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ON LINEAR PROJECTIONS OF QUADRATIC VARIETIES

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Abstract. We study simple outer linear projections of projective varieties whose homogeneous vanishing ideal is defined by quadrics which satisfy the condition K_2. We extend results on simple outer linear projections of rational normal scrolls.

1. Introduction

Throughout this paper, we work over an algebraically closed field $\Bbbk$ of arbitrary characteristic. We denote by $\mathbb{P}^r$ the projective r-space over $\Bbbk$.

For a non-degenerate irreducible projective variety $X \subset \mathbb{P}^r$ and a closed point $q \in \mathbb{P}^r$ outside of X, let $\pi_q : X \to \mathbb{P}^{r-1}$ be the linear projection of X from q and consider the subvariety $X_q = \pi_q(X) \subset \mathbb{P}^{r-1}$. One can naturally expect that algebraic and geometric properties of X_q may be described precisely in terms of those of X and the relative location of q with respect to X. For example, let $f_q : X \to X_q$ be the map induced by π_q and consider the coherent sheaf $\mathcal{F} := (f_q)_*\mathcal{O}_X/\mathcal{O}_{X_q}$ on X_q. Then the support of $\mathcal{F}$ is exactly the singular locus

$$\mathrm{Sing}(f_q) := \{x \in X_q \mid \mathrm{length}\big(f_q^{-1}(x)\big) \geq 2\}$$

of the morphism $f_q : X \to X_q$. Classically, the set $\mathrm{Join}\big(\mathrm{Sing}(f_q), q\big)$ with the reduced scheme structure is called the *secant cone of* X at q and is denoted by $\mathrm{Sec}_q(X)$. Also $\Sigma_q(X)$, the scheme-theoretic intersection of X and $\mathrm{Sec}_q(X)$, is called the *secant locus* (or *entry locus*) *of* X at q. These notions are related in an elementary way to the morphism $f_q : X \to X_q$ as follows:

(*i*) $f_q : X \to X_q$ is an isomorphism if and only if $\Sigma_q(X)$ is empty.
(*ii*) $f_q : X \to X_q$ is birational if and only if $\Sigma_q(X)$ is a proper subset of X.

In this paper we study the projected variety $X_q \subset \mathbb{P}^{r-1}$ in the case where X satisfies condition K_2, that is, it is scheme-theoretically cut out by some quadratic equations and the trivial syzygies among them are generated by linear syzygies (cf. Definition and Remark 3.1). Our main result in the present paper shows that various important properties of X_q are governed by the integer $s(q)$ defined as

$$s(q) := h^0(\mathbb{P}^r, \mathcal{I}_X(2)) - h^0(\mathbb{P}^{r-1}, \mathcal{I}_{X_q}(2)) - 1.$$

Thus we can say that $s(q)$ reflects the relative location of q with respect to X.

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Theorem 1.1. *Let $X \subset \mathbb{P}^r$ be a non-degenerate irreducible projective variety satisfying condition K_2, and let $q \in \mathbb{P}^r$ be a closed point outside of X. Then*

(a) *$s(q) > 0$ and the morphism $f_q : X \to X_q$ is birational.*

(b) *The secant cone $\mathrm{Sec}_q(X) \subset \mathbb{P}^r$ and the singular locus $\Lambda := \mathrm{Sing}(f_q) \subset \mathbb{P}^{r-1}$ are linear subspaces of dimension $(r - s(q))$ and $(r - s(q) - 1)$, respectively.*

(c) *The secant locus $\Sigma_q(X)$ is a quadratic hypersurface in $\mathrm{Sec}_q(X)$.*

(d) *Let A_X, A_{X_q} and A_Λ be respectively the homogeneous coordinate ring of $X \subset \mathbb{P}^r$, $X_q \subset \mathbb{P}^{r-1}$ and $\Lambda \subset \mathbb{P}^{r-1}$. Then there is an exact sequence of graded A_{X_q}-modules*

$$0 \longrightarrow A_{X_q} \longrightarrow A_X \longrightarrow A_\Lambda(-1) \longrightarrow 0. \tag{1.1}$$

(e) *The sheaf $(f_q)_*\mathcal{O}_X/\mathcal{O}_{X_q}$ is isomorphic to $\mathcal{O}_\Lambda(-1)$.*

We also illustrate this theorem by means of various simple exterior projections of the rational normal 3-fold scroll $S(1,1,4) \subset \mathbb{P}^8$.

Remark 1.2. (A) The statements of Theorem 1.1 are proved in [3] when X is a variety of minimal degree, in [5] when X is a projective normal variety satisfying condition $N_{2,2}$ and in [1] when X satisfies condition $N_{2,2}$. See Definition and Remark 3.1 for the definition of condition $K2$ and condition $N_{2,2}$.

(B) The sequence (1.1) allows us to compare algebraic properties of X_q and X. For example, the local properties of X and X_q are compared on use of this sequence. See Corollary 3.4.

(C) To the authors' best knowledge, there is no example of a variety $X \subset \mathbb{P}^r$ which satisfies condition K_2 but does not satisfy condition $N_{2,2}$. Nevertheless, the proof of Theorem 1.1 itself is interesting because it uses directly the definition of condition K_2. So, the rich structure of X_q stated in Theorem 1.1 and Corollary 3.4 is a direct consequence of condition K_2 of X.

(D) It seems natural to ask about the sets $\Phi_t := \{q \in \mathbb{P}^r \mid s(q) = t\}$. Theorem 1.1(b) says that $s(q) \le r-1$ if and only if the map $f_q : X \to X_q$ is singular. Thus Φ_t is contained in the secant variety of X whenever $t \le r-1$. This means that the Φ_t's for $t \le r-1$ constitute a stratification of the secant variety of X. When X is a smooth rational normal scroll, this stratification is understood very well (cf. [2]).

2. Quadratic varieties

Convention 2.1. (A) We write $S := \Bbbk[x_0, x_1, \cdots, x_r]$ for the homogeneous coordinate ring of $\mathbb{P}^r$. If $\mathfrak{a} \subseteq S$ is a graded ideal and $F_1, \ldots, F_n \in S$ are homogeneous polynomials, we write

$$\mathbb{V}(\mathfrak{a}) := \mathrm{Proj}(S/\mathfrak{a}) \quad \text{and} \quad \mathbb{V}(F_1, \ldots, F_n) := \mathbb{V}(\sum_{i=1}^n SF_i).$$

(B) Let $X \subset \mathbb{P}^r$ be a non-degenerate irreducible projective variety whose homogeneous vanishing ideal is $I_X \subset S$. Assume that the point $q = [0,0,\ldots,0,1] \in \mathbb{P}^n$ is outside of X. Then the linear projection map $\pi_q : \mathbb{P}^r \setminus \{q\} \to \mathbb{P}^{r-1}$ corresponds to the obvious inclusion of the homogeneous coordinate ring $S' := \Bbbk[x_0, x_1, \ldots, x_{r-1}]$ of $\mathbb{P}^{r-1}$ into S. Moreover, we always write

$$X_q := \mathrm{Proj}(S'/I_X \cap S') \quad \text{and} \quad I_{X_q} := I_X \cap S'$$

where I_{X_q} is the homogeneous vanishing ideal of X_q. In addition, we consider the induced finite projection morphism

$$f_q : X \to X_q, \quad [x_0, x_2, \ldots, x_r] \mapsto [x_0, x_1, \ldots, x_{r-1}].$$

(C) Let V be a $\Bbbk$-vector subspace of $\big(I_X\big)_2$ whose common zero locus does not contain q. Let $V_q := V \cap S' \subseteq \big(I_{X_q}\big)_2$ and write $\dim_{\Bbbk} V = t+1$ and $\dim_{\Bbbk} V_q = t-s$. Then we can choose a basis $\{Q_0, Q_1, \ldots, Q_t\}$ of V such that

$$(\dagger) \begin{cases} 1.\ Q_0 = G_0 + H_0 x_r + x_r^2, & \\ 2.\ Q_i = G_i + x_i x_r & \text{for } 1 \leq i \leq s, \text{ and} \\ 3.\ Q_i = G_i & \text{for } s+1 \leq i \leq t \end{cases}$$

where $H_0 \in S'$ and $G_0, G_1, \ldots, G_t \in S'$ are forms of degree 1 and 2 respectively.

Lemma 2.2. *Let the notation and hypotheses be as in Convention* 2.1 *(A), (B) and (C). Suppose that V cuts out X scheme-theoretically. Then*

(a) *For each closed point $p \in X_q$ it holds*

$$\operatorname{length}\big(f_q^{-1}(p)\big) = \begin{cases} 1, & \textit{if } p \notin \mathbb{V}(x_1, \ldots, x_s) \textit{ and} \\ 2, & \textit{if } p \in \mathbb{V}(x_1, \cdots x_s). \end{cases}$$

(b) $\operatorname{Sing}(f_q) = X_q \cap \mathbb{V}'(x_1, \ldots, x_s)$ *and* $\Sigma_q(X) = X \cap \mathbb{V}(x_1, \ldots, x_s)$.

(c) *Assume that I_X is generated by V and $s = 0$. Then $I_{X_q} = \sum_{i=1}^t S' Q_i$.*

Proof. For any point $p \in X_q$, consider the line $\langle p, q\rangle = \{\lambda p + \mu q \mid [\lambda, \mu] \in \mathbb{P}^1\}$. Note that $q \notin X \cap \langle p, q\rangle$ and so $X \cap \langle p, q\rangle$ is an affine subscheme of $\mathbb{A}^1 = \langle p, q\rangle \setminus \{q\} = \operatorname{Spec}(\Bbbk[\mu])$. Moreover, $X \cap \langle p, q\rangle$ in $\mathbb{A}^1$ is defined by the $s+1$ polynomials

$$\mu^2 + H_0(p)\mu + G_0(p),\ x_1(p)\mu + G_1(p), \ldots,\ x_s(p)\mu + G_s(p) \in \Bbbk[\mu]$$

since V cuts out X scheme-theoretically and the quadratic forms $Q_{s+1}, \cdots, Q_t$ vanish on the line $\langle p, q\rangle$. Therefore it holds that

$$\operatorname{length}\big(f_q^{-1}(p)\big) = \operatorname{length}\big(X \cap \langle p, q\rangle\big) = \begin{cases} 1, & \text{if } x_i(p) \neq 0 \text{ for some } i \geq 1, \text{ and} \\ 2, & \text{if } x_1(p) = \cdots = x_s(p) = 0. \end{cases}$$

This proves statement (a). The first part of (b) now follows by the definition of the singular locus $\operatorname{Sing}(f_q)$ of f_q. Then we can see that $\operatorname{Sec}_q(X)$ is equal to $\operatorname{Join}(X, q) \cap \mathbb{V}(x_1, \ldots, x_s)$. Therefore $\Sigma_q(X)$ is the scheme-theoretical intersection $X \cap \operatorname{Sec}_q(X) = X \cap \mathbb{V}(x_1, \ldots, x_s)$. In order to prove statement (d), we write $I := I_{X_q} = I_X \cap S'$ and $J := \sum_{i=1}^t S' Q_i$ and we show, by induction, that $I_d = J_d$ for all integers $d \geq 2$. For $d = 2$ this is clear by our choice of $Q_0, Q_1, \ldots, Q_t$. Moreover $J_d \subseteq I_d$ for all $d \geq 3$. So, let $F \in I$ be a homogeneous form of degree $d \geq 3$. Since $F \in I_X$, we have

$$F = Q_0 L_0 + Q_1 L_1 + \cdots + Q_t L_t.$$

For each form $L = L(x_0, \ldots, x_{r-1}, x_r) \in S_{d-2}$ we write $L' := L(x_0, \ldots, x_{r-1}, 0) \in S'_{d-2}$. Writing $Q_0 = G_0 + H_0 x_r + x_r^2$ and observing that $F, G_0, Q_1, \ldots, Q_t \in S'$ we thus get $F = G_0 L'_0 + Q_1 L'_1 + \cdots + Q_t L'_t$. It remains to show that $G_0 L'_0 \in J$. As $F, Q_1, \ldots Q_1 \in I$, we have $G_0 L'_0 \in I_d$. As I is a prime containing no linear form, we have $H_0 x_r + x_2 \notin I$, hence $G_0 \notin I$ and therefore $L'_0 \in I$. As $L'_0 \in S'_{d-2}$ it follows by induction that $L'_0 \in J$, so that indeed $G_0 L'_0 \in J$. $\square$

As an immediate application of the previous lemma, we get the following result.

Proposition 2.3. *Let the notation and hypotheses be as in Lemma* 2.2. *Then*

(a) *The morphism $f_q : X \to X_q$ is birational if and only if $s > 0$.*

(b) *Assume that I_X is generated by V and $s = 0$. Then X_q is a quadratic variety and X is the intersection of the cone* $\mathrm{Join}(q, X_q)$ *and a quadric. Furthermore, the morphism $f_q : X \to X_q$ is a double covering.*

3. The condition K_2

Definition and Remark 3.1. (A) Let the notation and hypotheses be as in Convention 2.1 and let $\underline{Q} := (Q_0, Q_1, \cdots, Q_t) \in S_2^{t+1}$ be a family of $\Bbbk$-linearly independent quadratic forms. We consider the module of syzygies

$$\mathrm{Syz}(\underline{Q}) := \{(F_0, F_2, \ldots, F_t) \in S^{t+1} \mid \sum_{i=0} F_i Q_i = 0\}$$

of the family $\underline{Q}$, furnished with its natural grading as a submodule of S^{t+1}. By a *linear syzygy* of $\underline{Q}$ we mean a homogeneous element of degree 1 in $\mathrm{Syz}(\underline{Q})$, hence an element of $\mathrm{Syz}(\underline{Q})_1$. We also introduce the graded submodule

$$\mathrm{Syz}_{\mathrm{lin}}(\underline{Q}) := \sum_{F \in \mathrm{Syz}(\underline{Q})_1} SF \quad \big(\subseteq \mathrm{Syz}(\underline{Q})\big)$$

generated by all linear syzygies of $\underline{Q}$.

For each $i \in \{0, 1, \ldots, t\}$, let $e_i := (0, \ldots, 0, 1, 0, \ldots 0) = (\delta_{i,j})_{j=0}^t$ denote the i-th canonical basis element of the S-module S^{t+1}. Whenever $0 \le i < j \le t$ we call the element

$$T_{i,j} := Q_j e_i - Q_i e_j = (0, \ldots, 0, Q_j, 0 \ldots, 0, -Q_i, 0, \ldots 0) \in \mathrm{Syz}(\underline{Q})_2$$

a *trivial syzygy* and we introduce the graded submodule

$$\mathrm{Syz}_{\mathrm{triv}}(\underline{Q}) := \sum_{0 \le i < j \le t} ST_{i,j} \quad \big(\subseteq \mathrm{Syz}(\underline{Q})\big)$$

generated by the trivial syzygies. Observe that $\mathrm{Syz}(\underline{Q}) = 0$ if $t = 0$.

(B) Let V be the $\Bbbk$-vector space spanned by $\{Q_0, Q_1, \cdots, Q_t\}$. If $\{Q'_0, Q'_1, \cdots, Q'_t\}$ is a basis for V, then there is a regular matrix $A := [a_{i,j} \mid 0 \le i, j \le t] \in \Bbbk^{(t+1)\times(t+1)}$ for which $Q_i = \sum_{j=0}^t \alpha_{i,j} Q'_j$ for all $i \in \{0, \ldots, t\}$. Then the family $\underline{Q'} := (Q'_0, Q'_1, \cdots, Q'_t) \in S_2^{t+1}$ and the automorphism $\phi : S^{t+1} \overset{\cong}{\to} S^{t+1}$, $e_i \mapsto \sum_{j=0}^t \alpha_{i,j} e_j$ for all $i \in \{0, \ldots, t\}$, induced by A have the property that

$$\phi\big(\mathrm{Syz}(\underline{Q})\big) = \mathrm{Syz}(\underline{Q'}), \quad \phi\big(\mathrm{Syz}_{\mathrm{lin}}(\underline{Q})\big) = \mathrm{Syz}_{\mathrm{lin}}(\underline{Q'}) \text{ and } \phi\big(\mathrm{Syz}_{\mathrm{triv}}(\underline{Q})\big) = \mathrm{Syz}_{\mathrm{triv}}(\underline{Q'}).$$

As a consequence, the two conditions

$$(K_2) \ : \ \mathrm{Syz}_{\mathrm{triv}}(\underline{Q}) \subseteq \mathrm{Syz}_{\mathrm{lin}}(\underline{Q}) \quad \text{and} \quad (N_{2,2}) \ : \ \mathrm{Syz}_{\mathrm{lin}}(\underline{Q}) = \mathrm{Syz}(\underline{Q})$$

do not depend on the choice of a basis for V and hence are intrinsic properties of V. Obviously, both conditions are satisfied if $t = 0$.

(C) Let $X \subset \mathbb{P}^r$ be the closed subscheme defined by a homogeneous ideal $I \subseteq S$. Following [6] we say that X *satisfies condition* K_2 if it is scheme-theoretically cut out by a subspace $V \subset I_2$ which satisfies condition K_2. Also, following [4] we say that X *satisfies condition* $N_{2,2}$ if I_2 generates I and satisfies condition $N_{2,2}$. Observe X satisfies condition K_2 if it satisfies condition $N_{2,2}$.

Lemma 3.2. *Keep the notation and hypotheses as in Convention* 2.1 *and Definition and Remark* 3.1. *Then*

(a) *If* $(F_0, F_1, \cdots, F_t) \in \mathrm{Syz}(\underline{Q})_1$, *then* $F_0 \in \sum_{i=1}^{s} \Bbbk x_i$.

(b) *Suppose that* V *cuts out* X *scheme-theoretically and satisfies the condition* K_2. *Then*

(1) $Q_1, \dots, Q_t \in \sum_{i=1}^{s} S_1 x_i$;

(2) $t > 0 \Longrightarrow s > 0$;

(3) $\Sigma_q(X) = \mathbb{V}(Q_0, x_1, \dots, x_s)$, $\mathrm{Sec}_q(X) = \mathbb{V}(x_1, \dots, x_s)$ *and* $\mathrm{Sing}(f_q) = \mathbb{V}'(x_1, \dots, x_s)$.

Proof. (a): Writing $F_i = \sum_{j=0}^{r} a_{i,j} x_j$ with $a_{i,j} \in \Bbbk$ for all $0 \le i \le t$, we have

$$F_0 Q_0 + F_1 Q_1 + \cdots + F_t Q_t = 0. \tag{3.1}$$

Also the left hand side of the equation (3.1) may be rewritten as

$$\sum_{i=0}^{t} F_i Q_i = a_{0,r} x_r^3 + (F_0 + a_{0,r} H_0 + a_{1,r} x_1 + \ldots + a_{s,r} x_s) x_r^2 + Q x_r + F$$

for some $Q \in S'_2$ and some $F \in S'_3$. Therefore the equation (3.1) implies that $a_{0,r} = 0$ and $F_0 + a_{1,r} x_1 + \ldots + a_{s,r} x_s = 0$, which completes the proof.

(b): Let $i \in \{1, \dots, t\}$. By condition K_2 we find some $n \in \mathbb{N}$, forms $L_j \in S_1$ and linear syzygies $\sum_{k=1}^{t} F_{j,k} e_k \in \mathrm{Syz}(\underline{Q})_1, \quad (j = 1, \dots, n)$ such that

$$T_{0,i} = Q_i e_0 - Q_0 e_i = \sum_{j=1}^{n} L_j \sum_{k=1}^{t} F_{j,k} e_k = \sum_{k=1}^{t} \Big(\sum_{j=1}^{n} L_j F_{j,k}\Big) e_k, \text{ whence } Q_i = \sum_{j=1}^{n} L_j F_{j,0}.$$

According to (a), we have $F_{j,0} \in \sum_{l=1}^{s} S x_l$, so that $Q_i \in \sum_{l=1}^{s} S_1 x_l$. This proves claim (1). The remaining claims (2) and (3) now follow easily from the use of Lemma 2.2 (b), (c). □

Notation and Remark 3.3. Let the notation and hypotheses be as in Convention 2.1. We consider the homogeneous coordinate rings

$$A_{X_q} := S'/I_{X_q} = S'/(I_X \cap S') \quad \text{and} \quad A_X = S/I_X$$

of X_q and of X, as well as the canonical map

$$\overline{\bullet} : S \to A_X, \text{ given by } F \mapsto \overline{F} := F + I_X.$$

As $S = S'[x_r]$, $\overline{x_r}^2 + \overline{H_0} x_r + \overline{G_0} = \overline{Q_0} = 0$, $\overline{x_i x_r} = \overline{x_i x_r} = \overline{Q_i - G_i} = \overline{G_i}$ for all $i \in \{1, \dots, s\}$, and $\overline{H_0}, \overline{G_0}, \dots, \overline{G_{s(q)}} \in A_{X_q}$, we obtain:

(a) $A_X = A_{X_q}[\overline{x_r}] = A_{X_q} + \overline{x_r} A_{X_q}$, with $\overline{x_r} \in (A_X)_1 \setminus A_{X_q}$, and

(b) $x_i A_X \subseteq A_{X_q}$ for all $i \in \{1, \dots, s\}$.

Proof of Theorem 1.1. Statement (a) follows immediately from Lemma 3.2 (b)(2) and Proposition 2.3 (a). Statement (b) is a consequence of Lemma 3.2 (b)(3). Statement (c) is immediate by Lemma 3.2 (b)(3). To prove statement (d), we set $s := s(q)$ and write $\Lambda := \mathrm{Sing}(f_q)$. We may assume that the notation and hypotheses are as in Convention 2.1 and Notation and Remark 3.3. Then, by Lemma 3.2 (b)(3) we have

$$\Lambda = \mathbb{V}'(x_1, \dots, x_s) = \mathrm{Proj}(\Bbbk[x_0, x_{s+1}, \dots, x_r]) = \mathbb{P}^{r-s-1} \subset \mathbb{P}^{r-1},$$

and the homogeneous vanishing ideal I_Λ of Λ in S' and the homogeneous coordinate ring A_Λ of Λ satisfy

$$I_{X_q} \subset I_\Lambda = \sum_{i=1}^{s} S' \quad \text{and} \quad A_\Lambda = S'/I_\Lambda.$$

According to statement (a) of Notation and Remark 3.3 we have

$$A_X/A_{X_q} \cong \big[S'/\mathrm{ann}_{S'}(A_X/A_{X_q})\big](-1).$$

So, it remains to show that $\mathrm{ann}_{S'}(A_X/A_{X_q}) = I_\Lambda$. According to statement (b) of Notation and Remark 3.3 it holds $I_\Lambda \subseteq \mathrm{ann}_{S'}(A_X/A_{X_q})$. As

$$\begin{aligned}\mathbb{V}'(I_\Lambda) = \Lambda = \mathrm{Sing}(f_q) &= \mathrm{Supp}_{\mathbb{P}^{r-1}}\big((f_q)_*\mathcal{O}_X/\mathcal{O}_{X_q}\big)\\ &= \mathrm{Supp}_{\mathbb{P}^{r-1}}\big(\widetilde{A_X/A_{X_q}}\big) = \mathbb{V}'\big(\mathrm{ann}_{S'}(A_X/A_{X_q})\big),\end{aligned}$$

it holds

$$\sqrt{\mathrm{ann}_{S'}(A_X/A_{X_q})} = \sqrt{I_\Lambda}.$$

As I_Λ is a prime ideal, it follows $\mathrm{ann}_{S'}(A_X/A_{X_q}) \subseteq I_\Lambda$, and this proves our claim.

Now, (e) follows immediately from statement (d) as $(f_q)_*\mathcal{O}_X/\mathcal{O}_{X_q} = \widetilde{A_X/A_{X_q}}$. $\square$

As an application of Theorem 1.1 we obtain the following result, in which

$$\mathrm{Nor}(Z),\ \mathrm{CM}(Z) \text{ and } S_2(Z)$$

respectively denote the locus of normal, Cohen-Macaulay and S_2-points of a locally Noetherian scheme Z.

Corollary 3.4. *Let $X \subset \mathbb{P}^r$ and $X_q \subset \mathbb{P}^{r-1}$ be as in Theorem 1.1. Then*

(a) *Each closed point in $\mathrm{Sing}(f_q)$ is a non-normal point of X_q. Therefore*

$$\mathrm{Nor}(X_q) = f_q(\mathrm{Nor}(X) \setminus \Sigma_q(X)) = f_q(\mathrm{Nor}(X)) \setminus \mathrm{Sing}(f_q).$$

In particular, if X is normal, then $f_q : X \to X_q$ is the normalization of X_q.

(b) *Assume that X is locally Cohen-Macaulay and $\dim\big(\Sigma_q(X)\big) < \dim(X) - 1$. Then, the generic point $\eta \in X_q$ of $\mathrm{Sing}(f_q)$ is a Goto point and*

$$\mathrm{CM}(X_q) = S_2(X_q) = X_q \setminus \mathrm{Sing}(f_q).$$

Proof. (a): Let $x \in \mathrm{Sing}(f_q)$. Then, the ring $\big((f_q)_*\mathcal{O}_X\big)_x$ is a finite birational integral extension of $\mathcal{O}_{X_q,x}$ such that $\big((f_q)_*\mathcal{O}_X\big)_x/\mathcal{O}_{X,x} \cong \mathcal{O}_{\Lambda,x} \neq 0$ by (1.1). Therefore $\mathcal{O}_{X,x}$ fails to be normal.

(b): Recall that $\eta \in X_q$ is said to be a Goto point if $\dim(\mathcal{O}_{X_q,\eta}) > 1$ and

$$H^i_{\mathfrak{m}_{X_q,\eta}}(\mathcal{O}_{X_q,\eta}) = \begin{cases} 0 & \text{if } i \neq 1, \dim(\mathcal{O}_{X_q,\eta}), \text{ and} \\ \kappa(\eta) & \text{if } i = 1. \end{cases}$$

In our case, we have $\dim(\mathcal{O}_{X_q,\eta}) > 1$, since we assume that $\dim\big(\Sigma_q(X)\big) < \dim(X) - 1$. Localizing the exact sequence (1.1) at η, we get the following exact sequence of $\mathcal{O}_{X_q,\eta}$-modules:

$$0 \to \mathcal{O}_{X_q,\eta} \to \big((f_q)_*\mathcal{O}_X\big)_\eta \to \kappa(\eta) \to 0.$$

Since X is locally Cohen-Macaulay, $\mathcal{O}_{X,y}$ is a Cohen-Macaulay local ring for each $y \in \pi_q^{-1}(\eta)$. Therefore $\left((f_q)_*\mathcal{O}_X\right)_\eta$ is a Cohen-Macaulay $\mathcal{O}_{X_q,\eta}$-module. So, the above exact sequence shows that

$$H^1_{\mathfrak{m}_{X_q,\eta}}(\mathcal{O}_{X_q,\eta}) \cong \kappa(\eta) \quad \text{and} \quad H^i_{\mathfrak{m}_{X_q,\eta}}(\mathcal{O}_{X_q,\eta}) = 0 \text{ for all } i \neq 1, \dim(\mathcal{O}_{X_q,\eta}).$$

As $\eta \in X_q$ is not an S_2-point, each $y \in \Lambda$ fails to be an S_2-point and a Cohen-Macaulay point of X_q. □

4. Examples

Example 4.1. Let $X \subset \mathbb{P}^8$ be the standard rational normal scroll $S(1,1,4)$ defined by the vanishing of the 2×2-minors of the matrix

$$M = \left(\begin{array}{c|c|cccc} x_0 & x_2 & x_4 & x_5 & x_6 & x_7 \\ x_1 & x_3 & x_5 & x_6 & x_7 & x_8 \end{array} \right).$$

Thus X is a quadratic variety and its homogeneous vanishing ideal is generated by the following set of 15 K-linearly independent quadrics:

$$\{Q_{i,j} \mid 1 \leq i < j \leq 6\}$$

where $Q_{i,j}$ is the determinant of the 2×2 matrix consisting of the i-th and j-th columns of M. We consider the following four points $q_i \in \mathbb{P}^8 \setminus X, \quad (i = 1, \ldots, 4)$:

$$q_1 = [0,0,0,0,0,0,1,0,0],\ q_2 = [0,0,0,0,0,1,0,0,0],$$
$$q_3 = [0,0,0,1,1,0,0,0,0],\ q_4 = [0,1,1,0,0,0,0,0,0].$$

Let $X_{q_i} \subset \mathbb{P}^7$ denote the image of $X \subset \mathbb{P}^8$ under the linear projection $\pi_{q_i} : \mathbb{P}^8 \setminus \{q_i\} \to \mathbb{P}^7$.

(A) When $i = 1$, the homogeneous vanishing ideal of q_1 is generated by all homogeneous coordinates of $\mathbb{P}^8$ except x_6. Also, among the above 15 quadrics, exactly the following 9 quadrics contain x_6:

$$Q_{1,4},\ Q_{1,5},\ Q_{2,4},\ Q_{2,5},\ Q_{3,4},\ Q_{3,5},\ Q_{4,5},\ Q_{4,6},\ Q_{5,6}.$$

This shows that $h^0(\mathbb{P}^7, \mathcal{I}_{X_{q_1}}(2)) = 15 - 9 = 6$ and $\Sigma_{q_1}(X)$ is empty since

$$\Sigma_{q_1}(X) = \mathbb{V}_{\mathbb{P}^8}(x_0, x_1, x_2, x_3, x_4, x_5, x_7, x_8, Q_{4,5}).$$

(B) When $i = 2$, the homogeneous vanishing ideal of q_2 is generated by all homogeneous coordinates of $\mathbb{P}^8$ except x_5. Also, among the above 15 quadrics, exactly the following 8 quadrics contain x_5:

$$Q_{1,3},\ Q_{1,4},\ Q_{2,3},\ Q_{2,4},\ Q_{3,4},\ Q_{3,5},\ Q_{3,6},\ Q_{4,6}.$$

This shows that $h^0(\mathbb{P}^7, \mathcal{I}_{X_{q_2}}(2)) = 15 - 8 = 7$ and $\Sigma_{q_2}(X)$ is a double point in $\mathbb{P}^1$ since

$$\Sigma_{q_2}(X) = \mathbb{V}_{\mathbb{P}^r}(x_0, x_1, x_2, x_3, x_6, x_7, x_8, Q_{3,4}).$$

(C) When $i = 3$, let $2y = x_3 - x_4$ and $2z = x_3 + x_4$. Then the homogeneous vanishing ideal of p_3 is generated by $\{x_0, x_1, x_2, y, x_5, x_6, x_7, x_8\}$. Also, among the above 15 quadrics, essentially the following 7 quadrics contain z since $Q_{2,5} + Q_{3,4}$ and $Q_{2,6} + Q_{3,5}$ are free with respect to z:

$$Q_{2,3},\ Q_{1,2},\ Q_{1,3},\ Q_{2,4},\ Q_{2,5},\ Q_{2,6},\ Q_{3,6}.$$

This shows that $h^0(\mathbb{P}^7, \mathcal{I}_{X_{q_3}}(2)) = 15 - 7 = 8$ and $\Sigma_{q_3}(X)$ is the union of two lines in $\mathbb{P}^2$ since

$$\Sigma_{q_3}(X) = \mathbb{V}_{\mathbb{P}^8}(x_0, x_1, x_5, x_6, x_7, x_8, Q_{2,3}).$$

(D) When $i = 4$, let $2y = x_1 - x_2$ and $2z = x_1 + x_2$. Then the homogeneous vanishing ideal of p_4 is generated by $\{x_0, y, x_3, x_4, x_5, x_6, x_7, x_8\}$. Also, among the above 15 quadrics equations, essentially the following 6 quadrics contain z since $Q_{1,4} + Q_{2,3}$, $Q_{1,5} + Q_{2,4}$, and $Q_{1,6} + Q_{2,5}$ are free with respect to z:

$$Q_{1,2},\ Q_{1,3},\ Q_{1,4},\ Q_{1,5},\ Q_{1,6},\ Q_{2,6}.$$

This shows that $h^0(\mathbb{P}^7, \mathcal{I}_{X_{q_4}}(2)) = 15 - 6 = 9$ and $\Sigma_{q_4}(X)$ is a smooth quadric in $\mathbb{P}^3$ since

$$\Sigma_{q_4}(X) = \mathbb{V}_{\mathbb{P}^r}(x_4, x_5, x_6, x_7, x_8, Q_{1,2}).$$

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