

## RAMSEY SHADOWING AND MINIMAL POINTS

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**ABSTRACT.** We say that a dynamical system  $X$  has the Ramsey shadowing property if an arbitrary sequence of points in  $X$  can be shadowed on a set that is “large” in the sense of Ramsey theory. Our main theorem states that this property is equivalent to the existence of a dense set of minimal points.

### 1. INTRODUCTION

In a recent paper [6], Meddaugh, Raines, and the author proved that, under certain circumstances, an arbitrary sequence in a dynamical system can be shadowed on a set that is “large” in the sense of Ramsey theory. The aim of this paper is to make a more thorough investigation of this property.

By a **dynamical system** we mean a pair  $(X, f)$ , where  $X$  is a compact Hausdorff space and  $f$  is a continuous map from  $X$  to itself. We freely abuse notation by writing  $X$  for  $(X, f)$  when the map  $f$  either has already been specified or need not be specified. A **metric dynamical system**, or simply a **metric system**, is a dynamical system  $(X, f)$  where  $X$  is metrizable.

Let  $X$  be a metric system (with some fixed metric  $d$ ) and let  $\xi = \langle x_n : n \in \mathbb{N} \rangle$  be a sequence of points in  $X$ . If  $\varepsilon > 0$  and  $x \in X$ , we say that  $x$   $\varepsilon$ -**shadows**  $\xi$  on a set  $A \subseteq \mathbb{N}$  if

$$\{n \in \mathbb{N} : d(f^n(x), x_n) < \varepsilon\} \supseteq A.$$

In other words,  $x$   $\varepsilon$ -shadows  $\xi$  on  $A$  whenever the orbit of  $x$  is a good approximation to  $\xi$  on the members of  $A$ .

A **Furstenberg family**, or simply a **family**, is a collection  $\mathcal{F}$  of subsets of  $\mathbb{N}$  that is closed under taking supersets: if  $A \in \mathcal{F}$  and  $A \subseteq B$ , then  $B \in \mathcal{F}$ . A family  $\mathcal{F}$  has the **Ramsey property** if whenever  $A \in \mathcal{F}$  and  $A = \bigcup_{i \leq n} A_i$ , then there is some  $i \leq n$  with  $A_i \in \mathcal{F}$ .

A metric system  $X$  has the **Ramsey shadowing property** if for any  $\varepsilon > 0$  and any family  $\mathcal{F}$  with the Ramsey property, any sequence of points of  $X$  can be  $\varepsilon$ -shadowed on a set in  $\mathcal{F}$ .

In a sense, Ramsey-type theorems like van der Waerden’s Theorem or Hindman’s Theorem tell us that there are no “completely random” finite partitions of  $\mathbb{N}$ : every finite partition will contain sets with nice structural properties. The Ramsey shadowing property tells us something similar about sequences in certain

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dynamical systems: they cannot be “completely random”, but must always be close (in an appropriate sense) to the orbit of some point. In [6] it is shown that every chain transitive system with shadowing has Ramsey shadowing.

The main theorem of this paper is that the Ramsey shadowing property is equivalent to the existence of a dense set of minimal points. Section 2 will contain a review of the standard material needed for this result: families and filters, ultrafilters, and minimality. In Section 3 we prove that the Ramsey shadowing property is equivalent to a local version of the same property. This will allow us to define Ramsey shadowing in arbitrary (not necessarily metric) systems naturally, without using the sometimes awkward language of uniformities. It will also facilitate the proof of the main result in Section 4.

## 2. PRELIMINARIES

A **filter** is a family  $\mathcal{F}$  with the additional property that if  $A, B \in \mathcal{F}$ , then  $A \cap B \in \mathcal{F}$ . An **ultrafilter** is a maximal filter, i.e., a filter that is not properly contained in any other filter. Equivalently, a filter  $\mathcal{F}$  is an ultrafilter if and only if for every  $A \subseteq \mathbb{N}$  either  $A \in \mathcal{F}$  or  $\mathbb{N} \setminus A \in \mathcal{F}$ .

A subset of  $\mathbb{N}$  is **thick** if it contains arbitrarily long intervals, and is **syndetic** if it has bounded gaps. That is,  $A \subseteq \mathbb{N}$  is syndetic if there is some  $k \in \mathbb{N}$  such that every interval of length  $k$  contains a point of  $A$ .  $A \subseteq \mathbb{N}$  is **piecewise syndetic** if it is the intersection of a thick set and a syndetic set.

If  $\mathcal{F}$  is a family, then the **dual** of  $\mathcal{F}$ , denoted  $\mathcal{F}^*$ , is the family of all sets that meet every element of  $\mathcal{F}$ . For example, the thick sets are dual to the syndetic sets, and every ultrafilter is dual to itself. The following simple but elegant result of Glasner establishes a correspondence between filters on  $\mathbb{N}$  and families with the Ramsey property:

**Proposition 2.1.** *A family  $\mathcal{F}$  has the Ramsey property if and only if  $\mathcal{F}^*$  is a filter.*

*Proof.* See [7, Proposition 1.2]. □

**Corollary 2.2.** *If  $\mathcal{F}$  is a family with the Ramsey property, then there is an ultrafilter  $p$  such that every element of  $p$  is in  $\mathcal{F}$ .*

*Proof.* By the previous proposition,  $\mathcal{F}^*$  is a filter. Let  $p$  be any ultrafilter extending  $\mathcal{F}^*$ . If  $A \in p$ , then  $A$  meets every member of  $\mathcal{F}^*$ , so  $A \in \mathcal{F}^{**} = \mathcal{F}$ . □

The set of all ultrafilters on  $\mathbb{N}$  is denoted  $\beta\mathbb{N}$ , and it has a natural topology making it a compact Hausdorff space. This topology is generated by the sets of the form  $\overline{A} = \{p \in \beta\mathbb{N} : A \in p\}$ .  $\mathbb{N}$  is naturally included in  $\beta\mathbb{N}$  if we are willing to identify each  $n \in \mathbb{N}$  with the principal ultrafilter  $\{A \subseteq \mathbb{N} : n \in A\}$ . The topology of  $\beta\mathbb{N}$  is a rich and subtle area of research that we do not go deeply into here; a good introduction can be found in [9].

As usual, we write  $A + n$  for  $\{m + n : m \in A\}$ . For each  $p \in \beta\mathbb{N}$ , define  $\sigma(p)$  to be the unique ultrafilter generated by  $\{A + 1 : A \in p\}$ . This is called the **shift map** on  $\beta\mathbb{N}$ , and whenever we speak of  $\beta\mathbb{N}$  as a dynamical system it is understood that we are talking about the shift map. The shift map is the unique continuous extension to  $\beta\mathbb{N}$  of the map on  $\mathbb{N}$  given by  $n \mapsto n + 1$ .

For a given  $p \in \beta\mathbb{N}$  and a sequence  $\langle x_n : n \in \mathbb{N} \rangle$  of points in some dynamical system  $X$ , we say that  $p\text{-}\lim_{n \in \mathbb{N}} x_n = y$  if and only if for every open  $U \ni y$  we have  $\{n : x_n \in U\} \in p$ .

For a fixed  $x \in X$ , the map  $p \mapsto p\text{-}\lim_{n \in \mathbb{N}} x_n$  is a quotient map (also known as a semi-conjugation) from  $\beta\mathbb{N} \setminus \mathbb{N}$  to  $\omega(x)$  (see Section 2 of [5] for details). This fact will not be used directly in what follows (Proposition 2.3 will be enough), but it does serve to motivate our use of ultrafilter limits in the proof of the main theorem.

Recall that a dynamical system is **minimal** if it admits no proper subsystem. A point  $x \in X$  is a **minimal point** if it belongs to some minimal subsystem of  $X$ . By a straightforward application of Zorn's Lemma, every dynamical system contains minimal subsystems, and hence minimal points. If  $p$  is a minimal point of  $\beta\mathbb{N}$ , we say that  $p$  is a **minimal ultrafilter**.

For any two subsets  $U$  and  $V$  of a dynamical system  $X$ , define  $N(U, V) = \{n \in \mathbb{N} : f^n(U) \cap V \neq \emptyset\}$ . If  $x \in X$  and  $U \subseteq X$  define  $N(x, U) = N(\{x\}, U) = \{n \in \mathbb{N} : f^n(x) \in U\}$ . For a given point  $x$ , we think of  $x$  as having strong recurrence properties whenever  $N(x, U)$  is “large” in some sense for every neighborhood  $U$  of  $x$ . The following proposition says that minimal points enjoy very strong recurrence properties:

**Proposition 2.3.** *Let  $X$  be any dynamical system. The following are equivalent:*

- (1)  $x$  is minimal.
- (2)  $N(x, U)$  is syndetic for every neighborhood  $U$  of  $x$ .
- (3) There is a minimal ultrafilter  $p$  with  $p\text{-}\lim_{n \in \mathbb{N}} f^n(x) = x$ .
- (4) There is some  $y \in X$  and some minimal ultrafilter  $p$  such that  $p\text{-}\lim_{n \in \mathbb{N}} f^n(y) = x$ .

*Proof.* It is well known that (1)  $\Rightarrow$  (2); see, e.g., Exercise 5 in [4]. (2)  $\Rightarrow$  (3) is given in Theorem 19.23 of [8]. (3)  $\Rightarrow$  (4) is trivial. That (4)  $\Rightarrow$  (1) follows from Theorem 3.5 in [4] (which in turn is adapted from [3]).  $\square$

### 3. A LOCAL VERSION OF THE RAMSEY SHADOWING PROPERTY

Let us say that a metric system  $X$  has **local Ramsey shadowing** if for every family  $\mathcal{F}$  with the Ramsey property, every  $\varepsilon > 0$ , and every  $x \in X$ , the constant sequence  $\langle x, x, x, \dots \rangle$  can be  $\varepsilon$ -shadowed on a set in  $\mathcal{F}$ . In other words, this is the definition of Ramsey shadowing, but we have replaced arbitrary sequences with constant sequences.

**Theorem 3.1.** *Ramsey shadowing is equivalent to local Ramsey shadowing in any metric system.*

*Proof.* The forward implication is obvious.

Let  $X$  be a metric system with local Ramsey shadowing. Let  $\xi = \langle x_n : n \in \mathbb{N} \rangle$  be a sequence in  $X$ , let  $\varepsilon > 0$ , and let  $\mathcal{F}$  be any family with the Ramsey property. By Corollary 2.2, there is an ultrafilter  $p$  with  $p \subseteq \mathcal{F}$ . Let  $\{y_i : i \leq n\}$  be a finite set of points in  $X$  such that every point of  $X$  is within  $\frac{\varepsilon}{2}$  of some  $y_i$  (such a set exists because  $X$  is compact). Because  $p$  is an ultrafilter, there is some  $i$  such that

$$A = \left\{ m \in \mathbb{N} : d(x_m, y_i) < \frac{\varepsilon}{2} \right\} \in p.$$

By assumption, there is some  $x \in X$  that  $\frac{\varepsilon}{2}$ -shadows the sequence  $\langle y_i, y_i, y_i, \dots \rangle$  on a set in  $p$ . That is,

$$B = \left\{ m \in \mathbb{N} : d(f^m(x), y_i) < \frac{\varepsilon}{2} \right\} \in p.$$

Because  $p$  is a filter,  $A \cap B \in p$ . Since

$$A \cap B \subseteq \{m \in \mathbb{N}: d(f^m(x), x_m) < \varepsilon\},$$

$x$   $\varepsilon$ -shadows  $\xi$  on a set in  $p$ . Since  $p \subseteq \mathcal{F}$ ,  $x$   $\varepsilon$ -shadows  $\xi$  on a set in  $\mathcal{F}$ .  $\square$

Observe now that the definition of local Ramsey shadowing does not require the mention of constant sequences or a metric:

**Proposition 3.2.** *A metric system  $X$  has Ramsey shadowing if and only if for every open  $U \subseteq X$  and any family  $\mathcal{F}$  with the Ramsey property, there is some  $x \in X$  such that  $N(x, U) \in \mathcal{F}$ .*

We take Proposition 3.2 as a definition of Ramsey shadowing for arbitrary dynamical systems. Explicitly, a dynamical system  $X$  has **Ramsey shadowing** if for every nonempty open set  $U$  and every family  $\mathcal{F}$  with the Ramsey property, there is a point  $x \in X$  such that  $N(x, U) \in \mathcal{F}$ .

At this point, the reader may object that there is already an obvious way to define Ramsey shadowing in arbitrary systems: simply replace the notion of a metric with the notion of a uniformity in the original (non-local) definition. The following proposition asserts that the definition obtained in this manner is equivalent to the one we have chosen. In other words, generalizing the local (rather than the global) definition of Ramsey shadowing is simply a matter of convenience.

**Proposition 3.3.** *The following are equivalent:*

- (1)  $X$  has Ramsey shadowing.
- (2) Let  $\xi$  be an arbitrary sequence in  $X$ , let  $\mathcal{U}$  be any open cover of  $X$ , and let  $\mathcal{F}$  be a family with the Ramsey property. There is a point  $x \in X$  such that

$$\{n \in \mathbb{N}: \text{for some } U \in \mathcal{U}, f^n(x) \in U \text{ and } x_n \in U\} \in \mathcal{F}.$$

*Proof.* To see (1)  $\Rightarrow$  (2), let  $\xi$  be an arbitrary sequence in  $X$ ,  $\mathcal{U}$  an open cover of  $X$ , and  $\mathcal{F}$  a family with the Ramsey property. Let  $p$  be an ultrafilter with  $p \subseteq \mathcal{F}$ . Because  $X$  is compact, there is a finite subcover  $\mathcal{U}'$  of  $\mathcal{U}$ . Because  $\mathcal{U}'$  is finite, there is some  $U \in \mathcal{U}'$  such that  $A = \{n \in \mathbb{N}: x_n \in U\} \in p$ . Because  $X$  has Ramsey shadowing and  $p$  has the Ramsey property, there is some  $x \in X$  such that  $N(x, U) = \{n \in \mathbb{N}: f^n(x) \in U\} \in p$ . Clearly,

$$B = \{n \in \mathbb{N}: \text{for some } U \in \mathcal{U}, f^n(x), x_n \in U\} \supseteq A \cap N(x, U) \in p,$$

so that  $B \in \mathcal{F}$  and (2) is satisfied.

To see (2)  $\Rightarrow$  (1), suppose  $X$  satisfies (2) and let  $U$  be any nonempty open subset of  $X$ . Because  $X$  is  $T_3$ , there is a nonempty open set  $V$  such that  $\overline{V} \subseteq U$ . Let  $y \in V$ , and apply (2) with  $\xi = \langle y, y, y, \dots \rangle$ ,  $\mathcal{U} = \{U, X \setminus \overline{V}\}$ , and  $\mathcal{F}$  any family with the Ramsey property. The point  $x$  guaranteed by (2) clearly satisfies  $N(x, U) \in \mathcal{F}$ .  $\square$

#### 4. A DENSE SET OF MINIMAL POINTS

In this section we prove that the Ramsey shadowing property is equivalent to the existence of a dense set of minimal points. First we need a lemma about the syndetic sets. Roughly, it states that, while the family of syndetic sets does not have the Ramsey property, nor does its dual, this family is nonetheless very special with respect to the Ramsey property.

**Lemma 4.1.** *The following are equivalent for any  $A \subseteq \mathbb{N}$ :*

- (1)  $A$  is syndetic.
- (2) If  $\mathcal{F}$  is any ultrafilter on  $\mathbb{N}$ , then there is some  $n \in \mathbb{N}$  such that  $A - n \in \mathcal{F}$ .
- (3) If  $\mathcal{F}$  is any family with the Ramsey property, then there is some  $n \in \mathbb{N}$  such that  $A - n \in \mathcal{F}$ .

*Proof.* Clearly (3) implies (2). If  $A$  is syndetic, then for some  $k$   $\mathbb{N} = \bigcup_{n \leq k} (A - n)$ . If  $\mathcal{F}$  has the Ramsey property, it follows that  $A - n \in \mathcal{F}$  for some  $n \leq k$ . Thus (1) implies (3).

Suppose that  $A$  is not syndetic. Then for every  $k$ ,  $\mathbb{N} \setminus \bigcup_{n \leq k} (A - n) \neq \emptyset$ . By de Morgan's Laws,  $\bigcap_{n \leq k} \mathbb{N} \setminus (A - n) \neq \emptyset$  for all  $k$ . In other words,  $\{\mathbb{N} \setminus (A - n) : n \in \mathbb{N}\}$  is a filter base. If  $p$  is any ultrafilter extending this filter base, there is no  $n$  such that  $A - n \in p$ .  $\square$

**Theorem 4.2.** *Let  $X$  be any dynamical system. The following are equivalent:*

- (1)  $X$  has Ramsey shadowing.
- (2)  $X$  has a dense set of minimal points.
- (3) For any open  $U \subseteq X$ , there is some  $x \in U$  such that  $N(x, U)$  is syndetic.
- (4) For any open  $U \subseteq X$ , there is some  $x \in X$  such that  $N(x, U)$  is piecewise syndetic.

*Proof.* We will show (2)  $\Rightarrow$  (3)  $\Rightarrow$  (1)  $\Rightarrow$  (4)  $\Rightarrow$  (2).

That (2)  $\Rightarrow$  (3) is easy: simply choose  $x$  to be a minimal point in  $U$  and apply Proposition 2.3.

Now suppose (3) holds. Let  $U$  be an open subset of  $X$  and fix  $x \in U$  with  $N(x, U)$  syndetic. Let  $\mathcal{F}$  be a family with the Ramsey property. By Lemma 4.1, there is some  $n$  such that  $N(x, U) - n \in \mathcal{F}$ . Since

$$\begin{aligned} N(x, U) - n &= \{m - n : f^m(x) \in U\} = \{m : f^{m+n}(x) \in U\} \\ &= \{m : (f^n(x))^m(x) \in U\} = N(f^n(x), U), \end{aligned}$$

we have  $N(f^n(x), U) \in \mathcal{F}$ . Thus  $X$  has Ramsey shadowing, proving that (3)  $\Rightarrow$  (1).

Assume (1). It is well known that the family of piecewise syndetic sets has the Ramsey property (see, e.g., p. 26 of [2]). It follows from the definition of Ramsey shadowing that (1)  $\Rightarrow$  (4).

Finally, suppose (4) holds and fix some open  $U \subseteq X$ . Since every compact Hausdorff space is regular, there is an open  $V \subseteq U$  such that  $\overline{V} \subseteq U$ . Using (4), fix  $x$  with  $N(x, V)$  piecewise syndetic. By Theorem 2.1 in [4] and Theorem 4.40 in [8], every piecewise syndetic set is a member of some minimal ultrafilter. In particular, there is some minimal ultrafilter  $p$  with  $N(x, V) \in p$ . It follows that  $p\text{-}\lim_{n \in \mathbb{N}} f^n(x) \in \overline{V}$ , from which we get  $p\text{-}\lim_{n \in \mathbb{N}} f^n(x) \in U$ . By Proposition 2.3,  $p\text{-}\lim_{n \in \mathbb{N}} f^n(x)$  is a minimal point. Since  $U$  was arbitrary,  $X$  has a dense set of minimal points. Thus (4)  $\Rightarrow$  (1).  $\square$

The property (3) in Theorem 4.2 is very close to the following property: for every open  $U \subseteq X$ ,  $N(U, U)$  is syndetic. It should be noted that these properties are not equivalent in general. For a counterexample, see Example 8.3 in [11], or see [13].

For a metric system  $X$  and  $\delta > 0$ , a  $\delta$ -**pseudo-orbit** in  $X$  is a sequence  $\langle x_n : n \in \mathbb{N} \rangle$  such that  $d(f(x_n), x_{n+1}) < \delta$  for all  $n$ .  $X$  has the **shadowing** property if for every  $\varepsilon$  there is a  $\delta$  such that every  $\delta$ -pseudo-orbit in  $X$  can be  $\varepsilon$ -shadowed on  $\mathbb{N}$ .

Shadowing is an important, well-studied, and ubiquitous property (for a formalization of its ubiquity, see [10] or [12]). Many variations of the shadowing property (e.g. in [6] or [1]) have the form: if  $\xi$  is a pseudo-orbit on a large enough set, then we can shadow it on a large set. Our Ramsey shadowing property has this form, with the loosest possible restrictions on  $\xi$ , namely none.

A metric system  $X$  is **chain transitive** if for every  $\delta > 0$  and every  $x \in X$ , there is a nontrivial  $\delta$ -pseudo-orbit from  $x$  to itself (where “nontrivial” means that the one element sequence  $\langle x \rangle$  does not count).

**Corollary 4.3.** *Let  $X$  be a metric system with shadowing. Then  $X$  has Ramsey shadowing if and only if  $X$  is chain recurrent.*

*Proof.* By the previous theorem, it suffices to show that chain recurrence together with shadowing implies that the set of minimal points is dense, which is well known. However, a short, direct proof of the corollary is also possible, which we give here for completeness.

Suppose  $X$  is chain recurrent and has shadowing, and let  $U \subseteq X$  be open. If  $x \in U$ , there is some  $\varepsilon > 0$  such that  $B_\varepsilon(x) \subseteq U$  and there is some  $\delta > 0$  such that every  $\delta$ -pseudo-orbit in  $X$  can be  $\varepsilon$ -shadowed. Fix a nontrivial  $\delta$ -chain  $\langle x, x_1, x_2, \dots, x_n, x \rangle$  from  $x$  to  $x$ . Then

$$\xi = \langle x, x_1, x_2, \dots, x_n, x, x_1, x_2, \dots, x_n, x, x_1, x_2, \dots, x_n, x, \dots \rangle$$

is a  $\delta$ -pseudo-orbit in  $X$ . There is some  $y \in X$  that  $\varepsilon$ -shadows  $\xi$ . In particular,  $d(f^m(y), x) < \varepsilon$  whenever  $m$  is a multiple of  $n + 1$ . It follows that  $N(y, U)$  is syndetic.  $\square$

In [6], Meddaugh, Raines, and the author prove that every chain transitive metric system with shadowing has Ramsey shadowing. Because every chain transitive system is chain recurrent, Corollary 4.3 strengthens this result.

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