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UNIQUENESS OF STABLE PROCESSES WITH DRIFT

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Abstract. Suppose that $d \geq 1$ and $\alpha \in (1,2)$. Let $\mathcal{L}^b = -(-\Delta)^{\alpha/2} + b \cdot \nabla$, where b is an $\mathbb{R}^d$-valued measurable function on $\mathbb{R}^d$ belonging to a certain Kato class of the rotationally symmetric α-stable process Y on $\mathbb{R}^d$. We show that the martingale problem for $(\mathcal{L}^b, C_c^\infty(\mathbb{R}^d))$ has a unique solution for every starting point $x \in \mathbb{R}^d$. Furthermore, we show that the stochastic differential equation $\mathrm{d}X_t = \mathrm{d}Y_t + b(X_t)\mathrm{d}t$ with $X_0 = x$ has a unique weak solution for every $x \in \mathbb{R}^d$.

1. Introduction

In this paper, unless otherwise stated, $d \geq 1$ and $\alpha \in (1,2)$. A rotationally symmetric α-stable process Y in $\mathbb{R}^d$ is a Lévy process with characteristic function given by

$$\mathbb{E}[\exp(\mathrm{i}\xi \cdot (Y_t - Y_0))] = \exp\left(-t|\xi|^\alpha\right), \quad \xi \in \mathbb{R}^d. \tag{1.1}$$

The infinitesimal generator of Y is the fractional Laplacian $\Delta^{\alpha/2} := -(-\Delta)^{\alpha/2}$. Here we use ":=" for definition. Denote by $B(x,r)$ the open ball in $\mathbb{R}^d$ centered at $x \in \mathbb{R}^d$ with radius $r > 0$ and $\mathrm{d}x$ the Lebesgue measure on $\mathbb{R}^d$.

Definition 1.1. For a real-valued function f on $\mathbb{R}^d$ and $r > 0$, define

$$M_f^\alpha(r) := \sup_{x\in\mathbb{R}^d} \int_{B(x,r)} \frac{|f(y)|}{|x-y|^{d+1-\alpha}} \mathrm{d}y. \tag{1.2}$$

A function f on $\mathbb{R}^d$ is said to belong to the *Kato class* $\mathbb{K}_{d,\alpha-1}$ if $\lim_{r\downarrow 0} M_f^\alpha(r) = 0$.

Using Hölder's inequality, it is easy to see that for every $p > d/(\alpha-1)$, $L^\infty(\mathbb{R}^d; \mathrm{d}x)$ $+L^p(\mathbb{R}^d; \mathrm{d}x) \subset \mathbb{K}_{d,\alpha-1}$. Throughout this paper we will assume $b = (b_1, \cdots, b_d)$ is an $\mathbb{R}^d$-valued function on $\mathbb{R}^d$ such that $|b| \in \mathbb{K}_{d,\alpha-1}$. For simplicity, sometimes we just denote it as $b \in \mathbb{K}_{d,\alpha-1}$.

Let $\mathcal{L}^b = \Delta^{\alpha/2} + b \cdot \nabla$. Recently Bogdan and Jakubowski [2] constructed *a particular* heat kernel (also called a fundamental solution) $q^b(t,x,y)$ for operator $\mathcal{L}^b$ using a perturbation argument; see Section 2 below for details.

Let $C_c^\infty(\mathbb{R}^d)$ be the space of smooth functions on $\mathbb{R}^d$ with compact support and $\mathbb{D}([0,\infty); \mathbb{R}^d)$ the space of right continuous $\mathbb{R}^d$-valued functions having left limits on $[0,\infty)$, equipped with Skorokhod topology. For $t \geq 0$, denote by X_t

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the projection coordinate map on $\mathbb{D}([0,\infty);\mathbb{R}^d)$. A Borel probability measure $\mathbf{Q}$ on the Skorokhod space $\mathbb{D}([0,\infty);\mathbb{R}^d)$ is said to be a solution to the martingale problem for $(\mathcal{L}^b, C_c^\infty(\mathbb{R}^d))$ with initial value $x\in\mathbb{R}^d$ if $\mathbf{Q}(X_0=x)=1$ and for every $f\in C_c^\infty(\mathbb{R}^d)$, $\int_0^t|\mathcal{L}^b f(X_s)|\mathrm{d}s<\infty$ $\mathbf{Q}$-a.s. for every $t>0$ and

$$M_t^f := f(X_t)-f(X_0)-\int_0^t \mathcal{L}^b f(X_s)\mathrm{d}s \tag{1.3}$$

is a $\mathbf{Q}$-martingale. The martingale problem for $(\mathcal{L}^b, C_c^\infty(\mathbb{R}^d))$ with initial value $x\in\mathbb{R}^d$ is said to be well-posed if it has a unique solution. It is easy to check (see [4, Proposition 2.3]) that the operators $\{T_t^b; t\ge 0\}$ determined by $q^b(t,x,y)$ form a Feller semigroup. So there exists an $\mathbb{R}^d$-valued conservative Feller process $\{X_t, t\ge 0, \mathbb{P}_x, x\in\mathbb{R}^d\}$ defined on the canonical Skorokhod space $\mathbb{D}([0,\infty);\mathbb{R}^d)$ having $q^b(t,x,y)$ as its transition density function. It is shown in Theorem 2.5 of Chen-Kim-Song [4] that $\mathbb{P}_x$ is a solution to the martingale problem for $(\mathcal{L}^b, C_c^\infty(\mathbb{R}^d))$ with $\mathbb{P}_x(X_0=x)=1$. However, in both [2] and [4], neither the uniqueness of a heat kernel of $\mathcal{L}^b$ nor the uniqueness of the martingale problem for $(\mathcal{L}^b, C_c^\infty(\mathbb{R}^d))$ are addressed. The main results of this paper, Theorems 1.2 and 1.3, in particular settle the uniqueness of a heat kernel of $\mathcal{L}^b$, and thus put the results of [2] and [4] in perspective.

Theorem 1.2. *For each $x\in\mathbb{R}^d$, the martingale problem for $(\mathcal{L}^b, C_c^\infty(\mathbb{R}^d))$ with initial value x is well-posed. These martingale problem solutions $\{\mathbb{P}_x, x\in\mathbb{R}^d\}$ form a strong Markov process, which has infinite lifetime and possesses $q^b(t,x,y)$ as its transition density function.*

Intuitively, $\mathcal{L}^b$ is the infinitesimal generator for stochastic differential equation (SDE)

$$\mathrm{d}X_t=\mathrm{d}Y_t+b(X_t)\mathrm{d}t,\quad X_0=x. \tag{1.4}$$

But does this SDE have a weak solution? Is the weak solution to (1.4) unique? We will answer these questions in this paper as well.

Recall that a weak solution to (1.4) is a process X defined on some probability space such that almost all paths are right continuous and admit left limits, $\int_0^t|b(X_s)|\,\mathrm{d}s<\infty$ a.s. for every $t>0$, and that X satisfies (1.4) for some symmetric α-stable process Y. If all the weak solutions (possibly defined on different probability spaces) to (1.4) have the same distribution, we say that (1.4) has a unique weak solution.

Theorem 1.3. *For each $x\in\mathbb{R}^d$, SDE* (1.4) *has a unique weak solution and the law of the unique weak solution to SDE* (1.4) *is the unique solution to the martingale problem for $(\mathcal{L}^b, C_c^\infty(\mathbb{R}^d))$.*

A solution to (1.4) will be called an α-stable process with drift b. When Y is a Brownian motion (which corresponds to $\alpha=2$), it is well known that Brownian motion with drift can be obtained from Brownian motion through a change of measure called a Girsanov transform. But for a symmetric α-stable process (where $0<\alpha<2$), SDE (1.4) cannot be solved by a change of measure. This is because Y is a purely discontinuous Lévy process and so the effect of a Girsanov transform can only produce a purely discontinuous "drift term"; see [5, 7].

The unique weak solutions of (1.4) form a strong Markov process X. Theorem 1.3 combined with the main result of [2] and [4] readily gives sharp two-sided estimates on the transition density $p^b(t,x,y)$ of X as well as on the transition density $p^b_D(t,x,y)$ of the subprocess $X^{b,D}$ of X killed upon leaving a bounded $C^{1,1}$ open set. See Corollary 3.2 below.

As mentioned above, the existence of the martingale solution to $(\mathcal{L}^b, C_c^\infty(\mathbb{R}^d))$ is established in [4, Theorem 2.5], using a (particular) heat kernel of $\mathcal{L}^b$ constructed in [2]. One deduces easily from Itô's formula that the uniqueness of the martingale problem for $(\mathcal{L}^b, C_c^\infty(\mathbb{R}^d))$ implies the weak uniqueness of SDE (1.4). So the main point of Theorems 1.2 and 1.3 is on the uniqueness of the martingale problem for $(\mathcal{L}^b, C_c^\infty(\mathbb{R}^d))$ and the existence of a weak solution to SDE (1.4). The novelty here is that the drift b is an $\mathbb{R}^d$-valued function in Kato class $\mathbb{K}_{d,\alpha-1}$, which in general is merely measurable and can be unbounded. Thus Picard's iteration method is not applicable either. Motivated by the approach in [1], we establish the uniqueness of solutions $\mathbf{Q}$ to the martingale problem for $(\mathcal{L}^b, C_c^\infty(\mathbb{R}^d))$ by showing that

$$\mathbb{E}_{\mathbf{Q}} \int_0^\infty \mathrm{e}^{-\lambda t} g(X_t)\mathrm{d}t = \mathbb{E}_x \int_0^\infty \mathrm{e}^{-\lambda t} g(X_t)\mathrm{d}t$$

for $g \in C_c^\infty(\mathbb{R}^d)$ and sufficiently large $\lambda > 0$.

The equivalence between weak solutions to SDE driven by Brownian motion and solutions to martingale problems for elliptic operators is well known. The crucial ingredient in this connection is a martingale representation theorem for Brownian motion. Such a martingale representation theorem is not available for stable processes. Recently, Kurtz [9, Theorem 2.3] studied equivalence between weak solutions to a class of SDEs driven by Poisson random measures and solutions to martingale problems for a class of non-local operators using a non-constructive approach. We point out that one cannot deduce the existence of weak solution to SDE (1.4) from the existence of the martingale problem for $(\mathcal{L}^b, C_c^\infty(\mathbb{R}^d))$ by applying results from [9] because $\mathcal{L}^b f$ is typically unbounded for $f \in C_c^\infty(\mathbb{R}^d)$. When the drift b is L^p-integrable for $p > d/(\alpha-1)$, Portenko [11] proposed a perturbation approach to construct a weak solution to SDE (1.4); see also [10]. In this paper we establish by using the arguments in [10, 11] that the Markov process X constructed above is the unique weak solution to SDE (1.4) when b belongs to the Kato class $\mathbb{K}_{d,\alpha-1}$.

Very recently, around the same time as the first version of this paper was completed, Kim and Song [8] studied stable process with singular drift, analogous to Brownian motion with singular drift introduced and studied in Bass and Chen [1]. Intuitively speaking, stable process with singular drift studied in [8] corresponds to SDE (1.4) with b being replaced by a suitable measure. However, following [1, Definition 2.5], the existence and uniqueness of the solution in [8] is formulated in a weaker sense as follows. Suppose $\mu = (\mu_1, \ldots, \mu_d)$ is a d-dimensional signed measure in Kato class $\mathbb{K}_{d,\alpha-1}$ and $\varphi \geq 0$ is a smooth radial function with compact support on $\mathbb{R}^d$ having $\int_{\mathbb{R}^d} \varphi(x)\mathrm{d}x = 1$. For $n \geq 1$, define $\varphi_n(x) = 2^{nd}\varphi(2^n x)$. For $x \in \mathbb{R}^d$, an α-stable process with drift μ on $\mathbb{R}^d$ starting from x is a probability measure $\mathbb{P}$ on $\mathbb{D}([0,\infty); \mathbb{R}^d)$ such that $X_t = x + Y_t + A_t$, where

(i) $A_t = \lim_{n\to\infty} \int_0^t b^n(X_s)\mathrm{d}s$ uniform in t over finite intervals, where the convergence is in probability and $b^n_j(x) := \int_{\mathbb{R}^d} \varphi_n(x-y)\mu_j(\mathrm{d}y)$;
(ii) there exists a subsequence n_k so that $\sup_{k\geq 1} \int_0^t |b^{n_k}(X_s)|\mathrm{d}s < \infty$ a.s. for every $t > 0$;
(iii) Y is a rotationally symmetric α-stable process in $\mathbb{R}^d$ with $Y_0 = 0$ under $\mathbb{P}$.

The existence of such a process is established in [8] through approximation by solutions of $X_t^n = x + Y_t + \int_0^t b^n(X_s^n)\mathrm{d}s$. The uniqueness proof in [8] (see Section 5 there), as our proof for the uniqueness of martingale problems in this paper, is also motivated by the approach of Bass and Chen [1, Section 5] by showing that $\mathbb{E}_{\mathbb{P}} \int_0^\infty \mathrm{e}^{-\alpha t} g(X_s)\mathrm{d}s = \mathbb{E}_x \int_0^\infty \mathrm{e}^{-\lambda s} g(X_s)\mathrm{d}s$ for $g \in C_c^\infty(\mathbb{R}^d)$ and sufficiently large $\lambda > 0$. When applying to the Kato function b considered in this paper, the results in [8] for $\mu(\mathrm{d}x) = b(x)\mathrm{d}x$ do not give the existence and uniqueness of weak solutions to SDE (1.4) nor the well-posedness of the martingale problem for $(\mathcal{L}^b, C_c^\infty(\mathbb{R}^d))$.

The approach of this paper is quite robust. It can be applied to study some other stochastic models. For example, it can be used to establish, for each $b = (b_1, \cdots, b_d) \in \mathbb{K}_{d,1}$, the well-posedness of the martingale problem for $(\Delta + b \cdot \nabla, C_c^\infty(\mathbb{R}^d))$ and to establish the weak existence and uniqueness of solutions to Brownian motion with singular drift: $\mathrm{d}X_t = \mathrm{d}B_t + b(X_t)\mathrm{d}t$. Recently, it has been applied in Chen and Hu [3] to establish, for each $b = (b_1, \cdots, b_d) \in \mathbb{K}_{d,1}$, the well-posedness of the martingale problem for $(\Delta + \Delta^{\alpha/2} + b \cdot \nabla, C_c^\infty(\mathbb{R}^d))$ and to establish the weak existence and uniqueness for solutions to SDE with singular drift: $\mathrm{d}X_t = \mathrm{d}(B_t + Y_t) + b(X_t)\mathrm{d}t$, where Y is a symmetric α-stable process independent of B.

The rest of the paper is organized as follows. The proof of the uniqueness of the martingale problem is given in Section 2, while the proof of Theorem 1.3 is presented in Section 3.

2. Uniqueness of the martingale problem

Recall that $\mathcal{L}^b = \Delta^{\alpha/2} + b \cdot \nabla$. When $b = 0$, we simply write $\mathcal{L}^0$ as $\mathcal{L}$; that is, $\mathcal{L} = \Delta^{\alpha/2}$. In this section, we establish the well-posedness of the martingale problem for $(\mathcal{L}^b, C_c^\infty(\mathbb{R}^d))$.

We first recall from Bogdan and Jakubowski [2] the construction of *a particular* fundamental solution $q^b(t, x, y)$ for the non-local operator $\mathcal{L}^b$ using a perturbation argument. It is based on the following heuristics: $q^b(t, x, y)$ of $\mathcal{L}^b$ can be related to the fundamental solution $p(t, x, y)$ of $\mathcal{L}$, which is the transition density of the symmetric stable process Y, by the following Duhamel's formula:

$$q^b(t, x, y) = p(t, x, y) + \int_0^t \int_{\mathbb{R}^d} q^b(s, x, z)\, b(z) \cdot \nabla_z p(t - s, z, y)\mathrm{d}z\mathrm{d}s. \tag{2.1}$$

Applying the above formula recursively, one expects $q^b(t, x, y)$ to be expressed as

$$q^b(t, x, y) = \sum_{k=0}^\infty q_k^b(t, x, y), \tag{2.2}$$

where $q_0^b(t, x, y) := p(t, x, y)$ and for $k \geq 1$,

$$q_k^b(t, x, y) := \int_0^t \int_{\mathbb{R}^d} q_{k-1}^b(s, x, z)\, b(z) \cdot \nabla_z p(t - s, z, y)\mathrm{d}z. \tag{2.3}$$

The following results come from [2, Theorem 1, Lemma 15, Lemma 23] and their proofs.

Proposition 2.1. *(i) There exist constants $T_0 > 0$ and $c_1 > 1$ depending only on d, α and on b only through the rate at which $M_{|b|}^\alpha(r)$ goes to zero so that*

$\sum_{k=0}^\infty q^b_k(t,x,y)$ converges locally uniformly on $(0,T_0]\times\mathbb{R}^d\times\mathbb{R}^d$ to a jointly continuous positive function $q^b(t,x,y)$ and that on $(0,T_0]\times\mathbb{R}^d\times\mathbb{R}^d$,

$$c_1^{-1}\left(t^{-d/\alpha}\wedge\frac{t}{|x-y|^{d+\alpha}}\right)\le q^b(t,x,y)\le c_1\left(t^{-d/\alpha}\wedge\frac{t}{|x-y|^{d+\alpha}}\right). \tag{2.4}$$

Moreover, the function $q^b(t,x,y)$ can be extended uniquely to a jointly continuous positive function on $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$ so that for all s, $t\in(0,\infty)$ and x, $y\in\mathbb{R}^d$, $\int_{\mathbb{R}^d}q^b(t,x,z)dz=1$ and

$$q^b(s+t,x,y)=\int_{\mathbb{R}^d}q^b(s,x,z)q^b(t,z,y)\mathrm{d}z. \tag{2.5}$$

(ii) Define $T^b_tf(x):=\int_{\mathbb{R}^d}q^b(t,x,y)f(y)\mathrm{d}y$. Then for any f, $g\in C^\infty_c(\mathbb{R}^d)$,

$$\lim_{t\downarrow 0}\frac{1}{t}\int_{\mathbb{R}^d}(T^b_tf(x)-f(x))g(x)\mathrm{d}x=\int_{\mathbb{R}^d}(\mathcal{L}^bf)(x)\,g(x)\mathrm{d}x.$$

Here and in the sequel, for a, $b\in\mathbb{R}$, $a\wedge b:=\min\{a,b\}$, $a\vee b:=\max\{a,b\}$, and the meaning of the phrase "depending on b only via the rate at which $M^\alpha_{|b|}(r)$ goes to zero" is that the statement is true for any $\mathbb{R}^d$-valued function $\widetilde{b}$ on $\mathbb{R}^d$ with $M^\alpha_{|\widetilde{b}|}(r)\le M^\alpha_{|b|}(r)$ for all $r>0$. Proposition 2.1(ii) indicates that $q^b(t,x,y)$ is a heat kernel (or fundamental solution) of $\mathcal{L}^b$ in the distributional sense.

Let $\{\mathbb{P}_x,x\in\mathbb{R}^d\}$ be the probability measures on $\mathbb{D}([0,\infty);\mathbb{R}^d)$ obtained from the kernel $q^b(t,x,y)$ in Proposition 2.1. The mathematical expectation taken under $\mathbb{P}_x$ will be denoted by $\mathbb{E}_x$. It was shown in [4, Theorem 2.5] that for each $x\in\mathbb{R}^d$, $\mathbb{P}_x$ solves the martingale problem for $(\mathcal{L}^b,C^\infty_c(\mathbb{R}^d))$ with initial value x. We will show in this section that $\mathbb{P}_x$ is in fact the unique solution. Our approach is motivated by that of Bass and Chen [1, Section 5].

Before presenting the proof of Theorem 1.2 we record two lemmas on the boundedness of the λ-resolvent operator R_λ corresponding to symmetric α-stable process Y. Denote by $p(t,x,y)=p(t,x-y)$ the transition density function of Y. Let $r_\lambda(x)=\int_0^\infty \mathrm{e}^{-\lambda t}p(t,x)\mathrm{d}t$ and define the resolvent operator R_λ by

$$R_\lambda g(x)=\int_{\mathbb{R}^d}r_\lambda(x-y)g(y)\mathrm{d}y=\int_{\mathbb{R}^d}r_\lambda(y)g(x-y)\mathrm{d}y,$$

for every $g\in C_b(\mathbb{R}^d)$ and $x\in\mathbb{R}^d$. Here $C_b(\mathbb{R}^d)$ (resp. $C_0(\mathbb{R}^d)$) denote the space of bounded continuous functions on $\mathbb{R}^d$ (resp. continuous functions on $\mathbb{R}^d$ that vanish at infinity). For $f\in C_b(\mathbb{R}^d)$, define $\|f\|_\infty=\sup_{x\in\mathbb{R}^d}|f(x)|$. Denote by $C^\infty_0(\mathbb{R}^d)$ the space of smooth functions on $\mathbb{R}^d$ that together with their partial derivatives of any order vanish at infinity. By [2, Lemma 7], for $(\lambda,x)\in(0,\infty)\times\mathbb{R}^d$,

$$r_\lambda(x)\asymp\left(\lambda^{(d-\alpha)/\alpha}\vee|x|^{\alpha-d}\right)\wedge\left(\lambda^{-2}|x|^{-d-\alpha}\right), \tag{2.6}$$

which can be rewritten as

$$r_\lambda(x)\asymp\begin{cases}\frac{1}{|x|^{d-\alpha}}\wedge\frac{\lambda^{-2}}{|x|^{d+\alpha}} & \text{when } d>\alpha,\\ \lambda^{(d-\alpha)/\alpha}\wedge\frac{\lambda^{-2}}{|x|^{d+\alpha}} & \text{when } d\le\alpha.\end{cases}$$

Here for any two positive functions f and g, $f\asymp g$ means that there is a positive constant $c\ge 1$ so that $c^{-1}g\le f\le c\,g$ on their common domain of definition.

Lemma 2.2. *For every $\lambda>0$, R_λ and ∇R_λ are bounded operators on $C_0(\mathbb{R}^d)$. Furthermore, $R_\lambda f\in C^\infty_0(\mathbb{R}^d)$ for every $f\in C^\infty_0(\mathbb{R}^d)$.*

Proof. It is known by [2, Lemma 9] that $r_\lambda(z)$ is continuously differentiable off the origin and there is a constant $c_1 > 1$ so that for every $\lambda > 0$ and $z \neq 0$,

$$c_1^{-1}\left(\frac{1}{|z|^{d+1-\alpha}} \wedge \frac{1}{\lambda^2|z|^{d+1+\alpha}}\right) \le |\nabla r_\lambda(z)| \le c_1\left(\frac{1}{|z|^{d+1-\alpha}} \wedge \frac{1}{\lambda^2|z|^{d+1+\alpha}}\right). \tag{2.7}$$

It follows that for $\lambda > 0$ and $f \in C_0(\mathbb{R}^d)$, $\nabla r_\lambda(x-y)$ is uniformly in x integrable against $f(y)\mathrm{d}y$. Thus $R_\lambda f$ is continuously differentiable and

$$\nabla R_\lambda f(x) = \int_{\mathbb{R}^d} \nabla r_\lambda(x-y) f(y)\mathrm{d}y = \int_{\mathbb{R}^d} \nabla r_\lambda(y) f(x-y)\mathrm{d}y.$$

Since both $r_\lambda(y)$ and $|\nabla r_\lambda(y)|$ are integrable over $\mathbb{R}^d$ and $f(x-y)$ converges to 0 as $|x| \to \infty$, we conclude from the dominated convergence theorem that both $R_\lambda f$ and $\nabla R_\lambda f$ are in $C_0(\mathbb{R}^d)$ with $\|R_\lambda f\|_\infty \le c_2\|f\|_\infty$ and $\|\nabla R_\lambda f\|_\infty \le c_2\|f\|_\infty$ for some constant $c_2 > 0$. Similarly, for $f \in C_0^\infty(\mathbb{R}^d)$, we have

$$\partial_{x_1}^{k_1}\cdots\partial_{x_d}^{k_d} R_\lambda f(x) = \int_{\mathbb{R}^d} r_\lambda(y)\, \partial_{x_1}^{k_1}\cdots\partial_{x_d}^{k_d} f(x-y)\,\mathrm{d}y,$$

and consequently $R_\lambda f \in C_0^\infty(\mathbb{R}^d)$. □

Lemma 2.3. *Let $b = (b_1, \cdots, b_d) \in \mathbb{K}_{d,\alpha-1}$. There exists $\lambda_0 > 0$ depending only on d, α and on b only via the rate at which $M_{|b|}^\alpha(r)$ goes to zero such that for every $\lambda > \lambda_0$ and $f \in C_0(\mathbb{R}^d)$,*

$$\|\nabla R_\lambda(bf)\|_\infty \le \frac{1}{2}\|f\|_\infty.$$

Proof. It follows from [2, Lemma 11 and Corollary 12] and their proof (with $\beta = 2$ there) that there exists a constant $c_1 > 0$ depending only on d and α such that for every $t > 0$,

$$\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}\left(\frac{1}{|x-y|^{d+1-\alpha}} \wedge \frac{t^2}{|x-y|^{d+1+\alpha}}\right)|b(y)|\mathrm{d}y \le c_1 M_{|b|}^\alpha(t^{1/\alpha}). \tag{2.8}$$

This together with (2.7) and [2, Lemma 16] implies that there exists a constant $c_2 > 0$ such that for every $\lambda > 0$,

$$\begin{aligned}\|\nabla R_\lambda(bf)\|_\infty &= \sup_{x\in\mathbb{R}^d}\left|\int_{\mathbb{R}^d} \nabla r_\lambda(x-y)\cdot b(y)f(y)\mathrm{d}y\right| \\ &\le c_2\|f\|_\infty \sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}\left(\frac{1}{|x-y|^{d-\alpha+1}} \wedge \frac{\lambda^{-2}}{|x-y|^{d+\alpha+1}}\right)|b(y)|\mathrm{d}y \\ &\le c_1c_2\|f\|_\infty M_{|b|}^\alpha(\lambda^{-1/\alpha}).\end{aligned}$$

Since $M_{|b|}^\alpha(\lambda^{-1/\alpha})$ tends to 0 as $\lambda \to \infty$, there exists some $\lambda_0 > 0$ so that

$$c_3 M_{|b|}^\alpha(\lambda^{-1/\alpha}) \le 1/2$$

for every $\lambda > \lambda_0$. This proves the lemma. □

It is well known that the transition density function $p(t, x, y)$ of the symmetric α-stable process Y on $\mathbb{R}^d$ has the two-sided estimates

$$p(t, x, y) \asymp t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}.$$

By (2.4) we have that there is a constant $C_1 \geq 1$ depending on b only through the rate at which $M^{\alpha}_{|b|}(r)$ goes to zero so that

$$C_1^{-1} p(t,x,y) \leq q^b(t,x,y) \leq C_1 p(t,x,y) \qquad \text{for every } (t,x,y) \in (0,T_0] \times \mathbb{R}^d \times \mathbb{R}^d.$$

It follows from (2.4) and the Chapman-Kolmogorov equation (2.5) that there are positive constants $C_2 \geq 1$ and $C_3 > 0$ depending on b only through the rate at which $M^{\alpha}_{|b|}(r)$ goes to zero so that

(2.9)

$$C_2^{-1} \mathrm{e}^{-C_3 t} p(t,x,y) \leq q^b(t,x,y) \leq C_2 \mathrm{e}^{C_3 t} p(t,x,y) \qquad \text{for every } t>0 \text{ and } x,\ y \in \mathbb{R}^d.$$

Thus for $\lambda > C_3$,

$$\mathbb{E}_x \left[\int_0^\infty \mathrm{e}^{-\lambda t} |b(X_t)| \mathrm{d}t \right] \leq C_2 \int_{\mathbb{R}^d} |b(y)| r_{\lambda - C_3}(x-y) \mathrm{d}y.$$

By [2, Lemma 16], there is a constant $C_4 > C_3$ depending on b only through the rate at which $M^{\alpha}_{|b|}(r)$ goes to zero such that for every $\lambda \geq C_4$,

$$\sup_{x \in \mathbb{R}^d} \mathbb{E}_x \left[\int_0^\infty \mathrm{e}^{-\lambda t} |b(X_t)| \mathrm{d}t \right] < \infty. \tag{2.10}$$

By increasing the value of λ_0 in Lemma 2.3 if needed, we may and do assume that $\lambda_0 \geq C_4$.

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Recall that for each $x \in \mathbb{R}^d$, $\mathbb{P}_x$ solves the martingale problem for $(\mathcal{L}^b, C_c^\infty(\mathbb{R}^d))$ with initial value x. The strong Markov property of $(X, \mathbb{P}_x, x \in \mathbb{R}^d)$ follows from the Feller property of $q^b(t,x,y)$, and the conservativeness of the Feller process X follows from Proposition 2.1. It remains to prove the uniqueness of the martingale problem.

Let $\mathbf{Q}$ be any solution to the martingale problem for $(\mathcal{L}^b, C_c^\infty(\mathbb{R}^d))$ with initial value x. We will show that $\mathbf{Q} = \mathbb{P}_x$. We divide the proof into five steps.

(i) We show that it suffices to consider the case that

$$\mathbb{E}_{\mathbf{Q}} \left[\int_0^\infty \mathrm{e}^{-\lambda t} |b(X_t)| \mathrm{d}t \right] < \infty \quad \text{for every } \lambda > \lambda_0, \tag{2.11}$$

where $\mathbb{E}_{\mathbf{Q}}$ is the mathematical expectation under the probability measure $\mathbf{Q}$ and λ_0 is the constant in Lemma 2.3.

Let $f \in C_c^\infty(\mathbb{R}^d)$. By the definition of the martingale problem solution, $\int_0^t |b(X_s) \cdot \nabla f(X_s)| \mathrm{d}s < \infty$ $\mathbf{Q}$-a.s. for every $t > 0$. Let

$$T_n(f) = \inf \left\{ t > 0 : \int_0^t |b(X_s) \cdot \nabla f(X_s)| \mathrm{d}s \geq n \right\}.$$

Then $\{T_n(f), n \geq 1\}$ is an increasing sequence of stopping times such that $\lim_{n\to\infty} T_n(f) = \infty$ $\mathbf{Q}$-a.s. with

$$\mathbb{E}_{\mathbf{Q}} \left[\int_0^{T_n(f)} |b(X_s) \cdot \nabla f(X_s)| \mathrm{d}s \right] \leq n. \tag{2.12}$$

Choose a sequence of functions $f_n^{(i)} \in C_c^\infty(\mathbb{R}^d)$ such that $f_n^{(i)}(x) = x_i$ for $x \in B(0,n)$ and $1 \leq i \leq d$. Define

$$S_n = \left(\min_{1 \leq i \leq d} T_n(f_n^{(i)}) \right) \wedge \inf \left\{ t :\ |X_t| > n \text{ or } |X_{t-}| > n \right\}.$$

Then S_n is an increasing sequence of stopping times with $\lim_{n\to\infty} S_n = \infty$. By (2.12),

$$\begin{aligned}\mathbb{E}_{\mathbf{Q}}\left[\int_0^{S_n} |b(X_s)|\mathrm{d}s\right] &\le \sum_{i=1}^d \mathbb{E}_{\mathbf{Q}}\left[\int_0^{S_n} |b_i(X_s)|\mathrm{d}s\right] \\ &\le \sum_{i=1}^d \mathbb{E}_{\mathbf{Q}}\left[\int_0^{S_n} |b(X_s)\cdot\nabla f_n^{(i)}(X_s)|\mathrm{d}s\right] \le nd.\end{aligned} \tag{2.13}$$

Now we construct a new probability measure $\widetilde{\mathbf{Q}}$ on $\mathbb{D}([0,\infty);\mathbb{R}^d)$ such that $\widetilde{\mathbf{Q}}$ is also a solution to the martingale problem for $(\mathcal{L}^b, C_c^\infty(\mathbb{R}^d))$ and that for every $\lambda > \lambda_0$,

$$\mathbb{E}_{\widetilde{\mathbf{Q}}}\left[\int_0^\infty \mathrm{e}^{-\lambda t}|b(X_t)|\mathrm{d}t\right] < \infty.$$

Let $\mathcal{F}_t$ be the filtration generated by $\{X_s; s\le t\}$. Fix $N \ge 1$. We specify $\widetilde{\mathbf{Q}}$ by

$$\widetilde{\mathbf{Q}}\left(B\cap(C\circ\theta_{S_N})\right) = \mathbb{E}_{\mathbf{Q}}\left[\mathbb{P}_{X_{S_N}}(C);\ B\right],$$

for $B\in\mathcal{F}_{S_N}$ and $C\in\mathcal{F}_\infty$. By the strong Markov property of $\{X_t,\mathbb{P}_x\}$ and the optimal stopping theorem, we have that for every $f\in C_c^\infty(\mathbb{R}^d)$ and stopping time T,

$$\begin{aligned}\mathbb{E}_{\widetilde{\mathbf{Q}}}\left[M_T^f\mathbb{1}_{\{T>S_N\}}\right] =& \mathbb{E}_{\mathbf{Q}}\left[\mathbb{E}_{X_{S_N}}\left[M^f_{T-S_N}\right]\mathbb{1}_{\{T>S_N\}}\right] \\ =& \mathbb{E}_{\mathbf{Q}}\left[\mathbb{E}_x\left[M_T^f|\mathcal{F}_{S_N}\right]\mathbb{1}_{\{T>S_N\}}\right] \\ =& \mathbb{E}\left[M^f_{S_N}\mathbb{1}_{\{T>S_N\}}\right].\end{aligned}$$

It follows that

$$\mathbb{E}_{\widetilde{\mathbf{Q}}}\left[M_T^f\right] = \mathbb{E}_{\mathbf{Q}}\left[M^f_{S_N}\mathbb{1}_{\{T>S_N\}}\right] + \mathbb{E}_{\mathbf{Q}}\left[M_T^f\mathbb{1}_{\{T\le S_N\}}\right] = \mathbb{E}_{\mathbf{Q}}\left[M^f_{T\wedge S_N}\right] = 0.$$

Therefore $\widetilde{\mathbf{Q}}$ is again a solution to the martingale problem for $(\mathcal{L}^b, C_c^\infty(\mathbb{R}^d))$. Furthermore,

$$\begin{aligned}&\mathbb{E}_{\widetilde{\mathbf{Q}}}\left[\int_0^\infty \mathrm{e}^{-\lambda t}|b(X_t)|\mathrm{d}t\right] \\ &\quad= \mathbb{E}_{\mathbf{Q}}\left[\int_0^{S_N} \mathrm{e}^{-\lambda t}|b(X_t)|\mathrm{d}t\right] + \mathbb{E}_{\mathbf{Q}}\left[\mathrm{e}^{-\lambda S_N}\mathbb{E}_{X_{S_N}}\left[\int_0^\infty \mathrm{e}^{-\lambda t}|b(X_t)|\mathrm{d}t\right]\right],\end{aligned}$$

which is finite by (2.10) and (2.13). Note that $\widetilde{\mathbf{Q}} = \mathbf{Q}$ on $\mathcal{F}_{S_N}$. If we can show $\widetilde{\mathbf{Q}} = \mathbb{P}_x$, then $\mathbf{Q} = \mathbb{P}_x$ on $\mathcal{F}_{S_N}$. Since $N\ge 1$ is arbitrary, we have that $\mathbf{Q} = \mathbb{P}_x$ on $\mathcal{F}_\infty$. So it suffices to consider the solution $\mathbf{Q}$ to the martingale problem satisfying (2.11).

(ii) We next show that for every $g\in C_0^\infty(\mathbb{R}^d)$ and $\lambda>\lambda_0$,

$$\mathbb{E}_{\mathbf{Q}}\left[\int_0^\infty \mathrm{e}^{-\lambda t}g(X_t)\mathrm{d}t\right] = R_\lambda g(x) + \mathbb{E}_{\mathbf{Q}}\left[\int_0^\infty \mathrm{e}^{-\lambda t}b(X_t)\cdot\nabla R_\lambda g(X_t)\mathrm{d}t\right]. \tag{2.14}$$

By (1.3), $f(X_t)$ is a semimartingale under $\mathbf{Q}$ for every $f \in C_c^\infty(\mathbb{R}^d)$. It follows by Itô's formula that

$$\begin{aligned}\mathrm{e}^{-\lambda t} f(X_t) = f(X_0) &+ \int_0^t \mathrm{e}^{-\lambda s} \mathrm{d}M_s^f \\ &+ \int_0^t \mathrm{e}^{-\lambda s}\left(\Delta^{\alpha/2} f(X_s) + b(X_s)\cdot\nabla f(X_s)\right)\mathrm{d}s - \lambda\int_0^t \mathrm{e}^{-\lambda s} f(X_s)\mathrm{d}s.\end{aligned}$$

Taking expectation with respect to $\mathbf{Q}$, we have

$$\begin{aligned}\mathbb{E}_{\mathbf{Q}}[\mathrm{e}^{-\lambda t} f(X_t)] = f(x) &- \mathbb{E}_{\mathbf{Q}}\left[\int_0^t \mathrm{e}^{-\lambda s}(\lambda f - \Delta^{\alpha/2} f)(X_s)\mathrm{d}s\right] \\ &+ \mathbb{E}_{\mathbf{Q}}\left[\int_0^t \mathrm{e}^{-\lambda s} b(X_s)\cdot\nabla f(X_s)\mathrm{d}s\right].\end{aligned} \tag{2.15}$$

Note that f, ∇f and $\Delta^{\alpha/2} f$ are all bounded. Taking limit $t \to \infty$ in both sides of (2.15) and using the fact (2.11), we obtain

$$\mathbb{E}_{\mathbf{Q}}\left[\int_0^\infty \mathrm{e}^{-\lambda t}(\lambda f - \Delta^{\alpha/2} f)(X_t)\mathrm{d}t\right] = f(x) + \mathbb{E}_{\mathbf{Q}}\left[\int_0^\infty \mathrm{e}^{-\lambda t} b(X_t)\cdot\nabla f(X_t)\mathrm{d}t\right]. \tag{2.16}$$

We want to show that (2.16) holds for all $f \in C_0^\infty(\mathbb{R}^d)$. In fact, for any $f \in C_0^\infty(\mathbb{R}^d)$, there exists a sequence of functions $f_n \in C_c^\infty(\mathbb{R}^d)$ such that $\|f_n - f\|_\infty \to 0$, $\|\Delta^{\alpha/2} f_n - \Delta^{\alpha/2} f\|_\infty \to 0$, $\|\nabla f_n - \nabla f\|_\infty \to 0$. Applying (2.11) again, we have

$$\lim_{n\to\infty} \mathbb{E}_{\mathbf{Q}}\left[\int_0^\infty \mathrm{e}^{-\lambda t}(\lambda f_n - \Delta^{\alpha/2} f_n)(X_t)\mathrm{d}t\right] = \mathbb{E}_{\mathbf{Q}}\left[\int_0^\infty \mathrm{e}^{-\lambda t}(\lambda f - \Delta^{\alpha/2} f)(X_t)\mathrm{d}t\right]$$

and

$$\lim_{n\to\infty} \mathbb{E}_{\mathbf{Q}}\left[\int_0^\infty \mathrm{e}^{-\lambda t} b(X_t)\cdot\nabla f_n(X_t)\mathrm{d}t\right] = \mathbb{E}_{\mathbf{Q}}\left[\int_0^\infty \mathrm{e}^{-\lambda t} b(X_t)\cdot\nabla f(X_t)\mathrm{d}t\right].$$

Thus (2.16) holds for $f \in C_0^\infty(\mathbb{R}^d)$.

By Lemma 2.2, $R_\lambda g \in C_0^\infty(\mathbb{R}^d)$ for $g \in C_0^\infty(\mathbb{R}^d)$. Taking $f = R_\lambda g$ in (2.16) and using the fact $(\lambda - \Delta^{\alpha/2})R_\lambda g = g$, we obtain (2.14).

(iii) We claim that

$$\mathbb{E}_{\mathbf{Q}}\left[\int_0^\infty \mathrm{e}^{-\lambda t}\mathbb{1}_A(X_t)\mathrm{d}t\right] = 0 \qquad \text{for any } A \subset \mathbb{R}^d \text{ with } m(A) = 0, \tag{2.17}$$

where m is the Lebesgue measure on $\mathbb{R}^d$.

To see this, suppose A is a bounded subset of $\mathbb{R}^d$ having $m(A) = 0$. Let ψ_n be a sequence of positive functions in $C_c^\infty(\mathbb{R}^d)$ so that $|\psi_n| \le 2$, $\lim_{n\to\infty}\psi_n = 0$ m-a.e. on $\mathbb{R}^d$ and $\lim_{n\to\infty}\psi_n \ge \mathbb{1}_A$. It follows from (2.7) and the dominated convergence theorem that $\nabla R_\lambda \psi_n(z) = \int_{\mathbb{R}^d}\nabla r_\lambda(z-y)\psi_n(y)\mathrm{d}y$ converges to 0 boundedly as $n \to \infty$. One concludes then from (2.11) and the dominated convergence theorem that

$$\lim_{n\to\infty} \mathbb{E}_{\mathbf{Q}}\left[\int_0^\infty \mathrm{e}^{-\lambda t} b(X_t)\cdot\nabla R_\lambda\psi_n(X_t)\mathrm{d}t\right] = 0.$$

Applying Fatou's lemma to (2.14) with ψ_n in place of g yields that

$$\mathbb{E}_{\mathbf{Q}}\left[\int_0^\infty \mathrm{e}^{-\lambda t}\mathbb{1}_A(X_t)\mathrm{d}t\right] \le \liminf_{n\to\infty}\mathbb{E}_{\mathbf{Q}}\left[\int_0^\infty \mathrm{e}^{-\lambda t}\psi_n(X_t)\mathrm{d}t\right] = \liminf_{n\to\infty} R_\lambda\psi_n(x) = 0,$$

where the last equality is due to (2.6) and the dominated convergence theorem. This establishes (2.17) for any bounded and hence for any subset $A \subset \mathbb{R}^d$ having $m(A) = 0$.

(iv) We now show that (2.14) holds for any function g on $\mathbb{R}^d$ with $|g| \le c|b|$ as well.

Let g be a function on $\mathbb{R}^d$ with $|g| \le c|b|$ for some $c > 0$. Fix $M > 0$ and define $g_M = (((-M) \vee g) \wedge M) \mathbb{1}_{B(0,M)}$. Let ϕ be a positive smooth function on $\mathbb{R}^d$ with compact support such that $\int_{\mathbb{R}^d} \phi(y) \mathrm{d}y = 1$. For $n \ge 1$, set $\phi_n(y) = n^d \phi(ny)$ and $f_n(z) := \int_{\mathbb{R}^d} \phi_n(z-y) g_M(y) \mathrm{d}y$. Then $f_n \in C_c^\infty(\mathbb{R}^d)$, $|f_n| \le M$, and f_n converges to g_M almost everywhere on $\mathbb{R}^d$ as $n \to \infty$. In view of (2.17) and the bounded convergence theorem, $\lim_{n\to\infty} R_\lambda f_n(x) = R_\lambda g_M(x)$ and

$$\lim_{n\to\infty} \mathbb{E}_{\mathbf{Q}} \left[\int_0^\infty \mathrm{e}^{-\lambda t} f_n(X_t) \mathrm{d}t \right] = \mathbb{E}_{\mathbf{Q}} \left[\int_0^\infty \mathrm{e}^{-\lambda t} g_M(X_t) \mathrm{d}t \right].$$

On the other hand, by (2.7) and the dominated convergence theorem, $\nabla R_\lambda f_n$ converges boundedly on $\mathbb{R}^d$ to $\nabla R_\lambda g_M$ as $n \to \infty$. Letting $n \to \infty$ in (2.14) with f_n in place of g, we deduce that

(2.18)

$$\mathbb{E}_{\mathbf{Q}} \left[\int_0^\infty \mathrm{e}^{-\lambda t} g_M(X_t) \mathrm{d}t \right] = R_\lambda g_M(x) + \mathbb{E}_{\mathbf{Q}} \left[\int_0^\infty \mathrm{e}^{-\lambda t} b(X_t) \cdot \nabla R_\lambda g_M(X_t) \mathrm{d}t \right].$$

Clearly by the dominated convergence theorem, (2.6) and (2.11),

$$\lim_{M\to\infty} \mathbb{E}_{\mathbf{Q}} \left[\int_0^\infty \mathrm{e}^{-\lambda t} g_M(X_t) \mathrm{d}t \right] = \mathbb{E}_{\mathbf{Q}} \left[\int_0^\infty \mathrm{e}^{-\lambda t} g(X_t) \mathrm{d}t \right]$$

and

$$\lim_{M\to\infty} R_\lambda g_M(x) = R_\lambda g(x),$$

while in view of (2.7) and (2.8), $\nabla R_\lambda g_M(z) = \int_{\mathbb{R}^d} \nabla r_\lambda(z-y) g_M(y) \mathrm{d}y$ converges boundedly on $\mathbb{R}^d$ to $\nabla R_\lambda g(z)$. Thus by (2.11) and the dominated convergence theorem,

$$\lim_{M\to\infty} \mathbb{E}_{\mathbf{Q}} \left[\int_0^\infty \mathrm{e}^{-\lambda t} b(X_t) \cdot \nabla R_\lambda g_M(X_t) \mathrm{d}t \right] = \mathbb{E}_{\mathbf{Q}} \left[\int_0^\infty \mathrm{e}^{-\lambda t} b(X_t) \cdot \nabla R_\lambda g(X_t) \mathrm{d}t \right].$$

The last two displays together with (2.18) establish the claim that (2.14) holds for any g with $|g| \le c|b|$.

(v) Define a linear functional V_λ by

$$V_\lambda f = \mathbb{E}_{\mathbf{Q}} \left[\int_0^\infty \mathrm{e}^{-\lambda t} f(X_t) \mathrm{d}t \right].$$

Then (2.14) can be rewritten as

(2.19)

$$V_\lambda g = R_\lambda g(x) + V_\lambda(B R_\lambda g) \quad \text{for } g \in C_0^\infty(\mathbb{R}^d) \cup \{g : |g| \le c|b| \text{ for some } c > 0\},$$

where B is the operator defined by

$$Bf(x) = b(x) \cdot \nabla f(x).$$

Fix $g \in C_0^\infty(\mathbb{R}^d)$. It follows from (2.7) that $|BR_\lambda g| \le c|b|$ for some constant $c > 0$. Applying (2.19) with $BR_\lambda g$ in place of g yields

$$V_\lambda(BR_\lambda g) = R_\lambda(BR_\lambda g)(x) + V_\lambda(BR_\lambda BR_\lambda g). \tag{2.20}$$

Repeating this procedure, we get that for every $g \in C_0^\infty(\mathbb{R}^d)$ and every integer $N \geq 1$,

$$V_\lambda g = \sum_{k=0}^{N} R_\lambda (BR_\lambda)^k g(x) + V_\lambda(B(R_\lambda B)^N R_\lambda g). \tag{2.21}$$

It follows from Lemma 2.3 that for $\lambda > \lambda_0$,

$$\begin{aligned} |V_\lambda(B(R_\lambda B)^N R_\lambda g)| \leq& \|(\nabla R_\lambda b)^N \nabla R_\lambda g\|_\infty \mathbb{E}_{\mathbf{Q}} \left[\int_0^\infty \mathrm{e}^{-\lambda t} |b(X_t)| \mathrm{d}t \right] \\ \leq& 2^{-N} \|\nabla R_\lambda g\|_\infty \mathbb{E}_{\mathbf{Q}} \left[\int_0^\infty \mathrm{e}^{-\lambda t} |b(X_t)| \mathrm{d}t \right], \end{aligned}$$

which tends to 0 as $N \to \infty$. Letting $N \to \infty$ in (2.21) gives

$$V_\lambda g = \sum_{k=0}^{\infty} R_\lambda (BR_\lambda)^k g(x). \tag{2.22}$$

Note that $\mathbb{P}_x$ is also a solution to the martingale problem for $(\mathcal{L}^b, C_c^\infty(\mathbb{R}^d))$ with initial value x. Then (2.22) also holds with $\mathbf{Q}$ replaced by $\mathbb{P}_x$, that is,

$$\mathbb{E}_x \left[\int_0^\infty \mathrm{e}^{-\lambda t} g(X_t) \mathrm{d}t \right] = \sum_{k=0}^{\infty} R_\lambda (BR_\lambda)^k g(x).$$

Consequently

$$\mathbb{E}_{\mathbf{Q}} \left[\int_0^\infty \mathrm{e}^{-\lambda t} g(X_t) \mathrm{d}t \right] = \mathbb{E}_x \left[\int_0^\infty \mathrm{e}^{-\lambda t} g(X_t) \mathrm{d}t \right] \tag{2.23}$$

for every $g \in C_c^\infty(\mathbb{R}^d)$ and $\lambda > \lambda_0$. By the uniqueness of the Laplace transform, we have $\mathbb{E}_{\mathbf{Q}}[g(X_t)] = \mathbb{E}_x[g(X_t)]$ for all t, or, the one-dimensional distributions of X_t under $\mathbf{Q}$ and $\mathbb{P}_x$ are the same. By [6, Theorem 4.4.2], one obtains equality of all finite-dimensional distributions and hence $\mathbf{Q} = \mathbb{P}_x$. The uniqueness for the martingale problem for $(\mathcal{L}^b, C_c^\infty(\mathbb{R}^d))$ is thus proved. □

3. Stochastic differential equation

It is known that for any $\alpha \in (0, 2)$ the fractional Laplacian $\Delta^{\alpha/2}$ can be written in the form

$$\Delta^{\alpha/2} u(x) = \int_{\mathbb{R}^d} \left(u(x+z) - u(x) - \nabla u(x) \cdot z \mathbb{1}_{\{|z| \leq 1\}} \right) \frac{\mathcal{A}(d, -\alpha)}{|z|^{d+\alpha}} \, \mathrm{d}z, \tag{3.1}$$

where $\mathcal{A}(d, -\alpha)$ is a normalizing constant so that

$$\int_{\mathbb{R}^d} \left(\mathrm{e}^{\mathrm{i}\xi \cdot z} - 1 - \mathrm{i}\xi \cdot z \mathbb{1}_{\{|z| \leq 1\}} \right) \frac{\mathcal{A}(d, -\alpha)}{|z|^{d+\alpha}} \mathrm{d}z = -|\xi|^\alpha, \qquad \xi \in \mathbb{R}^d. \tag{3.2}$$

In fact, $\mathcal{A}(d, -\alpha)$ can be computed explicitly in terms of Γ-function:

$$\mathcal{A}(d, -\alpha) = \alpha 2^{\alpha-1} \pi^{-d/2} \Gamma\left(\frac{d+\alpha}{2} \right) \Gamma\left(1 - \frac{\alpha}{2} \right)^{-1}.$$

Recall that $\{X, \mathbb{P}_x, x \in \mathbb{R}^d\}$ is the unique solution to the martingale problem for $(\mathcal{L}^b, C_c^\infty(\mathbb{R}^d))$ on the canonical Skorokhod space $\Omega := \mathbb{D}([0, \infty); \mathbb{R}^d)$. Following the arguments in [10, 11] we show in the following theorem that X is a weak solution to the SDE (1.4).

Theorem 3.1. *There exists a process Z defined on Ω so that all its paths are right continuous and admit left limits, and that under each $\mathbb{P}_x$, Z is a rotationally symmetric α-stable process on $\mathbb{R}^d$ and*

$$X_t = x + Z_t + \int_0^t b(X_s)\mathrm{d}s, \quad t \geq 0. \tag{3.3}$$

Proof. By Theorem 1.2, X is the Feller process determined by kernel $q^b(t,x,y)$ in Proposition 2.1. For $t \geq 0$, let

$$Z_t := X_t - X_0 - \int_0^t b(X_s)\mathrm{d}s. \tag{3.4}$$

Note that Z_t is well defined since we have from [4, Lemma 2.2] that

$$\mathbb{E}\left[\left|\int_0^t b(X_s)\mathrm{d}s\right|\right] \leq \int_0^t \int_{\mathbb{R}^d} q^b(s,x,y)|b(y)|\mathrm{d}y\mathrm{d}s < \infty.$$

The aim is to prove that under each $\mathbb{P}_x$, the process Z is a rotationally symmetric α-stable process on $\mathbb{R}^d$ starting at 0.

Fix $\xi \in \mathbb{R}^d$. Consider the bounded function

$$u(t,x,\xi) := \mathbb{E}_x\left[\exp\left(\mathrm{i}\xi \cdot \left(X_t - \int_0^t b(X_s)\mathrm{d}s\right)\right)\right], \quad t \geq 0,\ x \in \mathbb{R}^d. \tag{3.5}$$

We claim that $u(t,x,\xi)$ is the unique bounded solution to the integral equation

$$u(t,x,\xi) = \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}\xi\cdot y} q^b(t,x,y)\mathrm{d}y - \mathrm{i}\int_0^t \int_{\mathbb{R}^d} q^b(t-s,x,z)\left(\xi\cdot b(z)\right)u(s,z,\xi)\mathrm{d}z\mathrm{d}s. \tag{3.6}$$

In fact, it is easy to verify that

$$\mathrm{e}^{-\mathrm{i}\xi\cdot\int_0^t b(X_\tau)\mathrm{d}\tau} = 1 - \mathrm{i}\int_0^t \left(\xi\cdot b(X_s)\right)\mathrm{e}^{-\mathrm{i}\xi\cdot\int_s^t b(X_\tau)\mathrm{d}\tau}. \tag{3.7}$$

Thus by the Markov property we have

$$\begin{aligned} u(t,x,\xi) =& \mathbb{E}_x\left[\exp\left(\mathrm{i}\xi\cdot\left(X_t - \int_0^t b(X_s)\mathrm{d}s\right)\right)\right] \\ =& \mathbb{E}_x\left[\mathrm{e}^{\mathrm{i}\xi\cdot X_t}\right] - \mathrm{i}\int_0^t \mathbb{E}_x\left[\mathrm{e}^{\mathrm{i}\xi\cdot X_t}\left(\xi\cdot b(X_s)\right)\mathrm{e}^{-\mathrm{i}\xi\cdot\int_s^t b(X_\tau)\mathrm{d}\tau}\right]\mathrm{d}s \\ =& \mathbb{E}_x\left[\mathrm{e}^{\mathrm{i}\xi\cdot X_t}\right] - \mathrm{i}\int_0^t \mathbb{E}_x\left[\left(\xi\cdot b(X_s)\right)u(t-s,X_s,\xi)\right]\mathrm{d}s \\ =& \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}\xi\cdot y}q^b(t,x,y)\mathrm{d}y - \mathrm{i}\int_0^t\int_{\mathbb{R}^d} q^b(s,x,z)\left(\xi\cdot b(z)\right)u(t-s,z,\xi)\mathrm{d}z\mathrm{d}s. \end{aligned}$$

To prove the uniqueness of the bounded solution to (3.6), it suffices to show that the only bounded solution to the corresponding homogeneous equation

$$v(t,x,\xi) = -\mathrm{i}\int_0^t \int_{\mathbb{R}^d} q^b(t-s,x,z)\left(\xi\cdot b(z)\right)v(s,z,\xi)\mathrm{d}z\mathrm{d}s \tag{3.8}$$

is identically zero.

Since $|b| \in \mathbb{K}_{d,\alpha-1}$, we have from [4, Lemma 2.2] that for fixed $\xi \in \mathbb{R}^d$, there exists a constant $T_1 > 0$ such that

$$\sup_{x\in\mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} q^b(t-s,x,z)\,|\xi \cdot b(z)|\,\mathrm{d}z\mathrm{d}s < 1$$

for every $t \in (0, T_1]$. As a consequence, the map Γ on $L^\infty([0,T_1]\times\mathbb{R}^d)$ defined by

$$w \mapsto (\Gamma w)(t,x) = -\mathrm{i} \int_0^t \int_{\mathbb{R}^d} q^b(t-s,x,z)\,(\xi \cdot b(z))\,w(s,z)\mathrm{d}z\mathrm{d}s, \quad w \in L^\infty([0,T_1]\times\mathbb{R}^d)$$

is a contraction. Let v be a bounded solution to (3.6). Then v is a fixing point of Γ and hence $v(t,x,\xi) = 0$ for $t \le T_1$. Note that

$$\begin{aligned} v(T_1+t,x,\xi) &= -\mathrm{i} \int_0^{T_1+t} \int_{\mathbb{R}^d} q^b(T_1+t-s,x,z)\,(\xi \cdot b(z))\,v(s,z,\xi)\mathrm{d}z\mathrm{d}s \\ &= -\mathrm{i} \int_0^t \int_{\mathbb{R}^d} q^b(t-s,x,z)\,(\xi \cdot b(z))\,v(T_1+s,z,\xi)\mathrm{d}z\mathrm{d}s. \end{aligned}$$

We conclude that $v(t,x,\xi) = 0$ for all $t > 0$ by induction. This implies that the bounded solution to (3.6) is unique.

By Proposition 2.1, Duhamel's formula

$$q^b(t,x,y) = p(t,x,y) + \int_0^t \int_{\mathbb{R}^d} q^b(t-s,x,z) b(z) \cdot \nabla_z p(s,z,y)\mathrm{d}z\mathrm{d}s \tag{3.9}$$

holds true for $t \le T_0$. Integrating (3.9) against $\mathrm{e}^{\mathrm{i}\xi\cdot y}\mathrm{d}y$, we see that the function

$$\mathrm{e}^{\mathrm{i}\xi\cdot x - t|\xi|^\alpha} = \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}\xi\cdot y} p(t,x,y)\mathrm{d}y$$

is also a bounded solution to (3.6) for $t < T_0$. Therefore, by the uniqueness,

$$\mathbb{E}_x\left[\exp\left(\mathrm{i}\xi \cdot \left(X_t - X_0 - \int_0^t b(X_s)\mathrm{d}s\right)\right)\right] = \mathrm{e}^{-\mathrm{i}\xi\cdot x} u(t,x,y) = \mathrm{e}^{-t|\xi|^\alpha} \tag{3.10}$$

for $t \le T_0$ and, consequently,

$$Z_t = X_t - X_0 - \int_0^t b(X_s)\mathrm{d}s$$

is a rotationally symmetric α-stable process. The proof of this theorem is finished. $\square$

Now we are ready to complete the proof for the uniqueness of a weak solution to the SDE (1.4).

Proof of Theorem 1.3. Fix $x \in \mathbb{R}^d$. Theorem 3.1 gives the existence of a weak solution to SDE (1.4). If $\{X_t, \mathbb{P}_x\}$ is a weak solution to (1.4), then by Itô's formula (cf. [7]) and (3.1), we have for every $f \in C_c^\infty(\mathbb{R}^d)$,

$$f(X_t) - f(X_0) = \text{ a martingale} + \int_0^t \mathcal{L}^b f(X_s)\mathrm{d}s. \tag{3.11}$$

Thus every weak solution to (1.4) solves the martingale problem for $(\mathcal{L}^b, C_c^\infty(\mathbb{R}^d))$. So the uniqueness of a weak solution to (1.4) follows from the uniqueness of the martingale problem for $(\mathcal{L}^b, C_c^\infty(\mathbb{R}^d))$, which is established in Theorem 1.2. $\square$

The unique weak solutions $X = \{X_t, \mathbb{P}_x, x \in \mathbb{R}^d\}$ to SDE (1.4) form a strong Markov process. By Theorems 1.2 and 1.3, X has $q^b(t,x,y)$ of Proposition 2.1 as its transition density function. Thus one readily gets from Proposition 2.1 and [4, Theorem 1.3] the following sharp two-sided estimates on the transition density function of X and its subprocess in a bounded $C^{1,1}$ open set. Recall that an open set D in $\mathbb{R}^d$ (when $d \geq 2$) is said to be $C^{1,1}$ if there exist a localization radius $R > 0$ and a constant $\Lambda > 0$ such that for every $Q \in \partial D$, there exist a $C^{1,1}$-function $\phi = \phi_Q : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying $\phi(0) = \nabla\phi(0) = 0$, $\|\nabla\phi\|_\infty \leq \Lambda$, $|\nabla\phi(x) - \nabla\phi(y)| \leq \Lambda|x-y|$, and an orthonormal coordinate system CS_Q: $y = (y_1, \ldots, y_{d-1}, y_d) =: (\widetilde{y}, y_d)$ with its origin at Q such that

$$B(Q,R) \cap D = \{y = (\widetilde{y}, y_d) \in B(0,R) \text{ in } CS_Q : y_d > \phi(\widetilde{y})\}.$$

The pair (R, Λ) is called the characteristics of the $C^{1,1}$ open set D. Note that a $C^{1,1}$ open set can be unbounded and disconnected. For $x \in D$, let $\delta_D(x)$ denote the Euclidean distance between x and ∂D. The diameter of D will be denoted as $\mathrm{diam}(D)$.

Corollary 3.2. (i) *X has a jointly continuous transition density function $p(t,x,y)$ with respect to the Lebesgue measure on $\mathbb{R}^d$. Moreover, for every $T > 0$, there is a constant $c_1 > 1$ depending only on d, α, T and on b through the rate at which $M^\alpha_{|b|}(r)$ goes to zero so that for $(t,x,y) \in (0,T] \times \mathbb{R}^d \times \mathbb{R}^d$,*

$$c_1^{-1}\left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right) \leq p(t,x,y) \leq c_1\left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right).$$

(ii) *Let $d \geq 2$ and let D be a bounded $C^{1,1}$ open subset of $\mathbb{R}^d$ with $C^{1,1}$ characteristics (R_0, Λ_0). Define*

$$f_D(t,x,y) = \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right)\left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right).$$

For each $T > 0$, there are constants $c_2 = c_2(T, R_0, \Lambda_0, d, \alpha, \mathrm{diam}(D), b) \geq 1$ and $c_3 = c_3(T, d, \alpha, D, b) \geq 1$ with the dependence on b only through the rate at which $M^\alpha_{|b|}(r)$ goes to zero such that

(a) on $(0,T] \times D \times D$, $c_2^{-1} f_D(t,x,y) \leq p_D(t,x,y) \leq c_2 f_D(t,x,y)$;
(b) on $[T,\infty) \times D \times D$,

$$c_3^{-1}\, \mathrm{e}^{-t\lambda_1^{b,D}}\, \delta_D(x)^{\alpha/2}\, \delta_D(y)^{\alpha/2} \;\leq\; p_D(t,x,y) \;\leq\; c_3\, \mathrm{e}^{-t\lambda_1^{b,D}}\, \delta_D(x)^{\alpha/2}\, \delta_D(y)^{\alpha/2},$$

where $\lambda_1^{b,D} := -\sup \mathrm{Re}(\sigma(\mathcal{L}^b|_D)) > 0$. Here $\sigma(\mathcal{L}^b|_D)$ denotes the spectrum of the non-local operator $\mathcal{L}^b$ in D with zero exterior condition.

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