a morphism which intertwines the comultiplications

$$*_{i=1}^{k} C(\widehat{\Gamma} \wr_* \mathbb{G}^{aut}(B_i,\psi_i)) \longrightarrow C(\widehat{\Gamma} \wr_* \mathbb{G}^{aut}(B,\psi)).$$

Conversely, by joining the generators and the relations of the different factors of $*_{i=1}^{k} C(\widehat{\Gamma} \wr_* \mathbb{G}^{aut}(B_i,\psi_i))$, we exactly get the generators and the relations of $H^{+}_{(B,\psi)}(\widehat{\Gamma})$ (in the simplified version). Because of the universality of the free wreath product construction we get the inverse morphism which intertwines the comultiplications

$$C(\widehat{\Gamma} \wr_* \mathbb{G}^{aut}(B,\psi)) \longrightarrow *_{i=1}^{k} C(\widehat{\Gamma} \wr_* \mathbb{G}^{aut}(B_i,\psi_i))$$

and the quantum group isomorphism is proved. □

The non-trivial irreducible representations of $\widehat{\Gamma} \wr_* \mathbb{G}^{aut}(B,\psi)$ are then given by an alternating tensor product of non-trivial irreducible representations of the factors $\widehat{\Gamma} \wr_* \mathbb{G}^{aut}(B_i,\psi_i)$ (see [Wan95]).

We conclude this section with a remark about the spectral measure on a subalgebra of $H^{+}_{(B,\psi)}(\widehat{\Gamma})$ (ψ δ-trace) which will be useful in the following section.

Remark 2.14. It is well known that the character of the fundamental representation of a quantum symmetric group follows the free Poisson law (see e.g. [BC07]). Let $\chi(a(e)) := (\mathrm{Tr} \otimes \mathrm{id})(a(e))$ be the character of the representation $a(e)$ of $H^{+}_{(B,\psi)}(\widehat{\Gamma})$ (ψ is a δ-form). The character $\chi(a(e))$ is self-adjoint because of the relation (2.1). Therefore, in order to find its spectral measure, it is enough to compute the moments $h(\chi(a(e))^k)$. By denoting p_k the orthogonal projection onto the fixed points space $Hom(1, a(e)^{\otimes k})$ and by using some classic results of Woronowicz (see [Wor88]) we have $h(\chi(a(e))^k) = h((\mathrm{Tr} \otimes \mathrm{id})(a(e))^k) = \mathrm{Tr}((\mathrm{id} \otimes h)(a(e)^{\otimes k})) = \mathrm{Tr}(p_k) = \dim(Hom(1, a(e)^{\otimes k})) = \#NC(0,k) = C_k$ where C_k are the Catalan numbers. They are exactly the moments of the free Poisson law of parameter 1 so this is the spectral measure of $\chi(a(e))$.

3. Properties of $H^{+}_{(B,\psi)}(\widehat{\Gamma})$

3.1. **Simplicity and uniqueness of the trace for the reduced algebra.** We prove that, under certain conditions, $C_r(H^{+}_{(B,\psi)}(\widehat{\Gamma}))$ is simple with unique trace.

Remark 3.1. The free product decomposition given in Proposition 2.13 implies that the Haar measure of $\widehat{\Gamma} \wr_* \mathbb{G}^{aut}(B,\psi)$ is the free product of the Haar measures of its factors, by using a well-known result of Wang (see [Wan95]). It follows that the decomposition is still true at the level of the reduced C* and von Neumann algebras so the following isomorphisms hold:

$$(C_r(\widehat{\Gamma} \wr_* \mathbb{G}^{aut}(B,\psi)), h) \cong \mathop{*}_{\mathrm{red}\, i=1}^{\ k} (C_r(\widehat{\Gamma} \wr_* \mathbb{G}^{aut}(B_i,\psi_i)), h_i),$$

$$(L^\infty(\widehat{\Gamma} \wr_* \mathbb{G}^{aut}(B,\psi)), h) \cong *_{i=1}^{k} (L^\infty(\widehat{\Gamma} \wr_* \mathbb{G}^{aut}(B_i,\psi_i)), h_i),$$

where h and h_i are the Haar states on the respective C*-algebras.

Proposition 3.2. *Let (B,ψ) be a finite dimensional* C**-algebra endowed with a trace ψ. Let Γ be a discrete group, $|\Gamma| \geq 4$. Consider the free product decomposition of the reduced* C**-algebra $C_r(H^{+}_{(B,\psi)}(\widehat{\Gamma}))$ given in Remark* 3.1. *If there is either only one factor (i.e. ψ is a δ-trace) and* $\dim(B) \geq 8$ *or there are two or more factors with* $\dim(B_i) \geq 4$ *for all i, then $C_r(H^{+}_{(B,\psi)}(\widehat{\Gamma}))$ is simple with a unique trace given by the free product of the Haar measures.*

Proof. If there are two or more factors the result follows from a proposition of Avitzour (see [Avi82, Section 3]). It states that, given two C*-algebras A and B endowed with tracial Haar states h_A and h_B, the reduced free product C*-algebra $A \underset{\text{red}}{*} B$ is simple with unique trace, if there exist two unitary elements of $\mathrm{Ker}(h_A)$ which are orthogonal with respect to the scalar product induced by h_A and a unitary element in $\mathrm{Ker}(h_B)$. In order to show that, in our case, these elements exist we use a result from [DHR97, Proposition 4.1 (i)] according to which, if a C*-algebra A endowed with a normalized trace τ admits an abelian sub-C*-algebra B so that the spectral measure corresponding to $\tau_{|B}$ is diffuse, then there is a unitary element $u \in A$ such that $\tau(u^n) = 0$ for each $n \in \mathbb{Z}, n \neq 0$. We aim to apply such a proposition to every factor of the decomposition in order to satisfy the Avitzour condition. But this follows from Remark 2.14 where we observed that, when considering the generator $a(e)$ of an indecomposable free wreath product $\widehat{\Gamma} \wr_* \mathbb{G}^{aut}(B, \psi)$, ψ δ-trace, the spectral measure associated to its character $\chi(a(e))$ is the free Poisson law of parameter 1, which is diffuse. The simplicity and uniqueness of the trace in the multifactor case are then proved.

In the second case, when ψ is a δ-trace and there is not a free product decomposition, the proof is a generalization of the proof presented in [Lem14, Theorem 3.5] where $H_n^+(\Gamma)$ is replaced by $H^+_{(B,\psi)}(\widehat{\Gamma})$ and relies on the simplicity of $\mathbb{G}^{aut}(B, \psi)$ for a δ-trace proved in [Bra13]. $\square$

3.2. **Haagerup property.** The aim of this subsection is to prove that, under some hypothesis, the von Neumann algebra $L^\infty(H^+_{(B,\psi)}(\Gamma))$ has the Haagerup property.

Proposition 3.3. *Let (B, ψ) be a finite dimensional* C*-*algebra endowed with a trace ψ and Γ be a finite group. Consider the free product decomposition of the von Neumann algebra $L^\infty(\widehat{\Gamma} \wr_* \mathbb{G}^{aut}(B, \psi))$ given in Remark* 3.1. *If for each i, $\dim(B_i) \geq 4$, then $L^\infty(\widehat{\Gamma} \wr_* \mathbb{G}^{aut}(B, \psi))$ has the Haagerup property.*

Proof. If ψ is in particular a δ-trace, i.e., the decomposition contains only one factor, the proof used in [Lem14, Subsection 3.3] still applies. The multifactor case is easily reduced to the previous one recalling that the Haagerup property is stable under tracial free products (see [Boc93, Proposition 3.9]) and by using Remark 3.1. $\square$

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