

## COMPARISON OF $T_1$ CONDITIONS FOR MULTI-PARAMETER OPERATORS

ANA GRAU DE LA HERRÁN

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**ABSTRACT.** When Journé established the classical multi-parameter singular integral theory in 1985, it was formulated in the language of vector-valued Calderón-Zygmund theory. In 2011, Pott and Villarroya formulated a new  $T_1$  Theorem for product spaces. In their work, they replaced the vector-valued formulations for several mixed type conditions. Inspired by their work, Martikainen redefined the bi-parameter operators. Originally, it was thought that this new class of operators was larger (and contained the ones defined by Journé). In this paper we prove that both classes are actually equal.

### 1. INTRODUCTION

The  $T_1$  Theorem is an  $L^2$  boundedness criteria that was proven originally by David and Journé in 1984 [DJ] for generalized Calderón-Zygmund operators. When Journé extended such a result to the multi-parameter setting, some challenges arose that were not encountered in the classical case as, for example, that the singularities of multi-parameter operators lie not only at the origin (as in the case of standard Calderón-Zygmund kernels), but they spread over larger subspaces. Even though Journé was able to formulate the statement of the theorem in a way that a priori resembles the classical one, his formulation required a priori boundedness of some components of the operator, which differs from the classical setting.

In 2011, Pott and Villarroya [PV] modified the formulation of the  $T_1$  Theorem in product spaces so no a priori boundedness was required. Inspired by their work, Martikainen redefined the bi-parameter operators in [M] but the relationship between these two classes of operators was unclear. In this paper, we prove that both classes of operators are indeed equal.

One of the reasons that caused us to compare both formulations was to know if the  $L^2$  boundedness of a bi-parameter Calderón-Zygmund operator  $T$ , in the sense of Martikainen/Pott-Villarroya, implied that  $T_1 \in BMO_{prod}$ . Such a result was known for bi-parameter Calderón-Zygmund operators in the sense of Journé, but it remained unproven for this new class of operators.

We want to stress that, even when the two sets of conditions are found to be equivalent, the new Martikainen/Pott-Villarroya type of conditions are still useful since it may be easier to verify them in concrete cases than the vector-valued Journé type of conditions.

The layout of the paper proceeds as follows. We are going to state the classical conditions, as in [J], in Section 2 and we will introduce the new mixed type

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conditions, as they were defined in [M], in Section 3. Then, we will proceed to prove the relationship between such conditions in Section 4. Note that, in Sections 2 and 3, we are just stating the definitions and results that are convenient to our paper as they are in the original papers. We refer the reader to them for a deeper understanding of those.

## 2. CLASSICAL FORMULATION

In this section we are going to introduce the classical formulation as stated in Journé's original paper.

Let  $\Omega = \mathbb{R}^d \times \mathbb{R}^d \setminus \Delta$ , where  $\Delta = \{(x, y), x = y\}$  and let  $\delta \in (0, 1)$ .

**Definition 2.1.** Let  $K$  be a continuous function defined on  $\Omega$  and taking its values in a Banach space  $B$ . The function  $K$  is a  **$B$ - $\delta$ -standard kernel** if the following are satisfied, for some constant  $C > 0$ :

For all  $(x, y) \in \Omega$ ,

$$(2.1) \quad |K(x, y)|_B \leq \frac{C}{|x - y|^d}.$$

For all  $(x, y) \in \Omega$ , and  $x' \in \mathbb{R}^d$  such that  $|x - x'| < |x - y|/2$ ,

$$(2.2) \quad |K(x, y) - K(x', y)|_B \leq C \frac{|x - x'|^\delta}{|x - y|^{d+\delta}}$$

and

$$(2.3) \quad |K(y, x) - K(y, x')|_B \leq C \frac{|x - x'|^\delta}{|x - y|^{d+\delta}}.$$

The smallest constant  $C$ , for which (2.1) (2.2) and (2.3) hold, is denoted by  $|K|_{\delta, B}$ . When the Banach space is the complex plane  $\mathbb{C}$ , it is standard to omit the subscript  $B$  for simplicity.

**Definition 2.2.** Let  $T : \mathcal{C}_0^\infty(\mathbb{R}^d) \rightarrow [\mathcal{C}_0^\infty(\mathbb{R}^d)]'$  be a continuous linear mapping.  $T$  is a singular integral operator (SIO) if, for some  $\delta \in (0, 1)$ , there exists a  $\mathbb{C}$ - $\delta$ -standard kernel  $K$  such that for all functions  $f, g \in \mathcal{C}_0^\infty(\mathbb{R}^d)$  having disjoint supports

$$\langle g, Tf \rangle = \iint g(x)K(x, y)f(y)dydx.$$

We shall say that  $T$  is a  $\delta$ -SIO.

**Definition 2.3.** Let  $T$  be a  $\delta$ -SIO and  $K$  its kernel. We say that  $T$  is a  **$\delta$ -Calderón-Zygmund operator** ( $\delta$ -CZO) if it extends boundedly from  $L^2$  to itself. We also define the norm  $\|\cdot\|_{\delta CZ}$  by

$$(2.4) \quad \|T\|_{\delta CZ} = \|T\|_{2 \rightarrow 2} + |K|_\delta.$$

Note that the defined norm makes the set of  $\delta$ -CZO's a Banach space, which we denote by  $\delta CZ$ .

*Note 2.4.* We denote  $f = f_1 \otimes f_2$  (meaning  $f(x) = f_1(x_1) \cdot f_2(x_2)$  for  $x = (x_1, x_2) \in \mathbb{R}^d = \mathbb{R}^n \times \mathbb{R}^m$ ).

**Definition 2.5** ([J]). Let  $T : \mathcal{C}_0^\infty(\mathbb{R}^n) \otimes \mathcal{C}_0^\infty(\mathbb{R}^m) \rightarrow [\mathcal{C}_0^\infty(\mathbb{R}^n) \otimes \mathcal{C}_0^\infty(\mathbb{R}^m)]'$  be a continuous linear mapping. It is a **bi-parameter  $\delta$ -SIO on  $\mathbb{R}^n \times \mathbb{R}^m$**  if there exists a pair  $(K_1, K_2)$  of  $\delta CZ$ - $\delta$ -standard kernels (in  $\mathbb{R}^n$  and  $\mathbb{R}^m$  respectively) so that, for all  $f_1, g_1 \in \mathcal{C}_0^\infty(\mathbb{R}^n)$  and  $f_2, g_2 \in \mathcal{C}_0^\infty(\mathbb{R}^m)$ :

If  $\text{spt}f_1 \cap \text{spt}g_1 = \emptyset$  we have the kernel representation

$$(2.5) \quad \langle g_1 \otimes g_2, T(f_1 \otimes f_2) \rangle = \iint_{\mathbb{R}^n \times \mathbb{R}^n} g_1(x_1) \langle g_2, K_1(x_1, y_1) f_2 \rangle f_1(y_1) dy_1 dx_1,$$

and if  $\text{spt}f_2 \cap \text{spt}g_2 = \emptyset$  we have the kernel representation

$$(2.6) \quad \langle g_1 \otimes g_2, T(f_1 \otimes f_2) \rangle = \iint_{\mathbb{R}^m \times \mathbb{R}^m} g_2(x_2) \langle g_1, K_2(x_2, y_2) f_1 \rangle f_2(y_2) dy_2 dx_2.$$

Let  $\tilde{T}$  be defined by

$$(2.7) \quad \langle g \otimes k, \tilde{T}(f \otimes h) \rangle = \langle f \otimes k, T(g \otimes h) \rangle.$$

It is readily seen that  $\tilde{T}$  is a bi-parameter  $\delta$ -SIO if  $T$  is. Its kernels  $\tilde{K}_1$  and  $\tilde{K}_2$  will be given by  $\tilde{K}_1(x, y) = K_1(y, x)$  and  $\tilde{K}_2(x, y) = [K_2(x, y)]^*$ .

### 3. MIXED TYPE CONDITIONS FORMULATION

In this section we are going to introduce the mixed type conditions formulation introduced by Pott and Villarroya [PV] as reformulated by Martikainen [M].

**Definition 3.1.** Let  $V$  be a cube in  $\mathbb{R}^n$  (resp. in  $\mathbb{R}^m$ ). We say that a function  $u_V$  is **V-adapted with zero mean** in  $\mathbb{R}^n$  (resp. in  $\mathbb{R}^m$ ) if it satisfies that  $\text{spt}(u_V) \subset V$ ,  $|u_V| \leq 1$  and  $\int u_V = 0$ .

**Definition 3.2.** We say that the bi-parameter operator  $T$  has a **full kernel representation** with kernel  $K$  if the following holds. If  $f = f_1 \otimes f_2$  and  $g = g_1 \otimes g_2$  with  $f_1, g_1 : \mathbb{R}^n \rightarrow \mathbb{C}$ ,  $f_2, g_2 : \mathbb{R}^m \rightarrow \mathbb{C}$ ,  $\text{spt}f_1 \cap \text{spt}g_1 = \emptyset$  and  $\text{spt}f_2 \cap \text{spt}g_2 = \emptyset$ , we have the kernel representation

$$\langle Tf, g \rangle = \int_{\mathbb{R}^{n+m}} \int_{\mathbb{R}^{n+m}} K(x, y) f(y) g(x) dy dx,$$

where the kernel  $K$  is a function

$$K : (\mathbb{R}^{n+m} \times \mathbb{R}^{n+m}) \setminus \{(x_1, x_2; y_1, y_2) \in \mathbb{R}^{n+m} \times \mathbb{R}^{n+m} : x_1 = y_1 \text{ or } x_2 = y_2\} \rightarrow \mathbb{C}.$$

Note that this implies full kernel representations for  $T^*$ ,  $\tilde{T}$  and  $\tilde{T}^*$ . If we denote their kernels respectively by  $K^*$ ,  $\tilde{K}$  and  $\tilde{K}^*$ , then we have the formulae

$$K^*(x, y) = K(y_1, y_2; x_1, x_2), \tilde{K}(x, y) = K(y_1, x_2; x_1, y_2), \tilde{K}^*(x, y) = K(x_1, y_2; y_1, x_2).$$

**Definition 3.3.** The kernel  $K$  satisfies the **full standard estimates** if the following holds. We have the **size condition**

$$(3.1) \quad |K(x, y)| \leq C \frac{1}{|x_1 - y_1|^n} \frac{1}{|x_2 - y_2|^m},$$

the **Hölder condition**

$$(3.2) \quad |K(x, y) - K(x, (y_1, y'_2)) - K(x, (y'_1, y_2)) + K(x, y')| \leq C \frac{|y_1 - y'_1|^\delta}{|x_1 - y_1|^{n+\delta}} \frac{|y_2 - y'_2|^\delta}{|x_2 - y_2|^{m+\delta}}$$

whenever  $|y_1 - y'_1| \leq |x_1 - y_1|/2$  and  $|y_2 - y'_2| \leq |x_2 - y_2|/2$ , and the **mixed Hölder and size condition**

$$(3.3) \quad |K(x, y) - K(x, (y'_1, y_2))| \leq C \frac{|y_1 - y'_1|^\delta}{|x_1 - y_1|^{n+\delta}} \frac{1}{|x_2 - y_2|^m}$$

whenever  $|y_1 - y'_1| \leq |x_1 - y_1|/2$ .

The same conditions are imposed on  $K^*$ ,  $\tilde{K}$  and  $\tilde{K}^*$  as well.

**Definition 3.4.** We say that the bi-parameter operator  $T$  has **partial kernel representations** if the following holds. If  $f = f_1 \otimes f_2$  and  $g = g_1 \otimes g_2$  with  $\text{spt} f_1 \cap \text{spt} g_1 = \emptyset$ , we have

$$(3.4) \quad \langle Tf, g \rangle = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K_{f_2, g_2}^2(x_1, y_1) f_1(y_1) g_1(x_1) dy_1 dx_1.$$

Here  $K_{f_2, g_2}^2 : (\mathbb{R}^n \times \mathbb{R}^n) \setminus \{(x_1, y_1) \in \mathbb{R}^n \times \mathbb{R}^n : x_1 = y_1\} \rightarrow \mathbb{C}$ . Moreover, we assume that  $K_{f_2, g_2}^2$  satisfies the standard one-parameter kernel estimates and  $C(f_2, g_2)$  is the minimal constant (depending on  $f_2$  and  $g_2$ ) for which those estimates are satisfied. This constant is assumed to satisfy

$$(3.5) \quad C(\chi_V, \chi_V) + C(\chi_V, g_V) + C(g_V, \chi_V) \leq C|V|$$

whenever  $V \subset \mathbb{R}^m$  is a cube and  $g_V$  is a  $V$ -adapted function with zero mean in  $\mathbb{R}^m$ .

We assume the analogous representation and properties with a kernel  $K_{f_1, g_1}^1$  whenever  $\text{spt} f_2 \cap \text{spt} g_2 = \emptyset$ .

*Remark 3.5.* Note that by the linearity of  $T$  and the definition of the partial kernel above, given  $\alpha, \beta > 0$ ,  $f_2, g_2, h_2 : \mathbb{R}^m \rightarrow \mathbb{C}$ , the following equalities are satisfied:

$$\begin{aligned} K_{\alpha f_2 + \beta h_2, g_2}^2 &= \alpha K_{f_2, g_2}^2 + \beta K_{h_2, g_2}^2, \\ K_{f_2, \alpha g_2 + \beta h_2}^2 &= \alpha K_{f_2, g_2}^2 + \beta K_{f_2, h_2}^2. \end{aligned}$$

Same linearity observations can be done for the partial kernel  $K_{f_1, g_1}^1$ .

**Definition 3.6.** We say that a bi-parameter operator  $T$  is a bi-parameter SIO in the sense of Martikainen/Pott-Villarroya if the following holds:

- The operator  $T$  has a full kernel representation with a kernel  $K$  satisfying the full standard estimates.
- The operator  $T$  has a partial kernel representation.

*Remark 3.7.* It is worth noticing that Pott and Villarroya's original conditions differ slightly from the mixed type conditions that we have used in this paper. While we have used characteristic function and cube-adapted functions in condition (3.5), they used instead some bump functions which have not necessarily compact support.

That the above result can also be proven for the mixed type conditions as in [PV] is left for the reader. We would like to point out that, in the uniparametric setting, we can indiscriminately test our operator on characteristic functions or in bump functions (cf. [G]). Adding such an observation to the fact that we have used uniparametric results along the proof of this paper, one can get an idea of the blueprint for proving such results for the [PV] conditions.

**Lemma 3.8.** *Let  $T$  be a bi-parameter operator that has partial kernel representations with kernel  $K_{f_i, g_i}^i$  for  $i = 1, 2$  as defined in (3.4) and  $V$  a cube in  $\mathbb{R}^m$  (resp. in  $\mathbb{R}^n$ )*

$$(3.6) \quad C(\chi_V, g_V) + C(g_V, \chi_V) \leq C \|g_V\|_\infty |V|$$

whenever  $g_V \in L^\infty(V)$ .

*Proof.* Let  $g_V \in L^\infty(V)$  and rewrite it as follows:

$$g_V = \left( g_V - \left( \int_V g_V \right) \chi_V \right) + \left( \int_V g_V \right) \chi_V = g_V^1 + g_V^2.$$

It is trivial to check that  $\frac{1}{2\|g_V\|_\infty} g_V^1$  is  $V$ -adapted with zero mean and  $g_V^2$  is a constant between 0 and  $\|g_V\|_\infty$  multiplying the characteristic function restricted to  $V$ . By the remark above, the fact that  $C(\cdot, \cdot)$  is a minimal constant, and (3.5)

$$\begin{aligned} C(\chi_V, g_V) &\leq \left( 2\|g_V\|_\infty C(\chi_V, \frac{1}{2\|g_V\|_\infty} g_V^1) + \|g_V\|_\infty C(\chi_V, \chi_V) \right) \\ &\leq C \|g_V\|_\infty |V|. \end{aligned}$$

By symmetry, we get  $C(g_V, \chi_V) \leq C \|g_V\|_\infty |V|$ . □

#### 4. MAIN RESULT

Before stating the main result and proceeding to its proof, we are going to recall the following version of the uniparametric  $T(1)$  Theorem.

**Theorem 4.1.** *Let  $T$  be a  $\delta$ -SIO on  $\mathbb{R}^d$  and  $K$  its kernel. If there exists a constant  $A > 0$  such that for every cube  $V \subset \mathbb{R}^d$*

$$(4.1) \quad \|T\chi_V\|_{L^1(V)} \leq A|V|$$

and

$$(4.2) \quad \|T^* \chi_V\|_{L^1(V)} \leq A|V|,$$

then  $T$  is a bounded operator on  $L^2$  such that  $\|T\|_{2 \rightarrow 2} \leq C_{\delta, d} \cdot (A + |K|_\delta)$ .

*Remark 4.2.* This version of the  $T1$  Theorem is not as well known as some others, but it follows, by a standard localization argument, from the classical versions (see e.g. [Ho]).

**Theorem 4.3.** *If  $T$  is a bi-parameter SIO in the sense of Martikainen/Pott-Villarroya (as defined in Section 3), then  $T$  is a bi-parameter SIO in the sense of Journé (as defined in Section 2). The converse statement is clear.*

*Proof.* Let  $T$  be a bi-parameter SIO in the sense of Martikainen/Pott-Villarroya, i.e.,  $T$  has a full kernel representation, with a kernel  $K$  satisfying the full standard estimates, and  $T$  has partial kernel representations, with kernels  $(K_{\cdot, \cdot}^1, K_{\cdot, \cdot}^2)$ .

We want to prove that there exists a pair  $(K_1, K_2)$  of  $\delta CZ - \delta$ -standard kernels such that:

If  $f_1, g_1 \in C_0^\infty(\mathbb{R}^n)$ , with  $\text{spt} f_1 \cap \text{spt} g_1 = \emptyset$  and  $f_2, g_2 \in C_0^\infty(\mathbb{R}^m)$ , then

$$\langle T f_1 \otimes f_2, g_1 \otimes g_2 \rangle = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} g_1(x_1) \langle K_1(x_1, y_1) f_2, g_2 \rangle f_1(y_1) dy_1 dx_1.$$

Similarly, if  $f_1, g_1 \in C_0^\infty(\mathbb{R}^n)$  and  $f_2, g_2 \in C_0^\infty(\mathbb{R}^m)$ , with  $\text{spt} f_2 \cap \text{spt} g_2 = \emptyset$ , then

$$\langle T f_1 \otimes f_2, g_1 \otimes g_2 \rangle = \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} g_2(x_2) \langle K_2(x_2, y_2) f_1, g_1 \rangle f_2(y_2) dy_2 dx_2.$$

Fix  $x_1, y_1 \in \mathbb{R}^n$ ,  $x_1 \neq y_1$ . We define  $K_1(x_1, y_1)$  via the bi-linear form as

$$\langle K_1(x_1, y_1)f_2, g_2 \rangle := K_{f_2, g_2}^2(x_1, y_1), \quad f_2, g_2 \in \mathcal{C}_0^\infty(\mathbb{R}^m).$$

This guarantees the desired representation: if  $\text{spt}f_1 \cap \text{spt}g_1 = \emptyset$ , then

$$\begin{aligned} \langle T(f_1 \otimes f_2), g_1 \otimes g_2 \rangle &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K_{f_2, g_2}^2(x_1, y_1) f_1(y_1) g_1(x_1) dy_1 dx_1 \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \langle K_1(x_1, y_1)f_2, g_2 \rangle f_1(y_1) g_1(x_1) dy_1 dx_1. \end{aligned}$$

Next, we will show that  $K_1(x_1, y_1)$  is an SIO in  $\mathbb{R}^m$  with kernel  $(x_2, y_2) \rightarrow K(x_1, x_2; y_1, y_2)$ . To this end, fix  $f_2, g_2$  with  $\text{spt}f_2 \cap \text{spt}g_2 = \emptyset$ . For every  $f_1, g_1 : \mathbb{R}^n \rightarrow \mathbb{C}$  with  $\text{spt}f_1 \cap \text{spt}g_1 = \emptyset$  we have that

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} K(x_1, x_2; y_1, y_2) f_2(y_2) g_2(x_2) dy_2 dx_2 \right] f_1(y_1) g_1(x_1) dx_1 dy_1$$

equals

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K_{f_2, g_2}^2(x_1, y_1) f_1(y_1) g_1(x_1) dy_1 dx_1.$$

This is because they both are equal to  $\langle T(f_1 \otimes f_2), g_1 \otimes g_2 \rangle$  since  $\text{spt}f_i \cap \text{spt}g_i = \emptyset$  for  $i = 1$  and  $i = 2$ . Therefore, it is easy to conclude that

$$\begin{aligned} \langle K_1(x_1, y_1)f_2, g_2 \rangle &= K_{f_2, g_2}^2(x_1, y_1) \\ &= \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} K(x_1, x_2; y_1, y_2) f_2(y_2) g_2(x_2) dy_2 dx_2 \end{aligned}$$

if  $\text{spt}f_2 \cap \text{spt}g_2 = \emptyset$ . So  $K_1(x_1, y_1)$  has a kernel representation and the standard kernel estimates hold with a constant  $\lesssim |x_1 - y_1|^n$ . The latter fact uses the size condition of  $K$ , and the mixed Hölder and size estimates of  $K$  and  $K^*$ .

Next, we will show that

$$\|K_1(x_1, y_1)\|_{L^2(\mathbb{R}^m) \rightarrow L^2(\mathbb{R}^m)} \lesssim |x_1 - y_1|^{-n}.$$

Since  $K_1(x_1, y_1)$  is a  $\delta$ -SIO, it is enough to verify the conditions of the local T1 Theorem (Theorem 4.1). Note that, by duality, it is enough to verify that

$$|\langle K_1(x_1, y_1)\chi_V, g_V \rangle| + |\langle g_V, K_1(x_1, y_1)\chi_V \rangle| \leq \frac{C}{|x_1 - y_1|^n} |V|$$

for all  $g_V \in L^\infty(V)$  such that  $\|g_V\|_\infty \leq 1$ .

Let  $V \subset \mathbb{R}^m$  be a cube and  $g : \mathbb{R}^m \rightarrow \mathbb{C}$  be such that  $\text{spt}g \subset V$  and  $|g| \leq 1$ . Then, it holds that

$$\begin{aligned} |\langle K_1(x_1, y_1)\chi_V, g_V \rangle| + |\langle g_V, K_1(x_1, y_1)\chi_V \rangle| &= |K_{\chi_V, g_V}^2(x_1, y_1)| + |K_{g_V, \chi_V}^2(x_1, y_1)| \\ &\leq \frac{C}{|x_1 - y_1|^n} |V| \end{aligned}$$

as a consequence of Lemma 3.8. Therefore, we conclude that

$$\|K_1(x_1, y_1)\|_{L^2(\mathbb{R}^m) \rightarrow L^2(\mathbb{R}^m)}.$$

To show that  $(x_1, y_1) \mapsto K_1(x_1, y_1)$  is a vector-valued standard kernel with values in the  $L^2(\mathbb{R}^m)$  bounded SIOs, it remains to estimate the kernel constants and  $L^2(\mathbb{R}^m) \rightarrow L^2(\mathbb{R}^m)$  norm of  $K_1(x_1, y_1) - K_1(x_1, y'_1)$  and  $K_1(x_1, y_1) - K_1(x'_1, y_1)$  for  $|y_1 - y'_1| \leq |x_1 - y_1|/2$  and  $|x_1 - x'_1| \leq |x_1 - y_1|/2$  respectively. But the correct bounds

for these follow from the various kernel estimates using analogous deductions to the above.

Defining  $K_2(x_2, y_2)$  analogously ends the proof of the theorem.  $\square$

**Corollary 4.4.** *If  $T$  is an  $L^2$  bounded bi-parameter SIO in the sense of Martikainen/Pott-Villarroya, then  $T(1)$ ,  $T^*(1) \in BMO_{prod}$ .*

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#### REFERENCES

- [C] Michael Christ, *A  $T(b)$  theorem with remarks on analytic capacity and the Cauchy integral*, Colloq. Math. **60/61** (1990), no. 2, 601–628. MR1096400 (92k:42020)
- [DJ] Guy David and Jean-Lin Journé, *A boundedness criterion for generalized Calderón-Zygmund operators*, Ann. of Math. (2) **120** (1984), no. 2, 371–397, DOI 10.2307/2006946. MR763911 (85k:42041)
- [G] Loukas Grafakos, *Classical and modern Fourier analysis*, Pearson Education, Inc., Upper Saddle River, NJ, 2004. MR2449250
- [Ho] Steve Hofmann, *Local  $T_b$  theorems and applications in PDE*, International Congress of Mathematicians. Vol. II, Eur. Math. Soc., Zürich, 2006, pp. 1375–1392. MR2275650 (2007k:42030)
- [HoM] Steve Hofmann and Alan McIntosh, *Boundedness and applications of singular integrals and square functions: a survey*, Bull. Math. Sci. **1** (2011), no. 2, 201–244, DOI 10.1007/s13373-011-0014-3. MR2901001
- [HyM1] Tuomas Hytönen and Henri Martikainen, *On general local  $T_b$  theorems*, Trans. Amer. Math. Soc. **364** (2012), no. 9, 4819–4846, DOI 10.1090/S0002-9947-2012-05599-1. MR2922611
- [HyM2] Tuomas Hytönen and Henri Martikainen, *Non-homogeneous  $T_1$  theorem for bi-parameter singular integrals*, Adv. Math. **261** (2014), 220–273, DOI 10.1016/j.aim.2014.02.011. MR3213300
- [HyN] T. Hytönen and F. Nazarov, *The local  $T_b$  Theorem with rough test functions*, arxiv:1206.0907.
- [J] Jean-Lin Journé, *Calderón-Zygmund operators on product spaces*, Rev. Mat. Iberoamericana **1** (1985), no. 3, 55–91, DOI 10.4171/RMI/15. MR836284 (88d:42028)
- [M] Henri Martikainen, *Representation of bi-parameter singular integrals by dyadic operators*, Adv. Math. **229** (2012), no. 3, 1734–1761, DOI 10.1016/j.aim.2011.12.019. MR2871155
- [PV] S. Pott and P. Villarroya, *A  $T(1)$  theorem on product spaces*, arxiv:1105.2516

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF HELSINKI, FINLAND  
*E-mail address:* ana.grau@helsinki.fi