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NEVANLINNA COUNTING FUNCTION AND PULL–BACK MEASURE

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ABSTRACT. We give an explicit relation between the Nevanlinna counting function of an analytic self-map of the unit disk and its pull-back measure. This gives a simple proof of the results of Lefévre, Li, Queffélec and Rodrìguez–Piazza (2011).

1. INTRODUCTION

In this note we study the relationship between the Laplacian of the Nevanlinna counting function of an analytic self-map of the unit disk and its pull-back measure. Our results allow us to refine asymptotic identities involving the Nevanlinna counting function and the pull–back measure given recently by Lefèvre, Li, Queffélec and Rodrìguez–Piazza [3]. As consequences of their result, one easily recovers several previously known results about composition operators on the Hardy space, including the characterization of compact composition operators and composition operators belonging to Schatten classes (see [3–5, 7, 8]).

In order to state our main result, we need some notation. Let φ be a non–constant analytic map of the unit disk $\mathbb{D}$ of the complex plane into itself. The Nevanlinna counting function is defined for every $z \in \mathbb{D}\backslash\{\varphi(0)\}$ by

$$N_\varphi(z) = \begin{cases} \displaystyle\sum_{w\in\varphi^{-1}(z)} \log\frac{1}{|w|} & \text{if} \quad z \in \varphi(\mathbb{D}), \\ \\ 0 & \text{if} \quad z \notin \varphi(\mathbb{D}). \end{cases}$$

Let $\varphi^*(e^{it}) = \lim_{r\to 1-}\varphi(re^{it})$ be the radial limit of φ. The function φ^* maps the unit circle $\mathbb{T}$ into the unit disk $\overline{\mathbb{D}}$. The pull–back measure on $\overline{\mathbb{D}}$ of the Lebesgue measure induced by φ^* is given by

$$m_\varphi(B) := |\{\zeta \in \mathbb{T} : \varphi^*(\zeta) \in B\}|,$$

where B is a Borel subset of $\overline{\mathbb{D}}$ and $|E|$ denotes the normalized Lebesgue measure of a Borel subset E of the unit circle $\mathbb{T}$. For every $\zeta \in \mathbb{T}$ and $0 < h < 1$, the Carleson box $W(\zeta, h) \subset \overline{\mathbb{D}}$ centered at ζ and of size h is the set

$$W(\zeta, h) := \{z \in \overline{\mathbb{D}} : 1 - h \le |z| \le 1 \text{ and } |\arg(z\overline{\zeta})| \le \pi h\}.$$

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We use, inside the Carleson box $W(\zeta, h)$, the set

$$S(\zeta, h) := \{z \in \overline{\mathbb{D}} : |z - \zeta| \leq h\},$$

whose size is comparable to that of $W(\zeta, h)$. Denote by $D(a, r)$ the disk of radius r centered at a and as before $S(\zeta, h) := \{z \in \overline{\mathbb{D}} : |z - \zeta| \leq h\}$, for $\zeta \in \overline{\mathbb{D}}$.

Now we are able to state the main result. This is a generalization of a theorem that was obtained by Lefèvre, Li, Queffélec and Rodrìguez–Piazza in [3, Theorem 1.1] (see also Remark 5 below). In fact, our proof of this more general result is simpler than the original proof from [3].

Theorem 1.1. *For every* $0 < c < 1/8$ *and* $0 < h < (1 - |\varphi(0)|)/8$*, we have*

$$m_\varphi[S(\zeta, (1-c)h] \leq \frac{2}{c^2} \sup_{z \in S(\zeta, h) \cap \mathbb{D}} N_\varphi(z), \qquad \zeta \in \overline{\mathbb{D}} \tag{1.1}$$

and

$$\sup_{z \in W(\zeta, h) \cap \mathbb{D}} N_\varphi(z) \leq \frac{100}{c^2}\, m_\varphi[W(\zeta, (1+c)h)], \qquad \zeta \in \mathbb{T}. \tag{1.2}$$

2. Proof

Let H^2 be the Hardy space of the disk,

$$\mathrm{H}^2 = \Big\{ f = \sum_{n \geq 0} \widehat{f}(n) z^n \in \mathrm{Hol}(\mathbb{D}) : \sum_{n \geq 0} |\widehat{f}(n)|^2 < \infty \Big\}.$$

The normalized area measure on $\mathbb{D}$ will be denoted by dA and let $\Delta = 4\partial^2/\partial z \partial \overline{z}$ be the usual Laplacian. For the proof of the above theorem we need the following key lemma which gives a generalization of the classical Littlewood-Paley identity.

Lemma 2.1. *For every analytic non–constant self–map* $\varphi : \mathbb{D} \to \mathbb{D}$ *and every* $g \in \mathcal{C}^2$ *on* $\mathbb{C}$ *we have*

$$\int_{\overline{\mathbb{D}}} g(z) dm_\varphi(z) = g(\varphi(0)) + \frac{1}{2} \int_{\mathbb{D}} \Delta g(w) N_\varphi(w) dA(w).$$

Proof. Since Δg is bounded on $\mathbb{D}$, by the change of variable formula, we have

$$\begin{aligned}
\int_{\mathbb{D}} \Delta g(z) N_\varphi(z) dA(z) &= \int_{\mathbb{D}} \Delta g(\varphi(w)) |\varphi'(w)|^2 \log \frac{1}{|w|} dA(w) \\
&= \int_{\mathbb{D}} \Delta (g \circ \varphi)(w) \log \frac{1}{|w|} dA(w) \\
&= \lim_{r \to 1-} \int_{r\mathbb{D}} \Delta (g \circ \varphi)(w) \log \frac{1}{|w|} dA(w). \qquad (2.1)
\end{aligned}$$

Now by Green's formula, for $r < 1$,

$$\begin{aligned}
&\frac{1}{2} \int_{r\mathbb{D}} \Delta (g \circ \varphi)(w) \log \frac{r}{r|w|} dA(w) \qquad (2.2) \\
&\qquad = -g(\varphi(0)) + \int_{\mathbb{T}} g(\varphi(r\zeta)) \frac{|d\zeta|}{2\pi} + \frac{1}{2} \log \frac{1}{r} \int_{r\mathbb{D}} \Delta (g \circ \varphi)(w) dA(w).
\end{aligned}$$

Note that for $\rho < 1$,

$$\begin{aligned} \Big| \log \frac{1}{\rho} \int_{\rho\mathbb{D}} \Delta(g\circ\varphi)(w)dA(w) \Big| &= \Big| \log \frac{1}{\rho} \int_{\rho\mathbb{D}} \Delta g(\varphi(w))|\varphi'(w)|^2 dA(w) \Big| \\ &\leq \|\Delta g\|_\infty \log \frac{1}{\rho} \int_{\rho\mathbb{D}} |\varphi'(w)|^2 dA(w) \\ &\leq \|\Delta g\|_\infty \int_{\mathbb{D}} |\varphi'(w)|^2 \log \frac{1}{|w|} dA(w) \\ &= \frac{1}{2}\|\Delta g\|_\infty (\|\varphi\|^2_{\mathrm{H}^2} - |\varphi(0)|^2). \end{aligned}$$

Hence for $0 < \rho < r < 1$, we get

$$\begin{aligned} &\Big| \log \frac{1}{r} \int_{r\mathbb{D}} \Delta(g\circ\varphi)(w)dA(w) \Big| \\ &= \frac{\log(1/r)}{\log(1/\rho)} \log \frac{1}{\rho} \int_{\rho\mathbb{D}} |\Delta(g\circ\varphi)(w)|dA(w) + \log \frac{1}{r} \int_{r\mathbb{D}\setminus\rho\mathbb{D}} |\Delta(g\circ\varphi)(w)|dA(w) \\ &\leq \frac{\log(1/r)}{\log(1/\rho)} \|\Delta g\|_\infty \|\varphi\|^2_{\mathrm{H}^2} + \|\Delta g\|_\infty \int_{\mathbb{D}\setminus\rho\mathbb{D}} |\varphi'(w)|^2 \log \frac{1}{|w|} dA(w). \end{aligned} \tag{2.3}$$

Finally, if we choose $\rho = e^{-(\log 1/r)^{1/2}}$, by (2.1), (2.2) and (2.3) the result follows from $r \to 1$. □

Proof of Theorem 1.1. Let $\zeta \in \overline{\mathbb{D}}$ and consider the function g defined on $\mathbb{C}$ by

$$g(w) = \begin{cases} (h^2 - |w-\zeta|^2)^2 & \text{if} \quad |w-\zeta| \leq h, \\ 0 & \text{if} \quad |w-\zeta| \geq h. \end{cases}$$

First note that for $\epsilon > 0$, $g_\epsilon = g^{1+\epsilon} \in \mathcal{C}^2(\mathbb{C})$. Applying Lemma 2.1 with g_ϵ and letting ϵ go to 0, we obtain

$$\int_{\overline{\mathbb{D}}} g(z)dm_\varphi(z) = g(\varphi(0)) + \frac{1}{2}\int_{\mathbb{D}} \Delta g(w) N_\varphi(w) dA(w). \tag{2.4}$$

We have

$$\Delta g(w) = 8(2|w-\zeta|^2 - h^2).$$

Note that if $z \in S(\zeta, (1-c)h)$, then $g(z) \geq (7^2/2^4)c^2h^4$. By (2.4), we get

$$\begin{aligned} m_\varphi[S(\zeta,(1-c)h)] &\leq \frac{2^4}{7^2c^2h^4} \int_{S(\zeta,(1-c)h)} g(z)dm_\varphi(z) \\ &\leq \frac{2^3}{7^2c^2h^4} \int_{\mathbb{D}} \Delta g(z) N_\varphi(z) dA(z) \\ &\leq \frac{2^3}{7^2c^2h^4} \int_{h/\sqrt{2} \leq |z-\zeta| \leq h} \Delta g(z) N_\varphi(z) dA(z) \\ &\leq \frac{2^3}{7^2c^2h^4} \sup_{z\in S(\zeta,h)} N_\varphi(z) \int_{h/\sqrt{2} \leq |z-\zeta| \leq h} \Delta g(z) dA(z) \\ &\leq \frac{2^6}{7^2c^2} \sup_{z\in S(\zeta,h)} N_\varphi(z). \end{aligned}$$

For the second estimate, let $\zeta = z/|z|$, $h = 1-|z|$, $Z = (1+\eta h)\zeta$ and $\rho = 1+2c+\eta$ where $\eta = (1+2c)(1+\sqrt{2})$. Consider the function g defined by

$$g(w) = \begin{cases} (\rho^2 h^2 - |w-Z|^2)^2 & \text{if} \quad |w-Z| < \rho h, \\ 0 & \text{if} \quad |w-Z| \geq \rho h. \end{cases}$$

As before for $\epsilon > 0$, $g_\epsilon = g^{1+\epsilon} \in \mathcal{C}^2(\mathbb{C})$. Applying Lemma 2.1 with g_ϵ and letting ϵ go to 0 we get (2.4). We have

$$\Delta g(w) = 8(2|w-Z|^2 - \rho^2 h^2).$$

If $w \in \mathbb{D}$, then $|w-Z| \geq \eta h = \rho h/\sqrt{2}$ and so $\Delta g(w) \geq 0$. Note that $D(z, ch) \subset D(Z, \rho h)$. Also, if $|w-z| \leq ch$, then

$$|w-Z| \geq |Z-z| - |w-z| \geq (1-c+\eta)h,$$

hence

$$\Delta g(w) \geq 8(2(1-c+\eta)^2 - (1+2c+\eta)^2)h^2 \geq 2^3(\sqrt{2}+2)h^2, \qquad w \in D(Z, \rho h).$$

Note that $\varphi(0) \notin D(z, ch)$, by the sub-mean value property of N_φ (see Remark 3 below) and (2.4), we get

$$\begin{aligned} N_\varphi(z) &\leq \frac{2}{c^2h^2} \int_{D(z,ch)} N_\varphi(w) dA(w) \\ &\leq \frac{2}{2^3(\sqrt{2}+2)c^2h^4} \int_{D(z,ch)} N_\varphi(w)\Delta g(w) dA(w) \\ &\leq \frac{1}{2^2(\sqrt{2}+2)c^2h^4} \int_{\mathbb{D}} N_\varphi(w)\Delta g(w) dA(w) \\ &\leq \frac{2}{2^2(\sqrt{2}+2)c^2h^4} \int_{D(Z,\rho h)\cap\overline{\mathbb{D}}} g(w) dm_\varphi(w) \\ &\leq \frac{1}{2(\sqrt{2}+2)c^2h^4} \rho^4 h^4 \; m_\varphi(D(Z,\rho h)\cap\overline{\mathbb{D}}) \\ &\leq \frac{100}{c^2} \; m_\varphi(D(Z,\rho h)\cap\overline{\mathbb{D}}). \end{aligned}$$

We claim that

$$D(Z, \rho h) \cap \overline{\mathbb{D}} \subset W(\zeta, (1+2c)h).$$

Suppose that $\zeta = 1$ and let $e^{i\theta} \in \partial D(Z, \rho h) \cap \mathbb{T}$. We have

$$\begin{aligned} \sin^2\theta &= \frac{1}{4(1+\eta h)^2}[(1+\rho h)^2 - (1+\eta h)^2][(1+\eta h)^2 - (1-\rho h)^2] \\ &\leq (3+2\sqrt{2})(1+2c)^2 h^2. \end{aligned}$$

Note that $c < 1/8$ and $h < 1/8$, so we get $\theta \leq (1+c)h\pi$. Now the proof is complete. □

3. Remarks

1. The following corollary, first proved by Rudin [6], (see also Bishop [1]), is an immediate consequence of Lemma 2.1

Corollary 3.1 (Bishop & Rudin). *We have*

$$\Delta N_\varphi = -\delta_{\varphi(0)} + m_\varphi,$$

where Δ is the distributional Laplacian and δ_{z_0} is the Dirac mass at z_0. Furthermore N_φ is equal to a subharmonic function almost everywhere on $\mathbb{D}$.

2. The formula from Lemma 2.1 was given by Stanton [2, Theorem 2] when g is a subharmonic function instead of a smooth function. The proof of Stanton's formula is based on Jensen's formula.

3. By Corollary 3.1, the Nevanlinna counting function N_φ is equal to a subharmonic function almost everywhere on $\mathbb{D}$. In fact N_φ fails to be subharmonic only on a set of logarithmic capacity zero [2]. The Nevanlinna counting function satisfies the sub-mean value property

$$N_\varphi(z) \le \frac{2}{r^2}\int_{D(z,r)} N_\varphi(w)dA(w),$$

for every disk $D(z,r)$ of center z and radius r does not contain $\varphi(0)$; see [7,8].

4. In the proof in the first inequality (1.1), one can consider the following function $g(w) = (h^2 - \mathrm{Re}(w-\zeta)^2)^2(h^2 - \mathrm{Im}(w-\zeta)^2)^2$ if $\mathrm{Re}(w-\zeta) \le h$ and $\mathrm{Im}(w-\zeta) \le \pi h$ and $g(w) = 0$ otherwise. We obtain as before:

For every $0 < c < 1/8$ and $0 < h < (1-|\varphi(0)|)/8$, we have

$$m_\varphi[W(\zeta,(1-c)h] \le \frac{A}{c^4}\sup_{z\in W(\zeta,h)\cap\mathbb{D}} N_\varphi(z) \qquad \zeta\in\mathbb{T},$$

where A is an absolute constant.

5. The following result of Lefèvre, Li, Queffélec and Rodiguez–Piazza in [3] is based on Stanton's formula. Their proof is quite complicated. They showed that the classical Nevanlinna counting function and the pull-back measure are connected as follows:

For every $\zeta\in\mathbb{T}$ and $0 < h < (1-|\varphi(0)|)/16$, we have

$$\frac{1}{64}m_\varphi[\widetilde{W}(\zeta,h/64] \le \sup_{z\in W(\zeta,h)} N_\varphi(z) \le 196\, m_\varphi[\widetilde{W}(\zeta,24h)],$$

where $\widetilde{W}(\zeta,h) := \{z\in\overline{\mathbb{D}} : 1-h \le |z| \le 1 \text{ and } |\arg(z\overline{\zeta})| \le h\}$.

6. There are several different conditions which characterize the compactness of the composition operator $C_\varphi(f) = f\circ\varphi$ on the Hardy space H^2. For the holomorphic self-map φ of $\mathbb{D}$ the following conditions are equivalent:

(i) $N_\varphi(z) = o(\log 1/|z|)$ as $|z|\to 1-$,
(ii) $m_\varphi[W(\zeta,h)] = o(h)$ as $h\to 0$,
(iii) C_φ is compact on H^2.

The condition (ii) means that m_φ is a vanishing Carleson measure for H^2 [5]. For Hilbert-Schmidt class membership see [4]. We refer the reader to the monograph by Shapiro for an account of these problems [7].

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