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INCOHERENT COXETER GROUPS

KASIA JANKIEWICZ AND DANIEL T. WISE

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Abstract. We use probabilistic methods to prove that many Coxeter groups are incoherent. In particular, this holds for Coxeter groups of uniform exponent > 2 with sufficiently many generators.

1. Introduction

A *Coxeter group* G is given by the presentation

$$\langle a_1, \ldots, a_r \mid a_i^2, (a_i a_j)^{m_{ij}} : 1 \leq i < j \leq r \rangle$$

where $m_{ij} \in \{2, 3, \ldots, \infty\}$ and where $m_{ij} = \infty$ means no relator of the form $(a_i a_j)^{m_{ij}}$. Throughout this paper all presentations of Coxeter groups are of the above form. It is traditional to encode the above data for G in terms of an associated labelled graph Υ_G, whose vertices correspond to the generators and where an edge labelled by m_{ij} joins vertices a_i, a_j when $m_{ij} < \infty$. We omit an edge for $m_{ij} = \infty$.

Definition 1.1. A group G is *coherent* if every finitely generated subgroup of G is finitely presented. Otherwise, G is *incoherent*.

Our main result which is stated and proven as Theorem 3.3 is the following:

Theorem 1.2. *For each M there exists $R = R(M)$ such that if K is a Coxeter group with $3 \leq m_{ij} \leq M$ and rank $r \geq R$, then K is incoherent.*

Our result joins a similar result for groups acting properly and cocompactly on Bourdon buildings [Wis11] and we expect that there is more to come in this direction.

2. Preliminaries on Coxeter groups, walls, and Morse theory

2.1. **Euler characteristic and compression.** Let G be a Coxeter group given by

$$\langle a_1, \ldots, a_r \mid a_i^2, (a_i a_j)^{m_{ij}} : 1 \leq i < j \leq r \rangle$$

and let X be the standard 2-complex associated to this presentation. Consider an index d torsion-free subgroup G' of G. Let $\widehat{X} \to X$ be a cover of X corresponding to G'. All edges embed in $\widehat{X}$, since all generators are torsion elements, and all 2-cells embed since each proper subword of $(a_i a_j)^{m_{ij}}$ is a torsion element. Consider the complex $\overline{X}$ obtained from $\widehat{X}$ by first collapsing 2-cells corresponding to a_i^2 relators

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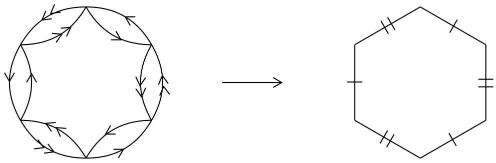

FIGURE 1. The compression $\widehat{X} \to \overline{X}$. Each bigon collapses to an edge and six 2-cells collapse to a single 2-cell.

to 1-cells and second collapsing $2m_{ij}$ copies of $2m_{ij}$-gons with the same boundary when $m_{ij} \neq \infty$. The complex $\overline{X}$ is the *compression* of $\widehat{X}$. See Figure 1 for the compression arising from $\langle a, b \mid a^2, b^2, (ab)^3 \rangle$.

We say G has *dimension* ≤ 2 when $\overline{X}$ is aspherical. This holds precisely when $\frac{1}{m_{ij}} + \frac{1}{m_{jk}} + \frac{1}{m_{ki}} \leq 1$ for each i, j, k. Indeed, there is then a natural metric of nonpositive curvature on $\overline{X}$ induced by metrizing each 2-cell as a regular Euclidean polygon. However if some 3-generator subgroup is finite, then $\overline{X}$ contains a copy of S^2. We focus on Coxeter groups of dimension ≤ 2, in which case the following discussion of $\chi(G)$ is sensible.

The complex X has one 0-cell, r 1-cells and one 2-cell for each pair of generators $\{i, j\}$ with $m_{ij} < \infty$. As degree$(\widehat{X} \to X) = d$, the complex $\widehat{X}$ has d 0-cells, dr 1-cells and d 2-cells for each pair $\{i, j\}$ with $m_{ij} < \infty$. The complex $\overline{X}$ has d 0-cells, $\frac{dr}{2}$ 1-cells and $\frac{d}{2m_{ij}}$ 2-cells for each pair $\{i, j\}$ with $m_{ij} < \infty$. The *Euler characteristic of* G is:

$$\chi(G) = \frac{\chi(\overline{X})}{[G : G']} = \frac{1}{d}\left(d - \frac{dr}{2} + \sum_{\{i,j\}} \frac{d}{2m_{ij}}\right) = 1 - \frac{r}{2} + \sum_{\{i,j\}} \frac{1}{2m_{ij}}. \tag{1}$$

This is independent of the choice of a finite index torison-free subgroup. We thus have:

$$\chi(G_{(r,m)}) = 1 - \frac{r}{2} + \frac{(r-1)r}{4m}.$$

Thus if m is fixed, then $\chi(G_{(r,m)}) > 0$ for all sufficiently large r.

2.2. **Walls.** Let K be a combinatorial 2-complex with the property that each 2-cell has an even number of sides (we have in mind $K = \overline{X}$ as defined in the previous section). Two 1-cells in the attaching map $\partial_{\mathsf{p}} C \to K^1$ of a 2-cell C are *parallel* if they are images of opposite edges in $\partial_{\mathsf{p}} C$. An *abstract wall* is an equivalence class of 1-cells in the equivalence relation generated by parallelism. A *wall* W associated to an abstract wall $\bar{W}$ is a graph with a locally injective map $\phi : W \to K$ defined as follows:

- for each 1-cell a in $\bar{W}$ there is a vertex v_a in W,
- $\phi(v_a)$ is the center of a,
- for each pair of 1-cells a, a' in $\bar{W}$ and each 2-cell C in which a, a' are parallel, there is an edge $(v_a, v_{a'})$ in W,
- the edge $(v_a, v_{a'})$ is sent by ϕ to an arc in C joining $\phi(v_a)$ and $\phi(v_{a'})$.

The wall W is *dual* to each 1-cell in $\bar{W}$. The wall W is *adjacent to x at a vertex v* of link(x), if it is dual to the 1-cell corresponding to v. The wall W is *adjacent to x at an edge e* of link(x), if W is not adjacent at either endpoint of e but is dual

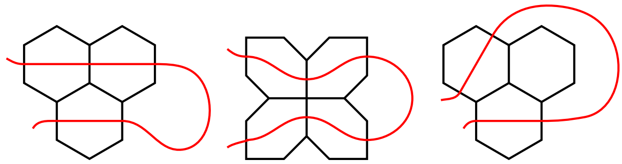

FIGURE 2. In each case above, the wall self-osculates at the central vertex.

to a pair of 1-cells in $\partial_{\mathsf{p}} C$ where C is a 2-cell corresponding to e. We say that the wall W

- *embeds* if $W \to K$ is injective,
- is *two-sided* if $W \to K$ extends to an embedding $W \times (-1, 1) \to K$,
- *self-osculates at* x if it is adjacent to x at more than one vertex and/or edge of $\operatorname{link}(x)$. See Figure 2.

2.3. **Orientation of walls.** An embedded wall $W \to K$ is two-sided if and only if there is a globally consistent orientation of its dual 1-cells such that parallel 1-cells in any 2-cell C have opposite orientations in $\partial_{\mathsf{p}} C$. An *orientation* of a two-sided wall W is one of two globally consistent orientations of its dual 1-cells. Let $\mathcal{W}$ be the set of all walls in K. An *orientation* on $\mathcal{W}$ is a choice of orientation on each $W \in \mathcal{W}$.

2.4. **Bestvina-Brady Morse theory.** An *affine complex* K has cells that are convex Euclidean polyhedra, which metrically agree on their faces. A map $f : K \to \mathbb{R}$ is a *Morse function* if it is linear on each cell C, constant on C if and only if $\dim C = 0$, and the image $f(K^0)$ of the 0-skeleton is a closed discrete subset of $\mathbb{R}$. It follows that the restriction of f to a cell has a unique minimum and maximum.

Let $x \in K^0$. A vertex $v \in \operatorname{link}(x)$ is *ascending* (resp. *descending*) if the corresponding 1-cell is oriented away from x (resp. toward x). An edge $e \in \operatorname{link}(x)$ is *ascending* (resp. *descending*) if each wall passing through the corresponding 2-cell is oriented away from x (resp. toward x). The *ascending link* $\operatorname{link}_{\uparrow}(x)$ (resp. *descending link* $\operatorname{link}_{\downarrow}(x)$) is the subgraph of $\operatorname{link}(x)$ consisting of all ascending (resp. descending) vertices and edges.

We will employ the following result of Bestvina-Brady proven in [BB97, Thm 4.1]:

Theorem 2.1. *Let K be a finite (aspherical) affine cell complex. Consider a map $K \to S^1$ that lifts to a Morse function $\widetilde{K} \to \mathbb{R}$. If $\operatorname{link}_{\uparrow}(x)$ and $\operatorname{link}_{\downarrow}(x)$ are nonempty and connected for each $x \in \widetilde{K}^0$, then $\ker(\pi_1 K \to \mathbb{Z})$ is finitely generated.*

2.5. **An orientation induces a combinatorial map $K^1 \to S^1$.** Let S^1 have a cell structure with one 0-cell and one (oriented) 1-cell. Each orientation on $\mathcal{W}$ determines an orientation preserving combinatorial map $K^1 \to S^1$. The map $\partial_{\mathsf{p}} C \to S^1$ is null-homotopic for each 2-cell C, since pairs of opposite 1-cells in $\partial_{\mathsf{p}} C$ travel in opposite directions around S^1. Thus the map $K^1 \to S^1$ extends to $K \to S^1$. The map $K \to S^1$ lifts to $\widetilde{K} \to \widetilde{S}^1 \simeq \mathbb{R}$, but the restriction of this map to a 2-cell does not necessarily have a unique minimum or maximum.

The *lawful subcomplex* $Y \subset K$ is the subcomplex of K obtained by discarding 2-cells whose attaching maps cannot be expressed as the concatenation $\alpha\beta^{-1}$ where $\alpha \to K^1$, $\beta \to K^1$ are positively directed paths. The restriction $Y \to S^1$ of the map $K \to S^1$ lifts to $\widetilde{Y} \to \mathbb{R}$ which is a Morse function in the sense of Bestvina-Brady.

3. Main theorem

The *Coxeter group of uniform exponent* m *and rank* r is the Coxeter group $G_{(r,m)}$ with the following presentation:

$$\langle a_1, \ldots, a_r \mid a_i^2, (a_i a_j)^m \ : \ 1 \leq i < j \leq r \rangle.$$

The standard 2-complex of the above presentation for $G_{(r,m)}$ is denoted by $X_{(r,m)}$.

Theorem 3.1. *For each $m \geq 3$ there exists R_m such that for all $r \geq R_m$ the group $G_{(r,m)}$ has a finite index torsion-free subgroup G' that admits an epimorphism $G' \to \mathbb{Z}$ whose kernel N is finitely generated.*

Corollary 3.2. *For $m \geq 3$, the group $G_{(r,m)}$ is incoherent for all sufficiently large r.*

Proof. A result of Bieri in [Bie81] states that a nontrivial finitely presented normal subgroup of a group of cohomological dimension ≤ 2 is either free or of finite index. Since $[G' : N] = \infty$ it remains to exclude the case where N is free, whence:

$$\chi(G') = \chi(N) \cdot \chi(\mathbb{Z}) = (1 - \operatorname{rank}(N)) \cdot 0 = 0.$$

This is impossible for all sufficiently large r, since then $\chi(G') > 0$ (see Section 2.1). $\square$

A *Coxeter subgroup* is generated by a subset of the generators of G. It is presented by those generators together with all relators in those generators appearing in the presentation of G [Dav08]. We now prove the main result stated in the introduction:

Theorem 3.3. *For each M there exists $R = R(M)$ such that if K is a Coxeter group with $3 \leq m_{ij} \leq M$ and rank $r \geq R$, then K is incoherent.*

Proof. The multi-color version of Ramsey's theorem [GRS80] states that given a number of colors c and natural numbers $n_1, \ldots, n_c$ there exists a number $R = R(n_1, \ldots, n_c)$ such that if the edges of a complete graph Γ of order at least R are colored with c colors, then for some i there exists a complete subgraph of Γ of order n_i with edges of color i. Let $c = M$ and $n_i = R_i$ of Theorem 3.1. Consequently there exists a uniform exponent Coxeter subgroup $G_{(r,m)}$ of K for some $m \leq M$ and $r = R_m$. By Corollary 3.2 the subgroup $G_{(r,m)}$ is incoherent and hence so is K. $\square$

The above results lend credence to the following:

Conjecture 3.4. *Let G be a finitely generated infinite Coxeter group of dimension ≤ 2. If $\chi(G) > 0$, then G is incoherent.*

3.1. **A polynomial degree finite cover of $X_{(r,m)}$ with good walls.** The goal of this subsection is to prove the following:

Proposition 3.5. *There is a homomorphism $\beta : G_{(r,m)} \to Q^{k(r)}$ such that the compression $\overline{X}_{(r,m)}$ of the induced cover $\widehat{X}_{(r,m)} \to X_{(r,m)}$ has the following property: each wall is 2-sided, embedded and has no self-osculation.*

Moreover $|\overline{X}^0_{(r,m)}|$ is at most $|Q|^{k(r)} \leq |Q| r^C$ for some constant C.

The proof of Proposition 3.5 appears at the end of this subsection.

A *partition* of a set S is a map $p : S \to \{1, 2, 3, 4\}$. The partition p *separates* a, b, c, d if $p(a), p(b), p(c), p(d)$ are distinct.

Lemma 3.6. *Let S have cardinality $r \geq 4$. There is a collection of*

$$k = k(r) = \left\lceil \frac{\log \binom{r}{4}}{\log \frac{32}{29}} \right\rceil$$

partitions such that each quadruple of distinct elements of S is separated by this collection.

Proof. Let M denote the set of all partitions of S, and note that $|M| = 4^r$. Let $\mathcal{M}_k$ denote the collection of cardinality k subsets of M and note that $|\mathcal{M}_k| = \binom{4^r}{k}$. Let $\mathcal{N}_k \subset \mathcal{M}_k$ be the subcollection consisting of sets of k partitions that do not separate some quadruple. We want to show that $|\mathcal{N}_k| < |\mathcal{M}_k|$. Let $\mathcal{N}_k(\{a,b,c,d\}) \subset \mathcal{M}_k$ be the subcollection of sets that fail to separate $a, b, c, d \in S$. We have

$$|\mathcal{N}_k| \leq \binom{r}{4} |\mathcal{N}_k(\{a,b,c,d\})|$$

since there are $\binom{r}{4}$ quadruples $\{a,b,c,d\}$ of distinct elements of S. There are $4! \cdot 4^{r-4} = 6 \cdot 4^{r-3}$ partitions that separate a, b, c, d. Thus there are $4^r - 6 \cdot 4^{r-3} = \frac{29}{32} \cdot 4^r$ partitions that do not separate a, b, c, d. We thus have

$$|\mathcal{N}_k(\{a,b,c,d\})| = \binom{\frac{29}{32} \cdot 4^r}{k}.$$

Observe that we have the following:

$$\binom{\frac{29}{32} \cdot 4^r}{k} < \left(\frac{29}{32}\right)^k \binom{4^r}{k}.$$

Since $k \geq \log \binom{r}{4} / \log \frac{32}{29}$ we have

$$\binom{r}{4} \left(\frac{29}{32}\right)^k \leq 1.$$

Altogether we have

$$|\mathcal{N}_k| \leq \binom{r}{4} |\mathcal{N}_k(\{a,b,c,d\})| < \binom{r}{4} \left(\frac{29}{32}\right)^k \binom{4^r}{k} \leq \binom{4^r}{k} = |\mathcal{M}_k(P)|. \qquad \square$$

Proof of Proposition 3.5. There is a finite quotient $\psi : G_{(4,m)} \twoheadrightarrow Q$ such that $\ker \psi$ is torsion-free, and the compression $\overline{X}_{(4,m)}$ of the induced cover $\widehat{X}_{(4,m)} \to X_{(4,m)}$ has the following property: each wall in $\overline{X}_{(4,m)}$ is two-sided, embedded and has no self-osculation. This follows from the separability of wall stabilizers [HW10].

Let $S = \{1, \ldots, r\}$. Each partition $p : S \to \{1, 2, 3, 4\}$ defines a homomorphism $\phi_p : G_{(r,m)} \to G_{(4,m)}$ induced by $\phi_p(a_i) = a_{p(i)}$. Let

$$\beta = (\psi \circ \phi_{p_1}, \ldots, \psi \circ \phi_{p_k}) : G_{(r,m)} \to Q^{k(r)}$$

where $(p_1, \ldots, p_k)$ is a collection of partitions from Lemma 3.6. For each partition p there is a map $\overline{\phi}_p : X_{(r,m)} \to X_{(4,m)}$ induced by ϕ_p. We will show that a "wall pathology" in $X_{(r,m)}$ would project to a wall pathology in $X_{(4,m)}$ for a suitable p and hence there are no such wall pathologies. Suppose there is a wall W in $X_{(r,m)}$ that self-intersects within a 2-cell C. Let a_i, a_j be the generators of $G_{(r,m)}$ labelling C. Let $p \in \{p_1, \ldots, p_k\}$ separate i and j. The image $\overline{\phi}_p(W)$ is a wall in $X_{(4,m)}$ that self-intersects, which is a contradiction. Thus walls in $X_{(r,m)}$ embed. We now show that no wall in $X_{(r,m)}$ has a self-osculation. Suppose W in $X_{(r,m)}$ has a self-osculation at some 0-cell x, and let C, C' be 2-cells adjacent to x such that

W is dual to edges in both C, C'. Let $a_i, a_j, a_{i'}, a_{j'}$ be generators that label the boundaries of C, C'. If i, j, i', j' are distinct consider a partition p that separates them. The image $\overline{\phi}_p(W)$ is a wall in $X_{(4,m)}$ that has a self-osculation, which is a contradiction. Otherwise C and C' share one label and we let p be a partition that separates the three distinct generators, and the argument is similar. Hence walls do not have self-osculations in $X_{(r,m)}$. Finally, the fact that all walls of $X_{(r,m)}$ are two-sided follows by considering a single $\overline{\phi}_p : X_{(r,m)} \to X_{(4,m)}$.

Finally, to see that the degree is bounded by a polynomial we observe that:

$$|Q|^k \leq |Q|^{\frac{\log \binom{r}{4}}{\log \frac{32}{29}}+1} = |Q|\binom{r}{4}^{\frac{\log |Q|}{\log \frac{32}{29}}} \leq |Q| r^{\frac{4 \log |Q|}{\log \frac{32}{29}}}. \qquad \square$$

3.2. **Probability of an empty or disconnected link is exponentially small.** Let Γ be a complete graph on r vertices. Consider assigning a vertex to be ascending [respectively descending] with probability $\frac{1}{2}$. Furthermore, for an edge whose vertices are ascending [descending] assign it to be ascending [descending] with probability $\frac{1}{2^{m-2}}$. Let $\Gamma_\uparrow$ [$\Gamma_\downarrow$] be the subgraph of Γ consisting of all ascending [descending] vertices and edges.

Observe that $\Gamma_\uparrow$ is assured to be nonempty and connected if

(1) there exists an ascending vertex in Γ, and
(2) for each pair of distinct vertices $v_1, v_2 \in \Gamma_\uparrow$ there is a third vertex $v_3 \in \Gamma_\uparrow$ such that (v_1, v_3) and (v_2, v_3) are edges in $\Gamma_\uparrow$.

Let P_i denote the probability that condition (1) fails to be satisfied. If Γ *fails* to be nonempty and connected, then at least one of (1), (2) is not satisfied. Consequently

$$\mathbb{P}(\Gamma_\uparrow \text{ fails}) \leq P_1 + P_2.$$

Similarly,

$$\mathbb{P}(\Gamma_\downarrow \text{ fails}) \leq P_1 + P_2.$$

Lemma 3.7. $P_1 = \frac{1}{2^r}$.

Proof. Since no wall in $\mathcal{W}$ has a self-osculation, each wall is adjacent to x at at most one vertex of Γ. Each of the r vertices in Γ is descending with probability $\frac{1}{2}$ and these probabilities are independent. Hence $P_1 = \frac{1}{2^r}$. $\square$

Lemma 3.8. $P_2 \leq \binom{r}{2}\frac{1}{4}(1 - \frac{1}{2^{2m-3}})^{r-2}$.

Proof. For distinct vertices $v_1, v_2 \in \Gamma_\uparrow$ the edge (v_1, v_2) is ascending with probability $\frac{1}{2^{m-2}}$. For a triple v_1, v_2, v_3 of distinct vertices in Γ, where v_1, v_2 are ascending the probability that v_3 is also ascending and both edges $(v_1, v_3), (v_2, v_3)$ are ascending is

$$\frac{1}{2^{2m-3}}.$$

For $v_1, v_2 \in \Gamma_\uparrow$ the probability that there is no connecting v_3 as above equals

$$(1 - \frac{1}{2^{2m-3}})^{r-2}.$$

Thus

$$P_2 \leq \sum_{v_1, v_2 \in \Gamma} \frac{1}{4}(1 - \frac{1}{2^{2m-3}})^{r-2} = \binom{r}{2}\frac{1}{4}(1 - \frac{1}{2^{2m-3}})^{r-2}. \qquad \square$$

Consider orientations on the set of all walls $\mathcal{W}$ of $\overline{X}_{(r,m)}$. We orient each wall randomly, assigning probability $\frac{1}{2}$ to each of two orientations for each wall $W \in \mathcal{W}$. For each 0-cell $x \in \overline{X}_{(r,m)}$ the graph $\text{link}(x)$ is complete on r vertices. No self-osculations in $\overline{X}_{(r,m)}$ provide that walls adjacent to two distinct edges and/or vertices of $\text{link}(x)$ are distinct. Thus every vertex of $\text{link}(x)$ is ascending [descending] with probability $\frac{1}{2}$ and each edge of $\text{link}(x)$ whose edges are ascending [descending] is ascending [descending] with probability $\frac{1}{2^{m-2}}$. We thus have the following:

Corollary 3.9. $\mathbb{P}(\text{link}_{\uparrow}(x)$ *or* $\text{link}_{\downarrow}(x)$ *fails) is exponentially decreasing. Specifically*

$$\begin{aligned}\mathbb{P}(\text{link}_{\uparrow}(x) \textit{ or } \text{link}_{\downarrow}(x) \textit{ fails}) \leq\ & \mathbb{P}(\text{link}_{\uparrow}(x) \textit{ fails}) + \mathbb{P}(\text{link}_{\downarrow}(x) \textit{ fails}) \\ \leq\ & 2(P_1 + P_2) \leq \frac{1}{2^{r-1}} + \binom{r}{2}\frac{1}{2}(1 - \frac{1}{2^{2m-3}})^{r-2}.\end{aligned}$$

3.3. **Proof of Theorem 3.1.**

Proof. Proposition 3.5 provides a finite cover $\widehat{X}_{(r,m)}$ whose degree is bounded by a polynomial in r, and such that the compression $\overline{X} = \overline{X}_{(r,m)}$ has the property that its walls are two-sided and have no self-osculations.

To apply Theorem 2.1 we need to find an orientation on $\mathcal{W}$ such that $\text{link}_{\uparrow}(x)$ and $\text{link}_{\downarrow}(x)$ are nonempty and connected for each $x \in \overline{X}^0$. We orient each $W \in \mathcal{W}$ randomly assigning probability $\frac{1}{2}$ to each of two orientations of W. We need to prove that

$$\mathbb{P}(\text{link}_{\uparrow}(x) \text{ or } \text{link}_{\downarrow}(x) \text{ fails for some } x \in \overline{X}^0) < 1.$$

Since the left hand side is bounded above by

$$\sum_{x \in \overline{X}^0} \mathbb{P}(\text{link}_{\uparrow}(x) \text{ or } \text{link}_{\downarrow}(x) \text{ fails})$$

it suffices to prove that for each $x \in \overline{X}^0$

$$(*) \qquad \mathbb{P}(\text{link}_{\uparrow}(x) \text{ or } \text{link}_{\downarrow}(x) \text{ fails}) < \frac{1}{|\overline{X}^0|}.$$

$|\overline{X}^0|$ is bounded by a polynomial in r, but by Corollary 3.9 the probability on the left decreases exponentially in r, hence the inequality $(*)$ holds for all r greater than some $R(m)$.

After finding an orientation on $\mathcal{W}$ such that $\text{link}_{\uparrow}(x)$ and $\text{link}_{\downarrow}(x)$ are nonempty and connected, we consider the lawful subcomplex $Y \subset \overline{X}$ and the map $\overline{X} \xrightarrow{\phi} S^1$ induced by the orientation whose restriction to Y lifts to a Morse function $\widetilde{Y} \to \mathbb{R}$. By Theorem 2.1 the group $\ker(\pi_1 Y \to \mathbb{Z})$ is finitely generated. Consequently, its quotient $N = \ker(\pi_1 \overline{X} \to \mathbb{Z})$ is also finitely generated. To see that $\pi_1 \overline{X} \to \mathbb{Z}$ is nontrivial, observe that X^1 has a positively directed closed path since $\overline{X}$ is compact and each $\text{link}_{\uparrow}(x)$ is nonempty. □

4. Local quasiconvexity and Coxeter groups with nonpositive sectional curvature

4.1. Negative sectional curvature and local quasiconvexity.

Definition 4.1 (Sectional curvature)**.** An *angled* 2*-complex* is a 2-complex Y with an *angle* $\measuredangle(e) \in \mathbb{R}$ assigned to each edge e of $\text{link}(y)$ for each $y \in Y^0$. As edges in $\text{link}(y)$ correspond to corners of 2-cells at y, we regard the angles as assigned to corners of 2-cells at y. The curvature at a 2-cell f of Y is given by

$$\kappa(f) = 2\pi - \sum_{e \in \text{Corners}(f)} \text{def}(e)$$

where $\text{def}(e) = \pi - \measuredangle(e)$. The *curvature* of Y at y is given by

$$\kappa(y) = 2\pi - \pi\chi(\text{link}(y)) + \sum_{e \in \text{Corners}(y)} \measuredangle(e) = (2-v)\pi + \sum \text{def}(e). \tag{2}$$

A *section* of a combinatorial 2-complex Y at the 0-cell y is a combinatorial immersion $(S, s) \to (Y, y)$. A section is *regular* if $\text{link}(s)$ is finite, connected, nonempty, with no valence ≤ 1 vertex. Pulling back the angles at a corner at y to corners at s, the *curvature* of a section $(S, s) \to (Y, y)$ is defined to be $\kappa(s)$. We say that Y has *sectional curvature* $\leq \alpha$ at y if all regular sections of Y at y have curvature $\leq \alpha$. Finally, Y has *sectional curvature* $\leq \alpha$ if each 2-cell has curvature $\leq \alpha$ and Y has sectional curvature $\leq \alpha$ at each 0-cell.

Definition 4.2 (Quasiconvexity)**.** Let G be a group with a finite generating set S and the Cayley graph $\Gamma(G, S)$. A subgroup H of G is *quasiconvex* if there is a constant $L \geq 0$ such that every geodesic in $\Gamma(G, S)$ between two elements of H lies in the L-neighborhood of H. When G is hyperbolic, the quasiconvexity of H is independent of the generating set of G [Sho91]. A group G is *locally quasiconvex* if every finitely generated subgroup of G is quasiconvex. Every quasiconvex subgroup of a hyperbolic group is finitely presented [Sho91]. Thus a locally quasiconvex hyperbolic group is coherent.

The main result about negative sectional curvature is as follows [Wis04, MPW13]:

Theorem 4.3. *If Y is a compact, piecewise Euclidean nonpositively curved 2-complex whose associated angles have negative sectional curvature, then $\pi_1 Y$ is locally quasiconvex.*

The following is known about locally quasiconvex Coxeter groups:

Proposition 4.4. *For each $r \geq 3$ there exists $N(r)$ such that for all $m > N(r)$ the group $G_{(r,m)}$ is locally quasiconvex.*

We briefly review two ways of proving Proposition 4.4. One method to prove Proposition 4.4 is from [MW05] or [Sch03, Thm IV] and shows that a Coxeter group $G_{(r,m)}$ is locally quasiconvex whenever $m \geq \frac{3}{2}r$. We shall focus on reviewing conditions ensuring negative sectional curvature so that Theorem 4.3 provides Proposition 4.4.

As in Section 2.1, let X be the standard 2-complex of the presentation of $G = G_{(r,m)}$ and let $\overline{X}$ be the compression of a finite cover of X corresponding to a finite index torsion-free subgroup of G. If each 3-generator Coxeter subgroup of G is infinite (i.e. $\frac{1}{m_{ij}} + \frac{1}{m_{jk}} + \frac{1}{m_{ki}} \leq 1$), then there is a natural metric of nonpositive curvature on $\overline{X}$ induced by metrizing each 2-cell as a regular Euclidean polygon. The previous condition is equivalent to the nonpositive curvature of all sections $(S,s) \to (\overline{X},x)$ where S is a disc. Thus we say that G has *nonpositive planar sectional curvature*, when all 3-generator Coxeter subgroups are infinite. Finally, if all exponents satisfy $m_{ij} > \frac{r(r-1)}{2(r-2)}$, then $\overline{X}$ has negative sectional curvature [Wis04, Thm 13.3].

4.2. **Nonpositive sectional curvature.** Let X denote the standard 2-complex of the presentation of Coxeter group G and let $\overline{X}$ denote the compression of a cover of X corresponding to a finite index torsion-free subgroup. There is a surprisingly elegant characterization of nonpositive sectional curvature of $\overline{X}$ in terms of the Euler characteristic of Coxeter subgroups of G.

Theorem 4.5. *The following are equivalent:*

(1) *$\overline{X}$ has nonpositive sectional curvature,*
(2) *$\chi(H) \leq 0$ for each nontrivial Coxeter subgroup $H \subset G$ whose associated graph Υ_H is connected but not a tree.*

Proof. (1) $\Rightarrow$ (2): Suppose $\chi(H) > 0$ and Υ_H is connected and not a tree. We can assume that Υ_H has no valence 1 vertex, since the Coxeter subgroup H' associated to the subgraph $\Upsilon_{H'}$ of Υ_H obtained by removing a valence 1 vertex satisfies $\chi(H') \geq \chi(H)$ by equation (1). A section at a 0-cell of $\overline{X}$ whose vertices correspond to the generators of H has curvature $2\pi\chi(H)$ by comparing equations (1) and (2).

(2) $\Rightarrow$ (1): Let x be a 0-cell of $\overline{X}$. It suffices to consider sections corresponding to the full subgraphs of $\text{link}(x)$. Indeed $\text{def}(e) > 0$ for each edge e since each angle is $< \pi$ and thus adding edges increases κ by the second part of equation (2). Any regular section corresponding to a full subgraph is isomorphic to the associated graph Υ_H of a Coxeter subgroup H and the curvature of the section equals $2\pi\chi(H)$. Thus if the section has positive curvature, then $\chi(H) > 0$. $\square$

Problem 4.6. Let G have a nonpositive planar sectional curvature with $\chi(G) > 0$ and Υ_G connected and not a tree. Is it true that $\pi_1 G$ is incoherent?

We hope that the methods used here can be applied to an appropriate finite index subgroup. An affirmative answer to Problem 4.6 would be a step in proving the following:

Conjecture 4.7. *If G has nonpositive planar sectional curvature, then the following are equivalent:*

(1) *G is coherent,*
(2) *$\overline{X}$ has nonpositive sectional curvature.*

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Department of Mathematics & Statistics, McGill University, Montreal, Quebec, Canada H3A 0B9

E-mail address: kasia@math.mcgill.ca

Department of Mathematics & Statistics, McGill University, Montreal, Quebec, Canada H3A 0B9

E-mail address: wise@math.mcgill.ca