PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 144, Number 6, June 2016, Pages 2521–2532
http://dx.doi.org/10.1090/proc/12906
Article electronically published on October 5, 2015

FINITE ERGODIC INDEX AND ASYMMETRY FOR INFINITE MEASURE PRESERVING ACTIONS

ALEXANDRE I. DANILENKO

(Communicated by Nimish Shah)

Abstract. Given $k > 0$ and an Abelian countable discrete group G with elements of infinite order, we construct (i) rigid funny rank-one infinite measure preserving (i.m.p.) G-actions of ergodic index k, (ii) 0-type funny rank-one i.m.p. G-actions of ergodic index k, (iii) funny rank-one i.m.p. G-actions T of ergodic index 2 such that the product $T\times T^{-1}$ is not ergodic. It is shown that $T \times T^{-1}$ is conservative for each funny rank-one G-action T.

0. Introduction

Let G be a discrete countable Abelian group and let $T = (T_g)_{g\in G}$ be a measure preserving action of G on an *infinite* σ-finite standard measure space $(X, \mathfrak{B}, \mu)$. By T^{-1} we denote the "inverse to T" action of G, i.e. $T^{-1} := (T_{-g})_{g\in G}$. Given two G-actions S and R of G, we denote by $S \times R$ and $S \otimes R$ the following product actions of G and $G\times G$ respectively on the product of the underlying measure spaces: $S\times R := (S_g\times R_g)_{g\in G}$ and $S\otimes R := (S_g\times R_h)_{g,h\in G}$. If S and R are both conservative or ergodic, then $S \otimes R$ is also conservative or ergodic respectively. However the product $S \times R$ can be neither ergodic nor conservative. If $T \times \cdots \times T\,(k$ times) is ergodic but $T \times \cdots \times T\,(k + 1$ times) is not, then T is said to *have ergodic index* k. In 1963, Kakutani and Parry [KaPa] constructed, for each k, an infinite Markov shift (i.e. $\mathbb{Z}$-action) of ergodic index k. In their examples, the product $T \times \cdots \times T\,(k$ times) is ergodic if and only if it is conservative. For a half-century their examples were the only examples of transformations of finite ergodic index $k > 1$. Recently another family of $\mathbb{Z}$-actions of an arbitrary finite ergodic index appeared in [AdSi] by Adams and Silva. That family consists of rank-one transformations T with *infinite conservative index*, i.e. $T \times \cdots \times T\,(l$ times) is conservative for each $l > 0$. We extend and refine their result to the Abelian groups containing elements of infinite order in the following way.

Theorem 0.1. *Let G have an element of infinite order. For each $k > 0$, there is a rigid funny rank-one*[1] *G-action T of ergodic index k. Moreover, the G-action*

Received by the editors December 13, 2014 and, in revised form, June 19, 2015 and July 11, 2015.
2010 *Mathematics Subject Classification.* Primary 37A40.

[1]Funny rank-one means that there is a sequence $(F_n)_{n=1}^{\infty}$ of finite subsets in G and a sequence $(A_n)_{n=1}^{\infty}$ of subsets of finite measure such that $T_gA_n \cap T_hA_n = \emptyset$ whenever $g \neq h \in F_n$ and the sequence of T-towers $\{T_gA_n \mid g \in F_n\}$ approximates the entire Borel σ-algebra as $n \to \infty$. In the case $G = \mathbb{Z}^k$, if each F_n is a cube, then T is said to be of rank one.

$\underbrace{T \times \cdots \times T}_{m\ times} \times \underbrace{T^{-1} \times \cdots \times T^{-1}}_{k-m\ times}$ *is ergodic for every* $m \in \{0, 1, \ldots, k-1\}$. *Furthermore, if* $G = \mathbb{Z}^d$ *for some* $d > 0$, *then* T *can be chosen in the class of rank-one actions.*

We note that T has infinite conservative index if T is rigid. We also note that while the proof of the first claim of Theorem 0.1 in [AdSi] (in the case $G = \mathbb{Z}$) is somewhat tricky, our proof is shorter and more direct.

In the next theorem we construct funny rank-one actions of finite ergodic index which are *mixing* (called also *zero type*; see [DaSi] and references therein), i.e. $\lim_{g\to\infty} \mu(T_g A \cap A) = 0$ for each subset A of finite measure. Thus these actions (in the case where $G = \mathbb{Z}$) are different from those constructed in [KaPa] and [AdSi].

Theorem 0.2. *Let* G *have an element of infinite order. For each* $k > 0$, *there is a mixing (zero type) funny rank-one* G*-action* T *of ergodic index* k *such that* $T \times \cdots \times T\,(k+1$ *times) is conservative but* $T \times \cdots \times T\,(k+2$ *times) is non-conservative. Moreover, the* G*-action* $\underbrace{T \times \cdots \times T}_{m\ times} \times \underbrace{T^{-1} \times \cdots \times T^{-1}}_{k-m\ times}$ *is ergodic for every* $m \in \{0, 1, \ldots, k-1\}$. *Furthermore, if* $G = \mathbb{Z}^d$ *for some* $d > 0$, *then* T *can be chosen in the class of rank-one actions.*

In a recent paper [Cl-Va], rank-one transformations T are constructed such that the product $T \times T$ is ergodic but $T \times T^{-1}$ is not. This is a partial answer to the following question of Bergelson (see problem P10 in [Da1]): is there a transformation T with infinite ergodic index and such that $T \times T^{-1}$ is non-ergodic? The next theorem extends this result to the actions of Abelian groups and simplifies the original proof. Moreover, we show (confirming a conjecture from [Cl-Va]) that these examples do not answer Bergelson's question completely because the G-action $T \times T \times T$ is not even conservative.

Theorem 0.3. *Let* G *have an element of infinite order. There is a funny rank-one action* T *of* G *of ergodic index* 2 *such that* $T \times T^{-1}$ *is non-ergodic but conservative and* $T \times T \times T$ *is non-conservative. Furthermore, if* $G = \mathbb{Z}^d$ *for some* $d > 0$, *then* T *can be chosen in the class of rank-one actions.*

It follows, in particular, that T is asymmetric, i.e. not isomorphic to T^{-1}. Thus, Theorem 0.3 illustrates that even such a simple invariant as "ergodicity of products" can distinguish between T and T^{-1}. Of course, this is impossible in the framework of finite measure preserving actions. For other, more involved examples of asymmetric infinite measure preserving systems we refer to [DaRy] and [Ry].

It was shown in [Cl-Va] that for each rank-one $\mathbb{Z}$-action T, the product $T \times T^{-1}$ is conservative. We generalize this result to the funny rank-one action of Abelian groups.

Theorem 0.4. *Let* T *be a funny rank-one action of* G. *Then the* G*-action* $T \times T^{-1}$ *is conservative.*

On the other hand, we show that for each infinite measure preserving Markov shift T of ergodic index 1, the product $T \times T^{-1}$ is not conservative (Corollary 3.3). This was also proved in [Cl-Va] under an additional assumption that T is "reversible" as a Markov shift. It follows from Corollary 3.2 that if an infinite Markov shift T has an ergodic index higher than 1, then $T \times T^{-1}$ is ergodic. Hence within the class of infinite Markov shifts, the answer to Bergelson's question is negative.

1. (C, F)-construction

All the examples in this paper are built via the (C, F)-construction which is an algebraic counterpart of the classical "geometric" cutting-and-stacking inductive construction process with a single tower on each step. In this section we briefly outline the basics of this construction. For a detailed exposition we refer the reader to [Da1] and [Da2]. Given two finite subsets $A, B \subset G$, we denote by $A + B$ the set of all sums $\{a + b \mid a \in A, b \in B\}$. If A is a singleton, say $A = \{a\}$, we write $a + B$ in place of $\{a\} + B$.

Let $(F_n)_{n\geq 0}$ and $(C_n)_{n\geq 1}$ be two sequences of finite subsets in G such that for each $n > 0$,

$$F_0 = \{0\}, \quad \#C_n > 1, \tag{1-1}$$

$$F_n + C_{n+1} \subset F_{n+1}, \tag{1-2}$$

$$(F_n + c) \cap (F_n + c') = \emptyset \quad \text{if } c, c' \in C_{n+1} \text{ and } c \neq c'. \tag{1-3}$$

We let $X_n := F_n \times C_{n+1} \times C_{n+2} \times \cdots$ and endow this set with the infinite product topology. Then X_n is a compact Cantor (i.e. totally disconnected perfect metric) space. The mapping

$$X_n \ni (f_n, c_{n+1}, c_{n+2}, \dots) \mapsto (f_n + c_{n+1}, c_{n+2}, \dots) \in X_{n+1}$$

is a topological embedding of X_n into X_{n+1}. Therefore an inductive limit X of the sequence $(X_n)_{n\geq 0}$ furnished with these embeddings is a well-defined locally compact Cantor space. Given a subset $A \subset F_n$, we let

$$[A]_n := \{x = (f_n, c_{n+1}, \dots) \in X_n \mid f_n \in A\}$$

and call this set an *n-cylinder* in X. It is open and compact in X. The collection of all cylinders coincides with the family of all compact open subsets in X. It is easy to see that

$$[A]_n \cap [B]_n = [A \cap B]_n, \quad [A]_n \cup [B]_n = [A \cup B]_n \quad \text{and}$$
$$[A]_n = [A + C_{n+1}]_{n+1}$$

for all $A, B \subset F_n$ and $n \geq 0$. For brevity, we will write $[f]_n$ for $[\{f\}]_n$, $f \in F_n$. Now we define the *(C, F)-measure* μ on X by setting

$$\mu([A]_n) = \frac{\#A}{\#C_1 \cdots \#C_n} \quad \text{for each subset } A \subset F_n, n > 0,$$

and then extending μ to the Borel σ-algebra on X. We note that μ is infinite if and only if

$$\lim_{n\to\infty} \frac{\#F_n}{\#C_1 \cdots \#C_n} = \infty. \tag{1-4}$$

Suppose that for each $g \in G$,

$$g + F_n + C_{n+1} \subset F_{n+1} \quad \text{eventually in } n. \tag{1-5}$$

We now define an action of G on X. Given $x \in X$ and $g \in G$, there is $n > 0$ such that $x = (f_n, c_{n+1}, c_{n+2}, \dots) \in X_n$ and $g + f_n \in F_n$. We let

$$T_g x := (g + f_n, c_{n+1}, \dots) \in X_n \subset X.$$

Then T_g is a well-defined homeomorphism of X and $T := (T_g)_{g\in G}$ is a continuous action of G on X. We call it *the (C, F)-action of G associated with the sequence* $(C_n, F_{n-1})_{n\geq 1}$. It is free. If $x, y \in X_n$, $x = (f_n, c_{n+1}, \dots)$, $x' = (f', c'_{n+1}, \dots)$ and

$y = T_g x$, then $g = (f' - f) + (c'_{n+1} - c_{n+1}) + \cdots$. Only finitely many parentheses in this infinite sum are different from 0. We note that T is of *funny rank-one along* $(F_n)_{n\geq 0}$, because the sequence of F_n-towers $\{T_f[0]_n \mid f \in F_n\} = \{[f]_n \mid f \in F_n\}$ approximates the entire Borel σ-algebra on X as $n \to \infty$. It is easy to see that T preserves μ. We note that the action $T \otimes T$ of $G \times G$ is also a (C, F)-action. It is associated with the sequence $(C_n \times C_n, F_{n-1} \times F_{n-1})_{n\geq 1}$. Therefore if A is a subset of $F_n \times F_n$, then we have $[A]_n = \bigsqcup_{(a,b)\in A} [a]_n \times [b]_n$.

To state the following lemma we recall the definition of *full groupoid*. Given a measure preserving action $R = (R_g)_{g\in G}$ on a standard measure space (X, μ) and a subset $A \subset X$, we say that a Borel map $\tau : A \to X$ belongs to the full groupoid of R (and write $\tau \in [[R]]$) if τ is one-to-one and $\tau x \in \{R_g x \mid g \in G\}$ for all $x \in A$. Equivalently, there is a partition of A into subsets A_g, $g \in G$, such that $\tau x = R_g x$ if $x \in A_g$ and $R_g A_g \cap R_h A_h = \emptyset$ if $g \neq h \in G$. Some of A_g can be of zero measure. It follows that τ preserves μ.

Lemma 1.1. *Let $\delta > 0$, let H be a subgroup of G and let $\mathcal{N}$ be an infinite subset of $\mathbb{N}$.*

(i) *If for each $n \in \mathcal{N}$ there is a subset $A \subset [0]_n$ and a map $\tau : A \to [0]_n$ such that $\tau \in [[(T_h)_{h\in H}]]$, $\mu(A) \geq \delta\mu([0]_n)$ and $\tau x \neq x$ for all $x \in A$, then the action $(T_h)_{h\in H}$ is conservative.*

(ii) *If for each $n \in \mathcal{N}$ and $v, w \in F_n$ there is a subset $A \subset [v]_n$ and a map $\tau : A \to [w]_n$ such that $\tau \in [[(T_h)_{h\in H}]]$ and $\mu(A) \geq \delta\mu([v]_n)$, then the action $(T_h)_{h\in H}$ is ergodic.*

Proof. (i) Let B be a subset of X of positive measure. Then there are $n \in \mathcal{N}$ and $f \in F_n$ with $\mu([f]_n \cap B) > (1 - \frac{\delta}{4})\mu([f]_n)$. By the assumption of the lemma, there is a subset $A \subset [0]_n$ and a map $\tau : A \to [0]_n$ such that $\tau \in [[(T_h)_{h\in H}]]$, $\mu(A) > \delta\mu([0]_n)$ and $\tau x \neq x$ for all $x \in A$. We define a new map $\varphi : T_f A \to [f]_n$ by setting $\varphi := T_f \tau T_f^{-1}$. Since G is Abelian, $\varphi \in [[(T_h)_{h\in H}]]$. Moreover, $\varphi x \neq x$ for all $x \in T_f A$ and

$$\mu(\varphi(T_f A \cap B) \cap B) > \frac{\delta}{2}\mu([f]_n) > 0.$$

Therefore, there is $h \in H$ such that $h \neq 0$ and $\mu(T_h(T_f A \cap B) \cap B) > 0$. Hence $(T_h)_{h\in H}$ is conservative.

(ii) Let B_1 and B_2 be subsets of X of positive measure. Then there are $n \in \mathcal{N}$ and elements $v, w \in F_n$ with $\mu([v]_n \cap B_1) > (1 - \frac{\delta}{4})\mu([v]_n)$ and $\mu([w]_n \cap B_2) > (1 - \frac{\delta}{4})\mu([w]_n)$. By the assumption of the lemma, there is a subset $A \subset [v]_n$ and a map $\tau : A \to [w]_n$ such that $\tau \in [[(T_h)_{h\in H}]]$ and $\mu(A) > \delta\mu([v]_n)$. It follows that $\mu(\tau(B_1 \cap [v]_n) \cap [w]_n \cap B_2) > 0$. Therefore there is $h \in H$ such that $\mu(T_h B_1 \cap B_2) > 0$. Hence $(T_h)_{h\in H}$ is ergodic. □

2. Proofs of the main results

Fix a Følner sequence $(\mathcal{F}_n)_{n=1}^{\infty}$ in G such that $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots$ and $\bigcup_n \mathcal{F}_n = G$. In the case where $G = \mathbb{Z}^d$, we choose $\mathcal{F}_n$ to be a cube for each n. The actions whose existence is stated in Theorems 0.1–03 will appear as (C, F)-actions associated with some sequences $(C_n, F_{n-1})_{n\geq 1}$. Moreover, $(F_n)_{n=1}^{\infty}$ will be a subsequence of $(\mathcal{F}_n)_{n=1}^{\infty}$. Therefore in the case $G = \mathbb{Z}^d$, the associated (C, F)-actions will be automatically of rank one. Hence we do not need to prove the final claims of Theorems 0.1–0.3.

Proof of Theorem 0.1. (i) Partition the natural numbers $\mathbb{N}$ into countably many subsets $\mathcal{N}_f$ indexed by elements $f \in G^k$ such that every $\mathcal{N}_f$ is an infinite arithmetic sequence. For each $f = (f_1, \dots, f_k) \in G^k$ and each $n \in \mathcal{N}_f$ there is a unique sequence $(d_{n,j})_{j=0}^k$ of non-negative integers such that $d_{n,0} = 0$ and $d_{n,j-1} - d_{n,j} = f_j$ for all $j = 1, \dots, k$. Fix an increasing sequence $(R_n)_{n \ge 0}$ of positive integers such that $\sum_{n \ge 0} R_n^{-k} = +\infty$ but $\sum_{n \ge 0} R_n^{-k-1} < +\infty$. We note that then $\sum_{n \in \mathcal{N}_f} R_n^{-k} = +\infty$ for each $f \in G^k$.

To construct T we have to define the corresponding sequence $(C_n, F_{n-1})_{n \ge 1}$. This will be done inductively. We let $F_0 = \{0\}$. Suppose that we have already determined the sequence $(C_j, F_j)_{j=1}^{n-1}$. Then we let

$$C_{n,0} := \{0, a_n + d_{n,1}, \dots, ka_n + d_{n,k}\}, \quad C_{n,1} := \{w_n, 2w_n, \dots, (R_n - k - 1)w_n\},$$

and $C_n := C_{n,0} \sqcup C_{n,1}$, where the elements $a_n, w_n \in G$ are chosen so that

$$(C_{n,1} - C_n - C_n + C_n) \cap (F_{n-1} - F_{n-1} + F_{n-1} - F_{n-1}) = \{0\}. \tag{2-1}$$

Now let F_n be an element of $(\mathcal{F}_j)_{j \ge 1}$ such that $F_{n-1} + F_{n-1} + C_n \subset F_n$. Continuing this process infinitely many times we obtain a sequence $(C_n, F_{n-1})_{n \ge 1}$ satisfying (1-1)–(1-5). Denote by T the associated (C, F)-action of G. Let (X, μ) stand for the space of T. It is easy to see that $\mu(T^{w_n} A \cap A) \to \mu(A)$ as $n \to \infty$ for each subset A of finite measure. Hence T is rigid.

Claim 1. $T \times \cdots \times T$ (k times) is ergodic.

Take $n > 0$ and let $v, w \in F_n^k$. We let $f := w - v$. Given $x \in [v]_n \subset X^k$, we write the expansion

$$x = (v, x_{n+1}, x_{n+2}, \dots) \in F_n^k \times C_{n+1}^k \times C_{n+2}^k \times \cdots$$

and set

$$\ell(x) := \min\{l \in \mathcal{N}_f \cap \{n+1, n+2, \dots\} \mid x_l \in (C_{l,0} \setminus \{ka_l + d_{l,k}\})^k\}.$$

Let A denote the subset of $[v]_n$ where the map ℓ is well defined. Then

$$\frac{\mu^k([v]_n \setminus A)}{\mu^k([v]_n)} = \prod_{l \in \mathcal{N}_f, l > n} \frac{(\#C_l)^k - (\#C_{l,0} - 1)^k}{(\#C_l)^k} = \prod_{l \in \mathcal{N}_f, l > n} \left(1 - \frac{k^k}{R_l^k}\right) = 0$$

because $\sum_{l \in \mathcal{N}_f} R_l^{-k} = \infty$. Thus ℓ is defined almost everywhere on $[v]_n$. For $l > n$, we set

$$A_l := \{x \in A \mid \ell(x) = l \text{ and } x_l = (0, a_l + d_{l,1}, \dots, (k-1)a_l + d_{l,k-1})\}.$$

Then $\mu^k(\bigsqcup_{l > n} A_l) = \mu^k(A)/(k+1)^k$. We now define a map $\tau : \bigsqcup_{l > n} A_l \to X^k$ by setting

$$\tau x := (T_{a_l} \times \cdots \times T_{a_l})x \quad \text{if } x \in A_l,\ l > n.$$

Of course, $\tau \in [[T \times \cdots \times T]]$. Since

$$a_l + ((j-1)a_l + d_{l,j-1}) = f_j + (ja_l + d_{l,j}) \quad \text{for } j = 1, \dots, k,$$

it follows that $(T_{a_l} \times \cdots \times T_{a_l})x = (T_{f_1} \times \cdots \times T_{f_k})y$, where $y = (y_i)_{i \ge n} \in F_n^k \times C_{n+1}^k \times C_{n+2}^k \times \cdots$, $y_i = x_i$ if $i \ge n$ and $i \ne l$ and $y_l = (a_l + d_{l,1}, \dots, ka_l + d_{l,k}) \in C_l^k$. Since $y \in [v]_n$ and $f = w - v$, we obtain that $(T_{f_1} \times \cdots \times T_{f_k})y \in [w]_n$ for each $x \in A_l$. Hence $\tau x \in [w]_n$ for each $x \in \bigsqcup_{l > n} A_l$. Therefore $T \times \cdots \times T$ (k times) is ergodic by Lemma 1.1(ii).

Claim 2. The G-action $\underbrace{T\times\cdots\times T}_{m \text{ times}}\times\underbrace{T^{-1}\times\cdots\times T^{-1}}_{k-m \text{ times}}$ is ergodic for every $m\in\{0,1,\dots,k-1\}$. The proof is similar to the proof of Claim 1. There are only a few points of difference which we specify now. Let $\widetilde{f}:=(f_1,\dots,f_m,-f_{m+1},\dots,-f_k)$. Replace $\mathcal{N}_f$ with $\mathcal{N}_{\widetilde{f}}$ in the definition of ℓ. Define

$$B_l:=\{x\in A\mid \ell(x)=l \text{ and } x_l=(0,a_l+d_{l,1},\dots,(m-1)a_l+d_{l,m-1},\\ (m+1)a_l+d_{l,m+1},\dots,ka_l+d_{l,k})\}.$$

Replace A_l with B_l and $T_{a_l}\times\cdots\times T_{a_l}$ with $\underbrace{T_{a_l}\times\cdots\times T_{a_l}}_{m \text{ times}}\times T_{-a_l}\times\cdots\times T_{-a_l}$ in the definition of τ.

Claim 3. $T\times\cdots\times T\,(k+1 \text{ times})$ is not ergodic. Choose $n>0$ such that $\sum_{j=n}^{\infty}(\frac{k+1}{R_j})^{k+1}<1$. We now let

$$W:=\{\mathbf{0}\}\times(C_n^{k+1}\setminus C_{n,0}^{k+1})\times(C_{n+1}^{k+1}\setminus C_{n+1,0}^{k+1})\times\cdots\subset[\mathbf{0}]_{n-1}\subset X^{k+1}.$$

Here $\mathbf{0}$ denotes zero in G^{k+1}. Then

$$\frac{\mu^{k+1}(W)}{\mu^{k+1}([\mathbf{0}]_{n-1})}=\prod_{j\ge n}\left(1-\left(\frac{k+1}{R_j}\right)^{k+1}\right)\ge 1-\sum_{j\ge n}\left(\frac{k+1}{R_j}\right)^{k+1}>0.$$

Fix $h\in F_{n-1}\setminus\{0\}$. We now show that the $(T\times\cdots\times T)$-orbit of W does not intersect the cylinder $B:=[0]_{n-1}\times\cdots\times[0]_{n-1}\times[h]_{n-1}\subset X^{k+1}$. If not, then there is $x=(x^1,\dots,x^{k+1})\in W$ and $g\in G$ such that

$$(T_gx_1,\dots,T_gx_{k+1})\in B. \tag{2-2}$$

Consider the expansions

$$\begin{aligned} x^l&=(0,x_n^l,x_{n+1}^l,\dots)\in\{0\}\times C_n\times C_{n+1}\times\cdots, \quad l=1,\dots,k+1,\\ T_gx^l&=(0,y_n^l,y_{n+1}^l,\dots)\in\{0\}\times C_n\times C_{n+1}\times\cdots, \quad l=1,\dots,k, \quad \text{and}\\ T_gx^{k+1}&=(h,y_n^{k+1},y_{n+1}^{k+1},\dots)\in\{h\}\times C_n\times C_{n+1}\times\cdots. \end{aligned}$$

It follows from (2-2) that there are integers $N_l\ge n$, $l=1,\dots,k+1$, such that

$$\begin{cases} g=\sum_{i=n}^{N_l}(y_i^l-x_i^l), \quad l=1,\dots,k,\\ g=h+\sum_{i=n}^{N_{k+1}}(y_i^{k+1}-x_i^{k+1}) \end{cases} \tag{2-3}$$

and $y_{N_l}^l\ne x_{N_l}^l$, $l=1,\dots,k+1$. Then (2-1) yields that $N_1=\cdots=N_{k+1}$. Since $x\in W$, there exists $l_0\in\{1,\dots,k+1\}$ with $x_{N_1}^{l_0}\in C_{N_1,1}$. It now follows from (2-1) that $y_{N_1}^l-x_{N_1}^l=y_{N_1}^{l_0}-x_{N_1}^{l_0}$ for all $l=1,\dots,k+1$. Hence we deduce from (2-3) that

$$\begin{cases} g-(y_{N_1}^1-x_{N_1}^1)=\sum_{i=n}^{N_1-1}(y_i^l-x_i^l), \quad l=1,\dots,k, \quad \text{and}\\ g-(y_{N_1}^1-x_{N_1}^1)=h+\sum_{i=n}^{N_1-1}(y_i^{k+1}-x_i^{k+1}). \end{cases}$$

Repeating this procedure at most N_1-n-1 times we obtain that $g=g-h$, a contradiction. □

Proof of Theorem 0.2. The desired action is constructed in the same way as in Theorem 0.1 however $C_{n,1}$ is different. We now set

$$C_{n,1}:=\{w_{n,1},\dots,w_{n,R_n-k-1}\},$$

where the elements $w_{n,j} \in G$ are chosen so that (2-1) is satisfied and

(2-4) $\quad$ the mapping $(C_{n,1} \times C_n) \setminus \mathcal{D} \ni (c, c') \mapsto c - c' \in G$ is one-to-one,

where $\mathcal{D}$ is the diagonal in $G \times G$. As in the proof of Theorem 0.1, we denote the corresponding (C, F)-action by T. Claims 1–3 from the proof of that theorem hold (verbally) for the "new" T as well.

Claim 4. $T \times \cdots \times T\,(k+1$ times) is conservative.

Take $n > 0$. Given $x = (x^1, \dots, x^{k+1}) \in [\mathbf{0}]_n$, we set

$$\ell(x) := \min\{l > n \mid x_l^1 = \cdots = x_l^{k+1}\}.$$

Let A denote the subset of $[\mathbf{0}]_n$ where ℓ is well defined. Then

$$\frac{\mu^{k+1}([0]_n \setminus A)}{\mu^{k+1}([0]_n)} = \prod_{l>n} \frac{(\#C_l)^{k+1} - \#C_l}{(\#C_l)^{k+1}} = \prod_{l \in \mathcal{N}_f, l>n} \left(1 - \frac{1}{R_l^k}\right) = 0.$$

For each $m \in \mathbb{N}$ and $c \in C_m$, we let $A_{m,c} := \{x \in A \mid \ell(x) = m, x_m^1 = c\}$ and fix an element c' from $C_m \setminus \{c\}$. We now define a map $\tau : A \to X^{k+1}$ by setting

$$\tau x = (T_{c'-c} \times \cdots \times T_{c'-c})x \quad \text{if } x \in A_{m,c},\ c \in C_m,\ m \in \mathbb{N}.$$

Since $A = \bigsqcup_{m \in \mathbb{N}} \bigsqcup_{c \in C_m} A_{m,c}$, it follows that τx is well defined, $\tau x \in [\mathbf{0}]_n$ and $\tau \in [[T \times \cdots \times T]]$. It remains to apply Lemma 1.1(i).

Claim 5. $T \times \cdots \times T\,(k+2$ times) is not conservative.

Choose $n > 0$ such that $\sum_{j=n}^{\infty} R_j^{-k-1} < 0.5$ and $R_n > (k+1)^{k+2}$. Denote by D_n the diagonal in $C_{n,1}^{k+2}$, i.e. $D_n := \{(c_1, \dots, c_{k+2}) \in C_{n,1}^{k+2} \mid c_1 = \cdots = c_{k+2}\}$. We now let

$$W := (C_n^{k+2} \setminus (C_{n,0}^{k+2} \cup D_n)) \times (C_{n+1}^{k+2} \setminus (C_{n+1,0}^{k+2} \cup D_{n+1})) \times \cdots \subset [\mathbf{0}]_{n-1},$$

where $\mathbf{0}$ stands now for the zero in G^{k+2}. Then

$$\begin{aligned}\frac{\mu^{k+2}(W)}{\mu^{k+2}([\mathbf{0}]_{n-1})} &= \prod_{j \ge n} \left(1 - \left(\frac{k+1}{R_j}\right)^{k+2} - \frac{R_j - k - 1}{R_j^{k+2}}\right) \\ &\ge \left(1 - \sum_{j \ge n} \frac{1}{R_j^{k+1}} \left(1 + \frac{(k+1)^{k+2}}{R_j}\right)\right) > 0.\end{aligned}$$

We now show that W is a $(T \times \cdots \times T)$-wandering set. If not, then there are $x = (x^1, \dots, x^{k+2}) \in W$ and $g \in G$ such that $(T_g x^1, \dots, T_g x^{k+2}) \in W$. Consider the expansions

$$\begin{aligned} x^l &= (0, x_n^l, x_{n+1}^l, \dots) \in \{0\} \times C_n \times C_{n+1} \times \cdots \quad \text{and} \\ T_g x^l &= (0, y_n^l, y_{n+1}^l, \dots) \in \{0\} \times C_n \times C_{n+1} \times \cdots, \end{aligned}$$

$l = 1, \dots, k+2$. Arguing in the same way as in the proof of Claim 3, we find N_1 such that $g = \sum_{i=n}^{N_1} (y_i^l - x_i^l)$, $0 \ne y_{N_1}^l - x_{N_1}^l = y_{N_1}^1 - x_{N_1}^1$ for all $l = 1, \dots, k+2$. Moreover, $x_{N_1}^l \in C_{N_1,1}$ for all $l = 1, \dots, k+2$ because $x \in W$. Then we deduce from (2-4) that $x_{N_1}^1 = \cdots = x_{N_1}^{k+2}$, i.e. $(x_{N_1}^1, \dots, x_{N_1}^{k+2}) \in D_{N_1}$. Therefore $x \notin W$, a contradiction.

Claim 6. T is mixing. Let $A \subset F_{n-1}$, $g \in (F_n - F_n) \setminus (F_{n-1} - F_{n-1})$ and $g + A + C_n \subset F_n$. Then we have

$$\mu(T_g[A]_{n-1} \cap [A]_{n-1}) = \sum_{c,c' \in C_n} \mu([g+A+c]_n \cap [A+c']_n)$$
$$\leq \frac{\mu([A]_{n-1}) \# \{(c,c') \in C_n \times C_n \mid g \in A - A + c - c'\}}{\# C_n}.$$

We first note that if

$$g \in A - A + c - c', \tag{2-5}$$

then $c \neq c'$ by the choice of g. If either $c \in C_{n,1}$ or $c' \in C_{n,1}$, then we deduce from (2-1) and (2-4) that at most one such pair (c, c') satisfies (2-5). If both $c \notin C_{n,1}$ and $c' \notin C_{n,1}$, then $c, c' \in C_{n,0}$ and hence there are no more than $k(k+1)$ such pairs (c, c') satisfying (2-5). Hence

$$\mu(T_g[A]_{n-1} \cap [A]_{n-1}) < \frac{(k+1)^2}{\# C_n} \mu([A]_{n-1}).$$

It follows that $\lim_{g \to \infty} \mu(T_g B \cap B) = 0$ for each cylinder B and hence for each subset of finite measure in X. □

Proof of Theorem 0.3. Let $(d_n)_{n=1}^\infty$ be a sequence of elements of G in which each element of G occurs infinitely many times. Let $(N_n)_{n=1}^\infty$ be a sequence of positive integers such that $\sum_{n>0} \frac{1}{N_n} < \frac{1}{4}$.

Suppose that we have already determined $(C_j, F_j)_{j=1}^{n-1}$. Suppose also that $d_n \in F_{n-1} - F_{n-1}$. We then set

$$C_n := \{e_{n,i}, -e_{n,i}, -l_{n,i}, l_{n,i} - d_n \mid i = 1, \ldots, N_n\},$$

for some elements $e_{n,i}, l_{n,i}$ of G, $1 \leq i \leq N_n$, such that

$$(C_n - C_n) \cap (F_{n-1} - F_{n-1} + F_{n-1} - F_{n-1}) = \{0\}. \tag{2-6}$$

We call $e_{n,i}$ and $-e_{n,i}$ as well as $l_{n,i}$ and $-l_{n,i} - d_n$ *antipodal*, $1 \leq i \leq N_n$. If $c_1, \ldots, c_4 \in C_n$, c_1 and c_4 are antipodal and c_2 and c_3 are antipodal, then

$$(c_1 - c_2) - (c_3 - c_4) \in \{0, d_n, -d_n\}.$$

We introduce the following conditions on C_n. Let $c_1, \ldots, c_4 \in C_n$.

(2-7) If $0 \neq (c_1 - c_2) - (c_3 - c_4) \in F_{n-1} - F_{n-1} + F_{n-1} - F_{n-1}$, then c_1 and c_4 (and c_2 and c_3) are antipodal, and

(2-8) the mapping $(C_n \times C_n) \setminus \mathcal{D} \ni (c, c') \mapsto c - c' \in G$ is one-to-one.

It is straightforward to verify that there exist $e_{n,i}, l_{n,i}$, $1 \leq i \leq N_n$, such that C_n satisfies (2-6)–(2-8). Now let F_n be an element of $(\mathcal{F}_j)_{j \geq 1}$ such that $F_n \supset F_{n-1} + F_{n-1} + C_n$ and $d_{n+1} \in F_n - F_n$. Continuing this construction process infinitely many times we obtain a sequence $(C_n, F_{n-1})_{n \geq 1}$ satisfying (1-1)–(1-4). Let T denote the (C, F)-action of G associated with $(C_n, F_{n-1})_{n \geq 1}$.

Claim 7. $T \times T$ is ergodic.

Take $m > 0$ and $v_1, v_2, w_1, w_2 \in F_m$. There is $n > m$ such that $d_n = w_2 - w_1 + v_1 - v_2$. Let

$$A := \bigsqcup_{i,j=1}^{N_n} [v_1 + D + l_{n,i}]_n \times [v_2 + D - e_{n,j}]_n \subset [v_1]_m \times [v_2]_m,$$

where $D := C_{m+1} + \cdots + C_{n-1}$. Define a map $\tau : A \to [w_1]_m \times [w_2]_m$ by setting

$$\tau(x, y) = (T_{w_1 - v_1 + e_{n,j} - l_{n,i}} x, T_{w_1 - v_1 + e_{n,j} - l_{n,i}} y)$$

if $x \in [v_1 + D + l_{n,i}]_n$ and $y \in [v_2 + D - e_{n,j}]_n$. Indeed, since

$$T_{w_1 - v_1 + e_{n,j} - l_{n,i}}[v_1 + D + l_{n,i}]_n = [w_1 + D + e_{n,j}]_n \text{ and}$$
$$T_{w_1 - v_1 + e_{n,j} - l_{n,i}}[v_2 + D - e_{n,j}]_n = [w_2 + D + (-l_{n,i} - d_n)]_n$$

for all $i, j = 1, \ldots, N_n$, it follows that τ is a bijection of A onto $\tau(A) \subset [w_1]_m \times [w_2]_m$. Of course, $\tau \in [[T \times T]]$. It is easy to compute that

$$(\mu \times \mu)(A) = (\mu \times \mu)([v_1]_m \times [v_2]_m)/16.$$

By Lemma 1.1(ii), the action $T \times T$ is ergodic.

Claim 8. $T \times T^{-1}$ is not ergodic. Fix $f_1 \in F_1 \setminus \{0\}$. We let

$$Z := \{(x, \widetilde{x}) \in [0]_1 \times [0]_1 \mid x_j \text{ and } \widetilde{x}_j \text{ are not antipodal for each } j > 0\},$$

where $x = (0, x_2, x_3, \ldots), \widetilde{x} = (0, \widetilde{x}_2, \widetilde{x}_3, \ldots) \in F_1 \times C_2 \times \cdots$. It is easy to compute that

$$(\mu \times \mu)(Z) = \left(1 - 4 \sum_{j>1} \frac{1}{N_j}\right)(\mu \times \mu)([0]_1 \times [0]_1).$$

Hence $(\mu \times \mu)(Z) > 0$. We show that $\bigcup_{g \in G}(T_g \times T_{-g})Z \cap ([0]_1 \times [f_1]_1) = \emptyset$. If not, there is $(x, \widetilde{x}) \in Z$ and $g \in G$ such that $T_g x \in [0]_1$ and $T_{-g}\widetilde{x} \in [f_1]_1$. Since $T_g x = (0, x_2', x_3', \ldots) \in F_1 \times C_2 \times C_3 \times \cdots$ and $T_{-g}\widetilde{x} = (f_1, \widetilde{x}_2', \ldots) \in F_1 \times C_2 \times \cdots$, there are integers M_1 and M_2 such that

$$\begin{aligned} g &= (x_2' - x_2) + \cdots + (x_{M_1}' - x_{M_1}) \quad \text{and} \\ -g &= f_1 + (\widetilde{x}_2' - \widetilde{x}_2) + \cdots + (\widetilde{x}_{M_2}' - \widetilde{x}_{M_2}) \end{aligned} \tag{2-9}$$

with $x_{M_1} \neq x_{M_1}'$ and $\widetilde{x}_{M_2} \neq \widetilde{x}_{M_2}'$. It follows from (2-6) that $M_1 = M_2$. Since x_{M_1} and $\widetilde{x}_{M_1}$ are not antipodal, we deduce from (2-7) and (2-9) that $x_{M_1} - x_{M_1}' = \widetilde{x}_{M_1}' - \widetilde{x}_{M_1}$. Hence (2-9) yields that

$$\begin{aligned} h &= (x_2' - x_2) + \cdots + (x_{M_1-1} - x_{M_1-1}') \quad \text{and} \\ -h &= f_1 + (\widetilde{x}_2' - \widetilde{x}_2) + \cdots + (\widetilde{x}_{M_1-1}' - \widetilde{x}_{M_1-1}), \end{aligned}$$

where $h := g + x_{M_1} - x_{M_1}'$. Continuing this way several times, we find $L \in \{2, 3, \ldots, M_1\}$ and $f \in G$ such that

$$\begin{aligned} f &= (x_2' - x_2) + \cdots + (x_L' - x_L) \quad \text{and} \\ -f &= f_1 + (\widetilde{x}_2' - \widetilde{x}_2) + \cdots + (\widetilde{x}_L' - \widetilde{x}_L) \end{aligned}$$

with $x_L - x_L' \neq \widetilde{x}_L' - \widetilde{x}_L$, $x_L - x_L' \neq 0$ and $\widetilde{x}_L' - \widetilde{x}_L \neq 0$. If such an L does not exists we then obtain that $f = 0$ and hence $f_1 = 0$, a contradiction. However then it follows from (2-7) that c_L and $\widetilde{c}_L$ are antipodal, a contradiction again.

Claim 9. $T \times T \times T$ is not conservative. Let

$$W := \{(x, y, z) \in [0]_0 \times [0]_0 \times [0]_0 \mid x_j \neq y_j, y_j \neq z_j, x_j \neq z_j \text{ for each } j > 0\},$$

where x_j, y_j and z_j are the j-th coordinates of x, y and z viewed as infinite sequences from $\{0\} \times C_1 \times C_2 \times \cdots$. Then

$$(\mu \times \mu \times \mu)(W) > 1 - \frac{3}{4} \sum_{j>0} \frac{1}{N_j} > 0.$$

We claim that W is a wandering subset for $T\times T\times T$. If not, there is $(x,y,z)\in W$ and $g\in G$ such that

$$T_gx, T_g\widetilde{y}, T_g\widetilde{z}\in[0]_1. \tag{2-10}$$

We write the expansions $x=(0,x_1,x_2,\dots)$, $y=(0,\widetilde{y}_1,\widetilde{y}_2,\dots)$, $z=(0,\widetilde{z}_1,\widetilde{z}_2,\dots)$, $T_gx=(0,x_1',x_2',\dots)$, $T_gy=(0,\widetilde{y}_1',\widetilde{y}_2',\dots)$ and $T_gz=(0,\widetilde{z}_1',\widetilde{z}_2',\dots)$ as infinite sequences from $\{0\}\times C_1\times C_2\times\cdots$. Then (2-10) and (2-6) yield that there is an integer M such that

$$\begin{aligned} g&=(x_1'-x_1)+\cdots+(x_M'-x_M),\\ g&=(y_1'-y_1)+\cdots+(y_M'-y_M)\quad\text{and}\\ g&=(z_1'-z_1)+\cdots+(z_M'-z_M)\end{aligned} \tag{2-11}$$

with $x_M'-x_M\neq 0$, $y_M'-y_M\neq 0$ and $z_M'-z_M\neq 0$. If $x_M'-x_M=y_M'-y_M$, then $x_M=y_M$ by (2-8) and hence $(x,y,z)\notin W$. Therefore $x_M'-x_M\neq y_M'-y_M$. In a similar way, $y_M'-y_M\neq z_M'-z_M$. However then (2-11) and (2-7) yield that x_{M_1} and y_{M_1}' are antipodal and z_{M_1} and y_{M_1}' are antipodal. This is only possible if $x_{M_1}=z_{M_1}$ and hence $(x,y,z)\notin W$, a contradiction.

The fact that $T\times T^{-1}$ is conservative follows from Theorem 0.4. □

Proof of Theorem 0.4. For each $n>0$, we let

$$A:=\bigsqcup_{c\neq c'\in C_{n+1}}[c]_{n+1}\times[c']_{n+1}\subset[0]_n\times[0]_n$$

and define a map $\tau:A\to[0]_n\times[0]_n$ by setting

$$\tau(x,y)=(T_{c'-c}x,T_{c-c'}y)\quad\text{if } x\in[c]_{n+1}, y\in[c']_{n+1}.$$

Then $\tau([c]_{n+1}\times[c']_{n+1})=[c']_{n+1}\times[c]_{n+1}$, $\tau\in[[T\times T^{-1}]]$ and

$$(\mu\times\mu)(A)=\left(1-\frac{1}{\#C_{n+1}}\right)(\mu\times\mu)([0]_n\times[0]_n)\geq\frac{1}{2}(\mu\times\mu)([0]_n\times[0]_n).$$

By Lemma 1.1(i), $T\times T^{-1}$ is conservative. □

3. On "symmetry" of Markov shifts

In this section we consider only the case where $G=\mathbb{Z}$. We first recall some basic definitions and properties of infinite measure preserving Markov shifts.

Let S be an infinite countable set and let $P=(P_{a,b})_{a,b\in S}$ be a stochastic matrix over S. Suppose that there is a strictly positive vector $\lambda=(\lambda_s)_{s\in S}$ which is a left eigenvector for P with eigenvalue 1, i.e. $\lambda P=\lambda$. Moreover, assume that $\sum_{s\in S}\lambda_s=\infty$. Consider the infinite product space $X:=S^{\mathbb{Z}}$ and endow X with the infinite product Borel structure. Let T denote the left shift on X. Given $s_0,\dots,s_k\in S$ and $l\in\mathbb{Z}$, we denote by $[s_0,\dots,s_k]_l^{l+k}$ the cylinder $\{x=(x_j)_{j\in\mathbb{Z}}\mid x_l=s_0,\dots,x_{l+k}=s_k\}$. Define a measure $\mu_{P,\lambda}$ on X by setting $\mu_{P,\lambda}([s_0,\dots,s_k]_l^{l+k})=\lambda_{s_0}P_{s_0,s_1}\cdots P_{s_{k-1},s_k}$ for each cylinder $[s_0,\dots,s_k]_l^{l+k}$. Then $\mu_{P,\lambda}$ extends uniquely to the Borel σ-algebra on X as a σ-finite infinite measure which is invariant under T. The dynamical system $(X,\mu_{P,\lambda},T)$ is called an *infinite Markov shift.*

Lemma 3.1 ([Aa], [KaPa])**.** *$(X, \mu_{P,\lambda}, T)$ is conservative and ergodic if and only if the following two conditions are satisfied:*

(i) *P is irreducible, i.e. for each $a, b \in S$, there is $n > 0$ such that $P_{a,b}^{(n)} > 0$, and*
(ii) *P is recurrent, i.e. $\sum_{n>0} P_{a,a}^{(n)} = \infty$ for some (and hence for each in view of (i)) $a \in S$.*

If (ii) *does not hold, then $(X, \mu_{P,\lambda}, T)$ is not conservative.*

Here $P^{(n)}$ means the usual matrix power $P \cdots P$ (n times).

Let $\sigma : X \to X$ denote the flip, i.e. $(\sigma x)_n := x_{-n}$ for $x \in X$ and $n \in \mathbb{Z}$. Denote by $\Lambda = (\Lambda_{a,b})_{a,b\in S}$ a matrix over S such that $\Lambda_{a,b} = \lambda_a$ if $a = b$ and $\Lambda_{a,b} = 0$ if $a \neq b$. It is straightforward to verify that $\sigma T \sigma^{-1} = T^{-1}$, $\Lambda^{-1} P^* \Lambda$ is a stochastic matrix and $\mu_{P,\lambda} \circ \sigma = \mu_{\Lambda^{-1}P^*\Lambda,\lambda}$. Given two infinite Markov shifts which are defined on the spaces $(S^{\mathbb{Z}}, \mu_{P,\lambda})$ and $(S_1^{\mathbb{Z}}, \mu_{P_1,\lambda_1})$, their Cartesian product is an infinite Markov shift defined on the space $((S \times S_1)^{\mathbb{Z}}, \mu_{P\otimes P_1, \lambda\times\lambda_1})$, where the matrix $P \otimes P_1$ is defined over $S \times S_1$ by $(P \otimes P_1)_{(a,a_1),(b,b_1)} := P_{a,b}(P_1)_{a_1,b_1}$.

Corollary 3.2. *Let $(X, \mu_{P,\lambda}, T)$ be an infinite Markov shift and let $0 \leq m \leq k$. Then the transformation $\underbrace{T \times \cdots \times T}_{m \text{ times}} \times \underbrace{T^{-1} \times \cdots \times T^{-1}}_{k-m \text{ times}}$ is conservative and ergodic if and only if $T \times \cdots \times T$ (k times) is conservative and ergodic.*

Proof. Fix $a \in S$. Then

$$(P^{\otimes m} \otimes (\Lambda^{-1}P^*\Lambda)^{\otimes(k-m)})^{(n)}_{(a,\ldots,a)} = (P_{a,a}^{(n)})^m (P_{a,a}^{(n)})^{k-m} = (P_{a,a}^{(n)})^k = (P^{\otimes k})^{(n)}_{(a,\ldots,a)}.$$

Hence by Lemma 3.1(ii), the stochastic matrix $P^{\otimes m} \otimes (\Lambda^{-1}P^*\Lambda)^{\otimes(k-m)}$ is recurrent if and only if the stochastic matrix $P^{\otimes k}$ is recurrent. In a similar way one can check that $P^{\otimes m} \otimes (\Lambda^{-1}P^*\Lambda)^{\otimes(k-m)}$ is irreducible if and only if so is $P^{\otimes k}$. □

The following assertion follows from Lemma 3.1 and Corollary 3.2.

Corollary 3.3. *Let T be an ergodic conservative infinite Markov shift of ergodic index one. Then $T \times T^{-1}$ is not conservative.*

We note that Corollary 3.3 was proved in [Cl-Va] under an extra assumption that $P = \Lambda^{-1}P^*\Lambda$.

4. Open problems and remarks

(1) Given $p \geq k \geq 1$, is there a mixing rank-one infinite measure preserving transformation of ergodic index k such that $T \times \cdots \times T$ (l times) is conservative if and only if $l \leq p$? Theorem 0.2 provides an affirmative answer to this question if $p = k + 1$.
(2) Is there a rank-one infinite measure preserving transformation T such that $T \times T^{-1}$ is ergodic but $T \times T$ is not?
(3) Is there a rank-one infinite measure preserving transformation T such that $T \times T \times T$ is ergodic but $T \times T^{-1}$ is not?
(4) We note that Theorem 0.4 extends naturally to the ergodic infinite measure preserving actions of *finite funny rank* (see [Da2] for the definition).

(5) It would be interesting to investigate which indexes of ergodicity and conservativeness are realizable on the infinite measure preserving transformations which are Maharam extensions of type III_1 ergodic non-singular transformations (see [DaSi] for the definitions).

References

[Aa] Jon Aaronson, *An introduction to infinite ergodic theory*, Mathematical Surveys and Monographs, vol. 50, American Mathematical Society, Providence, RI, 1997. MR1450400 (99d:28025)

[AdSi] T. M. Adams and C. E. Silva, *On infinite transformations with maximal control of ergodic two-fold product powers*, Isr. J. Math., to appear.

[Cl-Va] J. Clancy, R. Friedberg, I. Kasmalkar, I. Loh, T. Pădurariu, C. E. Silva, and S. Vasudevan, *Ergodicity and conservativity of products of infinite transformations and their inverses*, arXiv:1408.2445.

[Da1] Alexandre I. Danilenko, (C, F)*-actions in ergodic theory*, Geometry and dynamics of groups and spaces, Progr. Math., vol. 265, Birkhäuser, Basel, 2008, pp. 325–351, DOI 10.1007/978-3-7643-8608-5_6. MR2402408 (2009g:37003)

[Da2] Alexandre I. Danilenko, *Actions of finite rank: weak rational ergodicity and partial rigidity*, Ergod. Th. & Dyn. Syst.

[DaRy] Alexandre I. Danilenko and Valery V. Ryzhikov, *On self-similarities of ergodic flows*, Proc. Lond. Math. Soc. (3) **104** (2012), no. 3, 431–454, DOI 10.1112/plms/pdr032. MR2900232

[DaSi] A. I. Danilenko and C. E. Silva, *Ergodic theory: Nonsingular transformations*, Encyclopedia of Complexity and System Science, Springer **5** (2009), 3055–3083.

[KaPa] S. Kakutani and W. Parry, *Infinite measure preserving transformations with "mixing"*, Bull. Amer. Math. Soc. **69** (1963), 752–756. MR0153815 (27 #3776)

[Ry] V. V. Ryzhikov, *On the asymmetry of multiple asymptotic properties of ergodic actions*, Math. Notes **96** (2014), no. 3-4, 416–422. Translation of Mat. Zametki **96** (2014), no. 3, 432–439. MR3344315

Institute for Low Temperature Physics & Engineering of National Academy of Sciences of Ukraine, 47 Lenin Ave., Kharkov, 61164, Ukraine

E-mail address: alexandre.danilenko@gmail.com