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DOUBLE COSET SEPARABILITY OF ABELIAN SUBGROUPS OF HYPERBOLIC n-ORBIFOLD GROUPS

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Abstract. A subset X of a group G is said to be separable if it is closed in the profinite topology. Let $M = \mathbb{H}^n/\Gamma$ be a closed hyperbolic orbifold of dimension $n \geq 2$. We show that if H and K are abelian subgroups of Γ and $g \in \Gamma$, then the double coset HgK is separable in Γ. We generalize this result to cocompact lattices in linear, semisimple Lie groups of (real) rank one.

1. Introduction

In 1911, Max Dehn posed three important problems in combinatorial group theory: the word problem, the conjugacy problem, and the isomorphism problem. It is known that each problem is undecidable in general. However, algorithms have been developed for many well-known classes of groups. These decidability problems have been linked to subgroup separability.

Definition 1.1. The profinite topology on a group Γ is the coarsest topology in which every homomorphism from Γ to a finite group is continuous. If a subset X of Γ is closed in the profinite topology, then X is called *separable*. Equivalently, for every $\gamma \in \Gamma - X$ there is a homomorphism ϕ from Γ to a finite group such that $\phi(\gamma) \notin \phi(X)$.

1. A group Γ is *residually finite* if the trivial subgroup is separable in Γ.
2. A group Γ is *subgroup separable* or *locally extended residually finite (LERF)* if every finitely generated subgroup of Γ is separable in Γ.
3. A group Γ is *conjugacy separable* if every conjugacy class in Γ is separable.

Let Γ be a finitely presented group. If Γ is residually finite, then Γ has a solvable word problem, meaning that there exists an algorithm to decide if a given word in the presentation of Γ is trivial. If Γ is subgroup separable, then Γ has a solvable generalized word problem. Finally, if Γ is conjugacy separable, then Γ has a solvable conjugacy problem.

Subgroup separability is of great interest in low dimensional topology. If Γ is the fundamental group of a manifold M, then the profinite topology on Γ encodes the finite sheeted covering spaces of M. A precise connection between subgroup separability and geometric topology is given in [14]. In that paper, Scott proves that surface groups, finitely generated Fuchsian groups and fundamental groups of Seifert fibered 3-manifolds are subgroup separable. There has been recent dramatic progress in subgroup separability of hyperbolic 3-manifold groups. Agol proved

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that fundamental groups of closed hyperbolic 3-manifolds are subgroup separable [1]. Moreover, Wise proved that fundamental groups of hyperbolic 3-manifolds containing an embedded geometrically finite surface are subgroup separable [15]. The picture is more complicated for non-hyperbolic 3-manifolds. For example, Niblo-Wise showed that the fundamental group of a graph manifold M is subgroup separable if and only if M is geometric [12].

A related definition is *double coset separability.*

Definition 1.2. A group Γ is *double coset separable* if for every pair H, K of finitely generated subgroups of Γ, and every $g \in \Gamma$, the double coset HgK is separable in Γ.

It is known that free groups, finitely generated Fuchsian groups, surface groups, and fundamental groups of Seifert fibered 3-manifolds are double coset separable [11]. In [9], it is shown that if $M = \mathbb{H}^3/\Gamma$ is a hyperbolic 3-orbifold of finite volume, then for every pair H and K of abelian subgroups of Γ and every element $g \in \Gamma$, the double coset HgK is separable in Γ. This is used to prove that if M is a compact, orientable 3-manifold, then $\pi_1(M)$ is conjugacy separable [9]. In this paper, we consider double coset separability of fundamental groups of closed hyperbolic orbifolds of dimension $n \geq 2$. In particular, we prove the following theorem.

Theorem 1.3. *Let $M = \mathbb{H}^n/\Gamma$ be a closed hyperbolic orbifold of dimension $n \geq 2$. Let H and K be abelian subgroups of Γ, and let $g \in \Gamma$. Then the double coset $HgK = \{hgk \mid h \in H, k \in K\}$ is separable in Γ.*

To prove Theorem 1.3, we first reduce to the case where $M = \mathbb{H}^n/\Gamma$ is a closed, orientable hyperbolic manifold of dimension $n \geq 2$. Therefore, the non-trivial abelian subgroups of Γ will be free abelian of rank 1, generated by loxodromic isometries.

Theorem 1.3 does not follow from the recent work of Agol and Wise. Indeed, although many examples of closed hyperbolic n-orbifolds are known to be virtually special [4], it is as yet open as to whether every closed hyperbolic n-orbifold is virtually special.

This paper is organized as follows. In Section 2 we state algebraic preliminaries. In Section 3 we prove Theorem 1.3. In Section 4 we discuss some generalizations of Theorem 1.3. For example, we prove the following corollary to the proof of Theorem 1.3.

Theorem 1.4. *Let Γ be a cocompact lattice in $SO(n,1)$, $SU(n,1)$, $Sp(n,1)$, $n \geq 2$, or F_4^{-20}. Let H and K be abelian subgroups of Γ, and let $g \in \Gamma$. Then the double coset $HgK = \{hgk \mid h \in H, k \in K\}$ is separable in Γ.*

The virtually special technology does not apply in the setting of Theorem 1.4. Cocompact lattices in $Sp(n,1)$, $n \geq 2$, never virtually inject in a RAAG because they have Property (T). Cocompact lattices in $SU(n,1)$, $n \geq 2$, never virtually inject in a RAAG because they are Kahler [13].

2. Algebraic preliminaries

In this section we state three propositions that will be used in the proof of Theorem 1.3. The first proposition is well known. The second two propositions are from [8].

Proposition 2.1. *Let R be a finitely generated ring in a number field k, and let $x_1, x_2, \ldots, x_j$ be non-zero elements of R. Then there exist a finite field F and a ring homomorphism $\eta : R \to F$ such that $\eta(x_i) \neq 0$, for each $1 \leq i \leq j$.*

For a proof of Proposition 2.1 see Lemma 4.2 of [10]. We now state a generalization of Proposition 2.1.

Proposition 2.2. *Let R be a finitely generated ring in a number field k, let δ be a non-zero element of R that is not a root of unity, and let $x_1, x_2, \ldots, x_j$ be non-zero elements of R. Then there exists a positive integer n with the following property. For each integer $m \geq n$, there exist a finite field F and a ring homomorphism $\eta : R \to F$ such that the multiplicative order of $\eta(\delta)$ is equal to m and $\eta(x_i) \neq 0$, for each $1 \leq i \leq j$.*

For the proof of Proposition 2.2 see Corollary 2.5 of [8].

Proposition 2.3. *Let R be a finitely generated ring in a number field k. Let λ and ω be non-zero elements of R such that λ is not a multiplicative power of ω. Then there exist a finite ring S and a ring homomorphism $\eta : R \to S$ such that $\eta(\lambda)$ is not a multiplicative power of $\eta(\omega)$.*

For the proof of Proposition 2.3 see Corollary 2.8 of [8].

3. Proof of double-coset separability

In this section we prove that double cosets of abelian subgroups of hyperbolic n-orbifold groups are separable. For the proof will use the following proposition from [11].

Proposition 3.1. *Let G_0, H and K be finitely generated subgroups of a group G and set $H_0 = H \cap G_0$ and $K_0 = K \cap G_0$. If $[G : G_0]$ is finite and if H_0K_0 is separable in G_0, then HK is separable in G.*

For a proof of Proposition 3.1 see Proposition 2.2 of [11].

Theorem 3.2. *Let $M = \mathbb{H}^n/\Gamma$ be a closed hyperbolic orbifold of dimension $n \geq 2$. Let H and K be abelian subgroups of Γ, and let $g \in \Gamma$. Then the double coset $HgK = \{hgk \mid h \in H, k \in K\}$ is separable in Γ.*

Proof. Since finitely generated Fuchsian groups are double coset separable [11], we may assume that the dimension of M is greater than or equal to 3.

As noted in [11], since the profinite topology on Γ is equivariant under left and right multiplication, the double coset HgK is closed in Γ if and only if $H^gK = g^{-1}HgK$ is closed in Γ. Therefore, to prove the theorem, it suffices to show that if H and K are abelian subgroups of Γ, then the double coset HK is separable in Γ.

By Selberg's Lemma [2], Γ has a subgroup of finite index Γ_0 which is torsion free. We may assume that Γ_0 consists of orientation-preserving isometries of $\mathbb{H}^n$. Let $H_0 = H \cap \Gamma_0$ and $K_0 = K \cap \Gamma_0$. By Proposition 3.1, if H_0K_0 is separable in Γ_0, then HK is separable in Γ. Therefore, we may assume that Γ is torsion free, and thus is the fundamental group of an orientable, hyperbolic manifold of dimension $n \geq 3$. Since abelian subgroups of fundamental groups of closed hyperbolic n-manifolds are separable [7], we may assume that H and K do not commute. In particular, both H and K are non-trivial. Since M is closed and Γ is torsion free, the non-trivial abelian subgroups of Γ are free abelian of rank 1, generated by loxodromic isometries. Let

H' be the maximal abelian subgroup of Γ containing H. Then H has finite index in H'. Let $\{a_1, a_2, \ldots, a_{m-1}\}$ be a set of non-trivial coset representatives of H'/H. By [7], H is a separable subgroup of Γ. Therefore, there exists a subgroup Γ_H of finite index in Γ such that $H \subset \Gamma_H$ but $\Gamma_H \cap \{a_1, a_2, \ldots, a_{m-1}\} = \emptyset$. Then $\Gamma_H \cap H' = H$. In a similar way, define K' and choose Γ_K of finite index in Γ such that $\Gamma_K \cap K' = K$. Let $\Gamma_0 = \Gamma_H \cap \Gamma_K$ and set $H_0 = H \cap \Gamma_0$ and $K_0 = K \cap \Gamma_0$. Then Γ_0 is a subgroup of finite index in Γ, $H_0 = H' \cap \Gamma_0$ and $K_0 = K' \cap \Gamma_0$. By Proposition 3.1, by replacing Γ with Γ_0, if necessary, we may assume that $H = \langle h \rangle$ and $K = \langle k \rangle$ are maximal abelian subgroups of Γ.

Since M is a hyperbolic manifold of dimension n, there exists a discrete, faithful representation of Γ into $SO(n,1;\mathbb{R})$. Since $n \geq 3$, it follows from Mostow Rigidity that the trace field of Γ,

$$\mathbb{Q}(\mathrm{tr}\Gamma) = \mathbb{Q}(\mathrm{tr}\ (\gamma) \mid \gamma \in \Gamma),$$

is a number field. We may conjugate Γ in $\mathrm{GL}(n+1, \mathbb{C})$ to lie in a finite field extension of $\mathbb{Q}(\mathrm{tr}\Gamma)$ [3]. Therefore, we view $\Gamma \subset \mathrm{SL}(n+1, F)$ for some number field F. Let T_h and T_k be the linear transformations in $\mathrm{SL}(n+1, F)$ associated to h and k, respectively. Since h and k are loxodromic, T_h and T_k are semisimple. (See Proposition 3.2.3 of [5].) Let $\{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ and $\{\omega_0, \omega_1, \ldots, \omega_n\}$ denote the eigenvalues of T_h and T_k, respectively. There exists a natural number m such that the quotient of any two distinct elements of $\{\omega_0^m, \omega_1^m, \ldots, \omega_n^m\}$ is not a root of unity. Since K is a separable subgroup of Γ, there exists a subgroup Γ_0 of finite index in Γ such that $\Gamma_0 \cap K = \langle k^m \rangle$. By Proposition 3.1, by replacing Γ with Γ_0, if necessary, we may assume that the quotient of any two distinct eigenvalues of T_k is not a root of unity.

Let L be the field obtained by adjoining the eigenvalues of T_h and T_k to the field F. Since the eigenvalues of T_h and T_k are algebraic numbers, L is a number field. The transformations T_h and T_k are diagonalizable over L. Since T_h is diagonalizable over L, there exists $P \in \mathrm{GL}(n+1, L)$ such that $P^{-1}hP$ is a diagonal matrix. Denote this diagonal matrix by D_h. Since T_k is diagonalizable over L and $P^{-1}kP \in \mathrm{SL}(n+1, L)$, there exists $A \in \mathrm{GL}(n+1, L)$ such that $A^{-1}P^{-1}kPA = D_k$, where D_k is a diagonal matrix. Then $P^{-1}kP = AD_kA^{-1}$. Define an isomorphism ψ from Γ into $\mathrm{SL}(n+1, L)$ by $\psi(\Gamma) = P^{-1}\Gamma P$. Let R be the ring generated by the coefficients of the generators of $\psi(\Gamma)$ over $\mathbb{Z}$. Note that the eigenvalues of T_h are contained in R. By expanding R, if necessary, we may assume that the eigenvalues of T_k and the entries of A are contained in R. By replacing our original representation of Γ into $SO(1,n;\mathbb{R})$ with the new representation ψ of Γ into $\mathrm{SL}(n+1, R)$ we now have the following assumptions:

(1) The group $\Gamma \subset \mathrm{SL}(n+1, R)$, where R is a finitely generated ring in a number field L.
(2) The subgroup $H = \langle h \rangle = \langle D_h \rangle$, where D_h is a diagonal matrix with eigenvalues $\{\lambda_0, \lambda_1, \ldots, \lambda_n\}$.
(3) The subgroup $K = \langle k \rangle = \langle AD_kA^{-1} \rangle$, where $D_k \in \mathrm{SL}(n+1, R)$ is a diagonal matrix with eigenvalues $\{\omega_0, \omega_1, \ldots, \omega_n\}$ and $A \in \mathrm{GL}(n+1, R)$.
(4) The quotient of any two distinct elements of $\{\omega_0, \omega_1, \ldots, \omega_n\}$ is not a root of unity.
(5) H and K are distinct maximal abelian subgroups of Γ.

To prove that the double coset $HK = \langle D_h\rangle A\langle D_k\rangle A^{-1}$ is separable in Γ, let $B \in \Gamma - HK$. We will find a group homomorphism from Γ into a finite group such that the image of BA does not lie in the image of $\langle D_h\rangle A\langle D_k\rangle$. Then the image of B does not lie in the image of HK, as required.

Write

$$BA = \begin{pmatrix} x_{00} & x_{01} & \dots & x_{0n} \\ x_{10} & x_{11} & \dots & x_{1n} \\ \dots & \dots & \dots & \dots \\ x_{n0} & x_{n1} & \dots & x_{nn} \end{pmatrix},$$

and note that the set $\langle D_h\rangle A\langle D_k\rangle$ has the form

$$\begin{pmatrix} \lambda_0^s & 0 & 0 & \dots & 0 \\ 0 & \lambda_1^s & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \lambda_n^s \end{pmatrix} \begin{pmatrix} a_{00} & a_{01} & \dots & a_{0n} \\ a_{10} & a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots \\ a_{n0} & a_{n1} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} \omega_0^t & 0 & 0 & \dots & 0 \\ 0 & \omega_1^t & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \omega_n^t \end{pmatrix}$$

$$= \begin{pmatrix} a_{00}\lambda_0^s\omega_0^t & a_{01}\lambda_0^s\omega_1^t & \dots & a_{0n}\lambda_0^s\omega_n^t \\ a_{10}\lambda_1^s\omega_0^t & a_{11}\lambda_1^s\omega_1^t & \dots & a_{1n}\lambda_1^s\omega_n^t \\ \dots & \dots & \dots & \dots \\ a_{n0}\lambda_n^s\omega_0^t & a_{n1}\lambda_n^s\omega_1^t & \dots & a_{nn}\lambda_n^s\omega_n^t \end{pmatrix}, \text{ where } s, t \in \mathbb{Z}.$$

Since H and K do not commute, $k = AD_kA^{-1}$ is not a diagonal matrix. Since the set of generalized permutation matrices is the normalizer of the set of diagonal matrices, this implies that A is not a generalized permutation matrix. Therefore, there exists a row of A with at least two non-zero entries, a_{ij} and a_{il}, $j \neq l$. For any such pair, we prove the following.

Claim. Let a_{ij} and a_{il}, $j \neq l$, be two non-zero entries in a row of A. Then at least one of the following conditions is satisfied:

1. There exists a group homomorphism from Γ into a finite group such that the image of B is not contained in the image of HK, which completes the proof of the theorem.
2. The corresponding entries x_{ij} and x_{il} in BA are non-zero and
$$\frac{x_{ij}a_{il}}{x_{il}a_{ij}} \in \left\langle \frac{\omega_j}{\omega_l} \right\rangle.$$

Proof of the Claim. Suppose that $x_{ij} = 0$. Since the determinants of T_h and T_k are equal to 1, $\lambda_i \neq 0$ and $\omega_j \neq 0$. By Proposition 2.1, there exist a finite field F and a ring homomorphism $\eta : R \to F$ such that $\eta(a_{ij}) \neq 0$, $\eta(\lambda_i) \neq 0$ and $\eta(\omega_j) \neq 0$. The map η induces a group homomorphism

$$\overline{\eta} : \Gamma \hookrightarrow \mathrm{SL}(n+1, R) \to \mathrm{SL}(n+1, F).$$

Suppose that $\overline{\eta}(BA) \in \overline{\eta}(\langle D_h\rangle A\langle D_k\rangle)$. Then there exist integers s and t such that

$$\overline{\eta}(BA) = \overline{\eta}(D_h^s A D_k^t).$$

By equating coefficients,

$$0 = \eta(x_{ij}) = \eta(a_{ij}\lambda_i^s\omega_j^t),$$

a contradiction. We conclude that $\overline{\eta}(BA) \notin \overline{\eta}(\langle D_h\rangle A\langle D_k\rangle)$, and thus $\overline{\eta}(B) \notin \overline{\eta}(HK)$. Therefore, condition 1 is satisfied. For the remainder of the proof we may assume that $x_{ij} \neq 0$. Similarly, we may assume that $x_{il} \neq 0$.

Suppose that condition 2 is not satisfied, meaning that $x_{ij}a_{il}/x_{il}a_{ij}$ is not a multiplicative power of ω_j/ω_l. Then by Proposition 2.3, there exist a finite ring S and a ring homomorphism $\eta : R \to S$ such that $\eta(x_{ij}a_{il}/x_{il}a_{ij})$ is not a multiplicative power of $\eta(\omega_j/\omega_l)$. The map η induces a group homomorphism

$$\overline{\eta} : \Gamma \hookrightarrow \mathrm{SL}(n+1, R) \to \mathrm{SL}(n+1, S).$$

Suppose that $\overline{\eta}(BA) \in \overline{\eta}(\langle D_h \rangle A \langle D_k \rangle)$. Then there exist integers s and t such that

$$\overline{\eta}(BA) = \overline{\eta}(D_h^s A D_k^t).$$

By equating coefficients,

$$\eta(x_{ij}) = \eta(a_{ij}\lambda_i^s \omega_j^t) \text{ and } \eta(x_{il}) = \eta(a_{il}\lambda_i^s \omega_l^t).$$

By dividing,

$$\eta(x_{ij}/x_{il}) = \eta(a_{ij}/a_{il})\eta(\omega_j/\omega_l)^t,$$

a contradiction. We conclude that $\overline{\eta}(BA) \notin \overline{\eta}(\langle D_h \rangle A \langle D_k \rangle)$, and thus $\overline{\eta}(B) \notin \overline{\eta}(HK)$. This completes the proof of the claim.

For the remainder of the proof, we may assume that condition 2 of the claim holds for any pair of non-zero entries in any row of A. Therefore, given a pair of non-zero entries a_{ij} and a_{il}, $j \neq l$, there exists an integer $t(a_{ij}, a_{il})$, depending on a_{ij} and a_{il}, such that

$$\frac{x_{ij}a_{il}}{x_{il}a_{ij}} = \left(\frac{\omega_j}{\omega_l}\right)^{t(a_{ij},a_{il})}.$$

Case 1. There exists an integer t_0 such that, for any pair of non-zero entries, a_{ij} and a_{il}, $j \neq l$, in a row of A,

$$\frac{x_{ij}a_{il}}{x_{il}a_{ij}} = \left(\frac{\omega_j}{\omega_l}\right)^{t_0}.$$

The matrix $BAD_k^{-t_0}A^{-1} = Bk^{-t_0}$ is not diagonal. If Bk^{-t_0} were diagonal, then Bk^{-t_0} would commute with H. Since $Bk^{-t_0} \in \Gamma$ and H is a maximal abelian subgroup of Γ, this would imply that $Bk^{-t_0} = h^{s_0}$ for some $s_0 \in \mathbb{Z}$. But then $B = h^{s_0}k^{t_0} \in HK$, a contradiction. Let y_{ij}, $i \neq j$, be a non-zero entry of $BAD_k^{-t_0}A^{-1}$ that does not lie on the diagonal. By Proposition 2.1, there exist a finite field F and a ring homomorphism $\eta : R \to F$ such that $\eta(y_{ij}) \neq 0$. The map η induces a group homomorphism

$$\overline{\eta} : \Gamma \hookrightarrow \mathrm{SL}(n+1, R) \to \mathrm{SL}(n+1, F).$$

Since $\eta(y_{ij}) \neq 0$, $\overline{\eta}(BAD_k^{-t_0}A^{-1})$ is not a diagonal matrix. We will show that

$$\overline{\eta}(BAD_k^{-t_0}) \notin \overline{\eta}(\langle D_h \rangle A \langle D_k \rangle).$$

Suppose, to the contrary, that $\overline{\eta}(BAD_k^{-t_0}) = \overline{\eta}(D_h^s AD_k^t)$, for some integers s and t. Then

$$\begin{pmatrix} \eta(x_{00}\omega_0^{-t_0}) & \eta(x_{01}\omega_1^{-t_0}) & \dots & \eta(x_{0n}\omega_n^{-t_0}) \\ \eta(x_{10}\omega_0^{-t_0}) & \eta(x_{11}\omega_1^{-t_0}) & \dots & \eta(x_{1n}\omega_n^{-t_0}) \\ \dots & \dots & \dots & \dots \\ \eta(x_{n0}\omega_0^{-t_0}) & \eta(x_{n1}\omega_1^{-t_0}) & \dots & \eta(x_{nn}\omega_n^{-t_0}) \end{pmatrix}$$

$$= \begin{pmatrix} \eta(a_{00}\lambda_0^s\omega_0^t) & \eta(a_{01}\lambda_0^s\omega_1^t) & \dots & \eta(a_{0n}\lambda_0^s\omega_n^t) \\ \eta(a_{10}\lambda_1^s\omega_0^t) & \eta(a_{11}\lambda_1^s\omega_1^t) & \dots & \eta(a_{1n}\lambda_1^s\omega_n^t) \\ \dots & \dots & \dots & \dots \\ \eta(a_{n0}\lambda_n^s\omega_0^t) & \eta(a_{n1}\lambda_n^s\omega_1^t) & \dots & \eta(a_{nn}\lambda_n^s\omega_n^t) \end{pmatrix}.$$

Let $\{a_{ij}, a_{il}\},\ j \neq l$, be a pair of non-zero entries in any row of A. By equating coefficients,

$$\eta(x_{ij}\omega_j^{-t_0}) = \eta(a_{ij}\lambda_i^s\omega_j^t) \text{ and } \eta(x_{il}\omega_l^{-t_0}) = \eta(a_{il}\lambda_i^s\omega_l^t).$$

By dividing,

$$\frac{\eta(\omega_j^t)}{\eta(\omega_l^t)} = \frac{\eta(x_{ij}a_{il})}{\eta(x_{il}a_{ij})}\left(\frac{\eta(\omega_j)}{\eta(\omega_l)}\right)^{-t_0} = 1.$$

We conclude that for any pair $\{a_{ij}, a_{il}\}$ of non-zero entries in any row of A, $\eta(\omega_j^t) = \eta(\omega_l^t)$. For each row i of A, choose a non-zero entry a_{ij} and set

$$d_i = \eta(\lambda_i^s\omega_j^t).$$

The element d_i is well defined by the argument above. Then

$$\overline{\eta}(BAD_k^{-t_0}) = \overline{\eta}(D_h^s AD_k^t) = \begin{pmatrix} \eta(a_{00}\lambda_0^s\omega_0^t) & \eta(a_{01}\lambda_0^s\omega_1^t) & \dots & \eta(a_{0n}\lambda_0^s\omega_n^t) \\ \eta(a_{10}\lambda_1^s\omega_0^t) & \eta(a_{11}\lambda_1^s\omega_1^t) & \dots & \eta(a_{1n}\lambda_1^s\omega_n^t) \\ \dots & \dots & \dots & \dots \\ \eta(a_{n0}\lambda_n^s\omega_0^t) & \eta(a_{n1}\lambda_n^s\omega_1^t) & \dots & \eta(a_{nn}\lambda_n^s\omega_n^t) \end{pmatrix} =$$

$$\begin{pmatrix} d_0 & 0 & 0 & \dots & 0 \\ 0 & d_1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & d_n \end{pmatrix} \begin{pmatrix} \eta(a_{00}) & \eta(a_{01}) & \dots & \eta(a_{0n}) \\ \eta(a_{10}) & \eta(a_{11}) & \dots & \eta(a_{1n}) \\ \dots & \dots & \dots & \dots \\ \eta(a_{n0}) & \eta(a_{n1}) & \dots & \eta(a_{nn}) \end{pmatrix}.$$

Thus, $\overline{\eta}(BAD_k^{-t_0}A^{-1})$ is a diagonal matrix, a contradiction. We conclude that $\overline{\eta}(BAD_k^{-t_0}) \notin \overline{\eta}(\langle D_h\rangle A\langle D_k\rangle)$, and thus $\overline{\eta}(BA) \notin \overline{\eta}(\langle D_h\rangle A\langle D_k\rangle)$. This implies that $\overline{\eta}(B) \notin HK$, as required.

Case 2. There does not exist an integer t_0 such that, for any pair of non-zero entries, a_{ij} and $a_{il},\ j \neq l$, in a row of A,

$$\frac{x_{ij}a_{il}}{x_{il}a_{ij}} = \left(\frac{\omega_j}{\omega_l}\right)^{t_0}.$$

By assumption, given a pair of non-zero entries a_{ij} and $a_{il},\ j \neq l$, there exists an integer $t(a_{ij}, a_{il})$, depending on a_{ij} and a_{il}, such that

$$\frac{x_{ij}a_{il}}{x_{il}a_{ij}} = \left(\frac{\omega_j}{\omega_l}\right)^{t(a_{ij},a_{il})}.$$

If $\omega_j = \omega_l$, then $x_{ij}a_{il}/x_{il}a_{ij} = 1$. This implies that

$$\frac{x_{ij}a_{il}}{x_{il}a_{ij}} = \left(\frac{\omega_j}{\omega_l}\right)^t, \text{ for every } t \in \mathbb{Z}.$$

Therefore, in this case, there must exist two pairs $\{a_{ij}, a_{il}\}$ and $\{a_{uv}, a_{uw}\}$ of nonzero entries in rows of A such that $t(a_{ij}, a_{il}) \neq t(a_{uv}, a_{uw})$, $\omega_j \neq \omega_l$, and $\omega_v \neq \omega_w$. Our assumptions at the beginning of the proof then imply that ω_j/ω_l and ω_v/ω_w are not roots of unity.

Let p be a prime that does not divide $t(a_{ij}, a_{il}) - t(a_{uv}, a_{uw})$. If p is sufficiently large, then by Proposition 2.2, there exist finite fields F_1 and F_2, and ring homomorphisms

$$\eta_1 : R \to F_1 \text{ and } \eta_2 : R \to F_2,$$

such that the multiplicative orders of $\eta_1(\omega_j/\omega_l)$ and $\eta_2(\omega_v/\omega_w)$ are equal to p.

The ring homomorphism

$$\eta = \eta_1 \times \eta_2 : R \to F_1 \times F_2$$

induces a group homomorphism

$$\overline{\eta} : \Gamma \hookrightarrow \mathrm{SL}(n+1, R) \to \mathrm{SL}(n+1, F_1 \times F_2).$$

Suppose that $\overline{\eta}(BA) \in \overline{\eta}(\langle D_h\rangle A \langle D_k\rangle)$. Then there exist integers s and t such that

$$\overline{\eta}(BA) = \overline{\eta}(D_h^s A D_k^t).$$

By equating coefficients,

$$\eta(x_{ij}) = \eta(a_{ij}\lambda_i^s\omega_j^t), \ \eta(x_{il}) = \eta(a_{il}\lambda_i^s\omega_l^t)$$

and

$$\eta(x_{uv}) = \eta(a_{uv}\lambda_u^s\omega_v^t), \ \eta(x_{uw}) = \eta(a_{uw}\lambda_u^s\omega_w^t).$$

By dividing,

$$\left(\frac{\eta(\omega_j)}{\eta(\omega_l)}\right)^t = \frac{\eta(x_{ij}a_{il})}{\eta(x_{il}a_{ij})} = \left(\frac{\eta(\omega_j)}{\eta(\omega_l)}\right)^{t(a_{ij},a_{il})}$$

and

$$\left(\frac{\eta(\omega_v)}{\eta(\omega_w)}\right)^t = \frac{\eta(x_{uv}a_{uw})}{\eta(x_{uw}a_{uv})} = \left(\frac{\eta(\omega_v)}{\eta(\omega_w)}\right)^{t(a_{uv},a_{uw})}.$$

Therefore,

$$\eta_1(\omega_j/\omega_l)^t = \eta_1(\omega_j/\omega_l)^{t(a_{ij},a_{il})} \quad \text{and} \quad \eta_2(\omega_v/\omega_w)^t = \eta_2(\omega_v/\omega_w)^{t(a_{uv},a_{uw})}.$$

This implies that

$$t \equiv t(a_{ij}, a_{il}) \pmod p \quad \text{and} \quad t \equiv t(a_{uv}, a_{uw}) \pmod p.$$

Thus p divides $t(a_{ij}, a_{il}) - t(a_{uv}, a_{uw})$, a contradiction. We conclude that $\overline{\eta}(BA) \notin \overline{\eta}(\langle D_h\rangle A\langle D_k\rangle)$, implying that $\overline{\eta}(B) \notin HK$, as required. $\square$

4. Generalizations of Theorem 3.2

In the proof of Theorem 3.2, we used the assumption that $M = \mathbb{H}^n/\Gamma$ is a closed hyperbolic orbifold of dimension $n \geq 3$ to draw the following conclusions:

1. Γ is finitely generated and there exists a discrete, faithful representation ρ from Γ into $\mathrm{GL}(m, \mathbb{C})$, for some natural number m.
2. The trace field of $\rho(\Gamma)$ is a number field.
3. If Γ is torsion free, then the non-trivial abelian subgroups of $\rho(\Gamma)$ are free abelian of rank 1, generated by semisimple linear transformations.
4. Cyclic subgroups of $\rho(\Gamma)$ generated by semisimple linear transformations are separable in Γ.

Conditions 1, 2 and 4 are true under the weaker assumption that $M = \mathbb{H}^n/\Gamma$ has finite volume. Therefore, we have the following corollary to the proof of Theorem 3.2.

Theorem 4.1. *Let $M = \mathbb{H}^n/\Gamma$ be a finite volume hyperbolic orbifold of dimension $n \geq 2$. Let H and K be cyclic subgroups of Γ, generated by loxodromic isometries, and let $g \in \Gamma$. Then the double coset $HgK = \{hgk \ \mid h \in H, k \in K\}$ is separable in Γ.*

All four conditions are true for a class of cocompact lattices in semisimple Lie groups. Therefore, we have a second corollary to the proof of Theorem 3.2.

Theorem 4.2. *Let Γ be a cocompact lattice in $SO(n,1)$, $SU(n,1)$, $Sp(n,1)$, $n \geq 2$, or F_4^{-20}. Let H and K be abelian subgroups of Γ, and let $g \in \Gamma$. Then the double coset $HgK = \{hgk \ \mid h \in H, k \in K\}$ is separable in Γ.*

Proof. Cocompact lattices in $SO(2,1)$ are double coset separable [11]. Let G be one of $SO(n,1)$, $n \geq 3$, $SU(n,1)$, $n \geq 2$, $Sp(n,1)$, $n \geq 2$, or F_4^{-20}, and let Γ be a cocompact lattice in G. If suffices to verify the four conditions above. Since Γ is cocompact, Γ is finitely generated. Since G is a semisimple Lie group that is not locally isomorphic to $\mathrm{SL}(2,\mathbb{R})$, and Γ is a cocompact lattice in G, by work of Calabi-Vesentini, Selberg, Calabi and Weil, the defining embedding of Γ in G is locally rigid. (See Theorem 3.1 of [6].) It follows that the trace field of Γ is a number field. Since Γ is cocompact and G is of (real) rank one, the non-trivial, torsion free, abelian subgroups of Γ are free abelian of rank 1, generated by semisimple linear transformations. Condition 4 follows from the proof of Theorem 2 in [7]. $\square$

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