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SHIFT HARNACK INEQUALITY AND INTEGRATION BY PARTS FORMULA FOR FUNCTIONAL SDES DRIVEN BY FRACTIONAL BROWNIAN MOTION

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ABSTRACT. The shift Harnack inequality and the integration by parts formula for functional stochastic differential equations driven by fractional Brownian motion with Hurst parameter $\frac{1}{2} < H < 1$ are established by using a transformation formula for fractional Brownian motion and a new coupling argument.

1. Introduction

In stochastic analysis for diffusion processes, the Driver integration by parts formula [4] and the Bismut formula [2] (also known as the Bismut-Elworthy-Li formula due to [6]) are two fundamental tools. Based on the martingale method, a coupling argument or Malliavin calculus, the derivative formula has been widely studied and applied in various fields, such as heat kernel estimates, the strong Feller property and functional inequalities; see [7, 13, 15, 16] and references therein.

Recently, using a new coupling argument, Wang [14] derived the Driver integration by parts formula, and provided a new type of Harnack inequality, called the shift Harnack inequality. The main idea is to construct two processes which start from the same point and, at the expected time T, separate at a fixed vector almost surely. In [14], the Driver integration by parts formula and the shift Harnack inequality are also applied to various models including the degenerate diffusion process, delayed SDEs and semi-linear SPDEs. Using the new coupling argument, Zhang [17] established a shift Harnack inequality and an integration by parts formula for semi-linear SPDEs with delay, and Fan [8] established a shift Harnack inequality and an integration by parts formula for SDEs driven by additive fractional noise with Hurst parameter $\frac{1}{2} < H < 1$. However, in [8] the condition that b is Fréchet differentiable such that ∇b is bounded and Hölder continuous of order $1-\frac{1}{2H} < \rho < 1$ seems to be relatively strong. On the other hand, as far as we know, in the case that $\frac{1}{2} < H < 1$, using the approach of coupling for the segment process and Girsanov transformations, by virtue of irregularity of the operator K_H^{-1}, it is very difficult to obtain the shift Harnack inequality. In this paper, motivated mainly by [9], using a transformation formula for fractional Brownian motion, we

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establish the shift Harnack inequality and the integration by parts formula for the segment process in the stochastic functional differential equations

$$\begin{cases} dX(t) = b(X(t))dt + F(X_t)dt + dB_t^H, \\ X_0 = \xi, \end{cases}$$

driven by fractional Brownian motion with Hurst parameter $\frac{1}{2} < H < 1$.

The paper is organized as follows. In Section 2, we give some preliminaries on fractional Brownian motion. In Section 3, we establish the shift Harnack inequality and the integration by parts formula by using a transformation formula for fractional Brownian motion.

2. Preliminaries

Let $B^H = \{B_t^H, t \in [0,T]\}$ be a fractional Brownian motion with Hurst parameter $H \in (0,1)$ defined on the probability space $(\Omega, \mathcal{F}, P)$, i.e., B^H is a centered Gauss process with covariance function

$$R_H(t,s) = E(B_t^H B_s^H) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

In particular, if $H = \frac{1}{2}$, B is a Brownian motion. It is well known that if $H \neq \frac{1}{2}$, B^H does not have independent increments and has an α-order Hölder continuous path for all $\alpha \in (0,H)$.

For each $t \in [0,T]$, we denote by $\mathcal{F}_t$ the σ-algebra generated by the random variables $\{B_s^H : s \in [0,t]\}$ and the P-null sets.

We denote by $\mathcal{E}$ the set of step functions on $[0,T]$. Let $\mathcal{H}$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the scalar product

$$\langle I_{[0,t]}, I_{[0,s]} \rangle = R_H(t,s) = E(B_t^H B_s^H) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

The mapping $I_{[0,t]} \mapsto B_t^H$ can be extended to an isometry between $\mathcal{H}$ and the Gauss space $\mathcal{H}_1$ associated with B^H. Denote this isometry by $\phi \mapsto B^H(\phi)$. For more details, see [10]. On the other hand, from [5], we know the covariance kernel $R_H(t,s)$ can be written as

$$R_H(t,s) = \int_0^{t\wedge s} K_H(t,r)K_H(s,r)dr,$$

where K_H is a square integrable kernel given by

$$K_H(t,s) = \Gamma(H + \frac{1}{2})^{-1}(t-s)^{H-\frac{1}{2}} F(H - \frac{1}{2}, \frac{1}{2} - H, H + \frac{1}{2}, 1 - \frac{t}{s}),$$

in which $F(\cdot,\cdot,\cdot,\cdot)$ is the Gauss hypergeometric function.

Define the linear operator $K_H^* \mathcal{E} \to L^2([0,T]; R)$ as follows:

$$(K_H^*\phi)(s) = K_H(T,s)\phi(s) + \int_s^T (\phi(r) - \phi(s))\frac{\partial K_H}{\partial r}(r,s)dr.$$

By [1], we know that for all $\phi, \psi \in \mathcal{E}$, $\langle K_H^*\phi, K_H^*\psi\rangle_{L^2[0,T]} = \langle \phi, \psi \rangle$ holds. From the B.L.T. theorem, K_H^* can be extended to an isometry between $\mathcal{H}$ and $L^2[0,T]$. Therefore, according to [1], the process $\{W_t = B((K_H^*)^{-1}(I_{[0,t]})), t \in [0,T]\}$ is a Wiener process, and B^H has the integral representation

$$B_t^H = \int_0^t K_H(t,s)dW_s.$$

By [5], the operator $K_H : L^2[0,T] \to I_{0+}^{H+\frac{1}{2}}(L^2[0,T])$ associated with the square integrable kernel $K_H(\cdot,\cdot)$ is defined as

$$(K_H f)(t) := \int_0^t K_H(t,s)f(s)ds,\ f \in L^2[0,T],$$

where $I_{0+}^{H+\frac{1}{2}}$ is the $H+\frac{1}{2}$-order left-fractional Riemann-Liouville operator on $[0,T]$; see [12]. It is an isomorphism and for each $f \in L^2[0,T]$,

$$(K_H f)(s) = I_{0+}^{2H} s^{\frac{1}{2}-H} I_{0+}^{\frac{1}{2}-H} s^{H-\frac{1}{2}} f,\ \ H \le \frac{1}{2},$$

$$(K_H f)(s) = I_{0+}^{1} s^{H-\frac{1}{2}} I_{0+}^{H-\frac{1}{2}} s^{\frac{1}{2}-H} f,\ \ H \ge \frac{1}{2}.$$

As a consequence, for every $h \in I_{0+}^{H+\frac{1}{2}}(L^2[0,T])$, the inverse operator K_H^{-1} is of the form

$$(K_H^{-1}h)(s) = s^{H-\frac{1}{2}} D_{0+}^{H-\frac{1}{2}} s^{\frac{1}{2}-H} h',\ \ H > \frac{1}{2},$$

$$(K_H^{-1}h)(s) = s^{\frac{1}{2}-H} D_{0+}^{\frac{1}{2}-H} s^{H-\frac{1}{2}} D_{0+}^{2H} h,\ \ H < \frac{1}{2},$$

where $D_{0+}^{H-\frac{1}{2}}(D_{0+}^{\frac{1}{2}-H})$ is the $H-\frac{1}{2}(\frac{1}{2}-H)$-order left-sided Riemann-Liouville derivative; see [12].

In particular, if h is absolutely continuous, we have

$$(K_H^{-1}h)(s) = s^{H-\frac{1}{2}} I_{0+}^{\frac{1}{2}-H} s^{\frac{1}{2}-H} h',\ \ H < \frac{1}{2}. \tag{1}$$

Let $r > 0$ be fixed, and let $\mathcal{L} = C([-r,0];R)$ be equipped with the uniform norm $\|\cdot\|_\infty$. We consider the following functional stochastic differential equation driven by fractional Brownian motion on R,

$$\begin{cases} dX(t) = b(X(t))dt + F(X_t)dt + dB_t^H, \\ X_0 = \xi, \end{cases} \tag{2}$$

where $\xi \in \mathcal{L}$ and, for each $t \ge 0$, $X_t \in \mathcal{L}$ is fixed as $X_t(u) = X(t+u), u \in [-r,0]$.

The aim of the paper is to consider the shift Harnack inequality and the integration by parts formula for equation (2) in case $\frac{1}{2} < H < 1$. We define

$$P_t f(\xi) := Ef(X_t^\xi),\ \ t \in [0,T],\ \ f \in \mathcal{B}_b(\mathcal{L}), \tag{3}$$

where X_t^ξ is the solution to equation (2) and $\mathcal{B}_b(\mathcal{L})$ denotes the set of all bounded measurable functions on $\mathcal{L}$.

3. Shift Harnack inequality

Let us start with the following hypotheses:

(H1) $|b(x) - b(y)| \le K_1|x-y|, \forall x, y \in R$, where $K_1 > 0$ is a constant;

(H2) F is globally Lipschitz on $\mathcal{L}$, i.e., for some $K_2 > 0$,

$$|F(x) - F(y)| \le K_2\|x-y\|_\infty \quad \forall x, y \in \mathcal{L},\ t \in [0,T].$$

According to [3], conditions (H1) and (H2) ensure that equation (2) has a unique solution.

Theorem 3.1. *Assume that* (H1) *and* (H2) *hold. Then, there exists a unique strong solution to equation* (2).

Now, we aim to establish the integration by parts formula and the shift Harnack inequality for P_T. For any $p>1$, we introduce the space

$$\mathcal{H}_p := \{\eta \in \mathcal{L} : \|\eta\|_{\mathcal{H}}^p := \int_{-r}^{0} |\eta'(t)|^p dt < \infty\}.$$

Theorem 3.2. *Assume* (H1) *and* (H2) *hold and let* $T>r$, $\frac{1}{2}<H<1$. *Then for any* $p>1$, $\max\{0, 2H-\frac{3}{2}\} < \frac{1}{\sigma} < H-\frac{1}{2}$, $\frac{\sigma}{\sigma-1} < \sigma' < \frac{2}{3-2H}$, $\eta \in \mathcal{H}_{\frac{\sigma\sigma'}{\sigma\sigma'-\sigma-\sigma'}}$ *and* $f \in B_b(\mathcal{L})$, *the operator* P_T *satisfies*

$$(P_T f(\xi))^p \le P_T f^p(\eta+\cdot)(\xi)\exp\Big[\frac{p}{p-1}C(H,\sigma,\sigma')\rho(T,r,H,\eta)\Big], \qquad \forall \xi \in \mathcal{L}. \tag{4}$$

Here, in the relation (4),

$$C(H,\sigma,\sigma') = \Big[\frac{B(-\sigma'/2+H\sigma'+1, -3\sigma'/2+H\sigma'+1)^{1/\sigma'}}{\tilde{c}_H\Gamma(H-\frac{1}{2})|\sigma(1-2H)+1|^{\frac{1}{\sigma}}}\Big]^2, \tag{5}$$

where $B(\cdot,\cdot)$ *and* $\Gamma(\cdot)$ *are the standard Beta and Gamma functions,* $\tilde{c}_H = \left(\frac{2H}{\Gamma(2H)\Gamma(3-2H)}\right)^{\frac{1}{2}}$ *and*

$$\begin{aligned}
&\rho(T,r,H,\eta)\\
=&T^{\frac{2}{\sigma}+\frac{2}{\sigma'}-2H}\Big[\frac{1}{2H+\frac{2}{\sigma'}-2} + B\Big(2H+\frac{2}{\sigma'}-2, 3-4H+\frac{2}{\sigma}\Big)\Big]\\
&\cdot\Big\{(1+(TK_1)^q)[(\frac{2}{T-r})^{q-1}|\eta(-r)|^q + 2^{q-1}\|\eta\|_{\mathcal{H}}^q] + 2^{q-1}K_2^q(T-r)|\eta(-r)|^q\Big\}^{2/q},
\end{aligned}$$

where $q = \frac{\sigma\sigma'}{\sigma\sigma'-\sigma-\sigma'}$.

Proof. For any $\phi \in \mathcal{B}_b([0,T-r])$ such that $\int_0^{T-r}\phi(t)dt = 1$, let

$$\Lambda(t) = \begin{cases} \phi(t)\eta(-r), & \text{if } t\in[0,T-r],\\ \eta'(t-T), & \text{if } t\in[T-r,T],\end{cases} \tag{6}$$

and

$$\Theta(t) = \int_0^{t\vee 0}\Lambda(s)ds, \quad t\in[-r,T]. \tag{7}$$

For any $\varepsilon\in[0,1]$, let $X^\varepsilon(t)$ solve the equation

$$dX^\varepsilon(t) = \{b(X(t)) + F(X_t) + \varepsilon\Lambda(t)\}dt + dB_t^H, \ \ t\ge 0, \ \ X_0^\varepsilon = \xi. \tag{8}$$

Then, it is easy to see that

$$X_t^\varepsilon = X_t + \varepsilon\Theta_t, \ \ t\in[0,T]. \tag{9}$$

In particular, $X_T^\varepsilon = X_T + \varepsilon\eta(T)$. Next, for any $t\in[0,T]$, define

$$\begin{aligned}
d\widetilde{B}_t^H &:= dB_t^H + [\varepsilon\Lambda(t) - b(X^\varepsilon(t)) + b(X(t)) - F(X_t^\varepsilon) + F(X_t)]dt\\
&= dB_t^H + h_\varepsilon(t)dt,
\end{aligned}$$

where

$$h_\varepsilon(t) = \varepsilon\Lambda(t) - b(X^\varepsilon(t)) + b(X(t)) - F(X_t^\varepsilon) + F(X_t), \ \ t\in[0,T]. \tag{10}$$

As a direct consequence, we can rewrite equation (8) as

$$dX^\varepsilon(t) = \{b(X^\varepsilon(t)) + F(X_t^\varepsilon)\}dt + d\widetilde{B}_t^H, \ \ t\ge 0, \ \ X_0^\varepsilon = \xi.$$

Then by virtue of (6), (9) and (10), we have

$$\begin{aligned}\int_0^T |h_\varepsilon(t)|^q dt \leq{} & 2^{q-1}\varepsilon^q \int_0^{T-r} |\phi(t)\eta(-r)|^q dt + 2^{q-1}\varepsilon^q \int_{T-r}^T |\eta'(t-T)|^q dt \\ & + 2^{q-1}(\varepsilon K_1)^q \int_0^T |\Theta(t)|^q dt + 2^{q-1}(\varepsilon K_2)^q \int_0^{T-r} |\Theta(t)|^q dt.\end{aligned} \tag{11}$$

Taking $\phi(t) = \frac{1}{T-r}$, we have

$$\begin{aligned}\int_0^T |h_\varepsilon(t)|^q dt \leq{} & (\frac{2}{T-r})^{q-1}\varepsilon^q|\eta(-r)|^q + 2^{q-1}\varepsilon^q\|\eta\|^q_{\mathcal{H}} \\ & + 2^{q-1}(\varepsilon K_1 T)^q \int_0^T |\Lambda(t)|^q dt + 2^{q-1}(\varepsilon K_2(T-r))^q \int_0^{T-r} |\Lambda(t)|^q dt \\ \leq & (1+(TK_1)^q)[(\frac{2}{T-r})^{q-1}\varepsilon^q|\eta(-r)|^q \\ & + 2^{q-1}\varepsilon^q\|\eta\|^q_{\mathcal{H}}] + 2^{q-1}(\varepsilon K_2)^q(T-r)|\eta(-r)|^q \\ < & +\infty.\end{aligned} \tag{12}$$

On the other hand, we have

$$\begin{aligned} & \int_0^T (T-u)^{1-2H} h_\varepsilon(u) du \\ = & \frac{1}{\Gamma(\frac{3}{2}-H)} \int_0^T (T-u)^{\frac{1}{2}-H}\Gamma\Big(\frac{3}{2}-H\Big)(T-u)^{\frac{1}{2}-H} h_\varepsilon(u) du \\ = & (I_{0+}^{\frac{3}{2}-H}\overline{h_\varepsilon})(T),\end{aligned}$$

where

$$\overline{h_\varepsilon}(u) = \Gamma\Big(\frac{3}{2}-H\Big)(T-u)^{\frac{1}{2}-H} h_\varepsilon(u).$$

Note that $\frac{q-1}{q}\frac{1}{2H-1} = (\frac{1}{\sigma}+\frac{1}{\sigma'})\frac{1}{2H-1} > 1$ and $\frac{q}{q-1}(1-2H) > -1$. We have

$$\begin{aligned}\int_0^T |(T-u)^{\frac{1}{2}-H} h_\varepsilon(u)|^2 du \leq & \Big(\int_0^T |h_\varepsilon(u)|^q du\Big)^{\frac{2}{q}} \cdot \Big(\int_0^T (T-u)^{(1-2H)\frac{q}{q-1}} du\Big)^{\frac{q-1}{q}} \\ < & +\infty.\end{aligned}$$

Then $\overline{h_\varepsilon}(u) \in L^2([0,T];R)$. Thus, we have that $\int_0^\cdot \tilde{c}_H^{-1}(T-u)^{1-2H}h_\varepsilon(u)du \in I_{0+}^{\frac{3}{2}-H}(L^2([0,T];R))$. Note that by means of the integral representation of fractional Brownian motion, the definition of the operator K_H and transformation formulas for fractional Brownian motion (see [9]), we get that for the any $T > r$,

$$\begin{aligned}\widetilde{B}^H_T &= \int_0^T h_\varepsilon(s)ds + B^H_T \\ &= \int_0^T h_\varepsilon(s)ds + \tilde{c}_H \int_0^T (T-s)^{2H-1} dB^{1-H}_s \\ &= \int_0^T \tilde{c}_H(T-s)^{2H-1}[\tilde{c}_H^{-1}(T-s)^{1-2H}h_\varepsilon(s)ds + dB^{1-H}_s].\end{aligned} \tag{13}$$

For any $t \in [0,T]$, let

$$\begin{aligned}\widetilde{B}_t^{1-H} &= \int_0^t \tilde{c}_H^{-1}(T-s)^{1-2H}h_\varepsilon(s)ds + B_t^{1-H}\\ &= \int_0^t \tilde{c}_H^{-1}(T-s)^{1-2H}h_\varepsilon(s)ds + \int_0^t K_{1-H}(t,s)dW_s\\ &= \int_0^t K_{1-H}(t,s)\Big[\Big(K_{1-H}^{-1}\int_0^\cdot \tilde{c}_H^{-1}(T-u)^{1-2H}h_\varepsilon(u)du\Big)(s)ds + dW_s\Big],\end{aligned}$$

where $\tilde{c}_H = \left(\frac{2H}{\Gamma(2H)\Gamma(3-2H)}\right)^{\frac{1}{2}}$.

Now, let

$$\begin{aligned}R_\varepsilon(T) = \exp\Big[&-\int_0^T \left(K_{1-H}^{-1}\int_0^\cdot \tilde{c}_H^{-1}(T-v)^{1-2H}h_\varepsilon(v)dv\right)(s)dW_s\\ &-\frac{1}{2}\int_0^T \left(K_{1-H}^{-1}\int_0^\cdot \tilde{c}_H^{-1}(T-v)^{1-2H}h_\varepsilon(v)dv\right)^2(s)ds\Big].\end{aligned}$$

Using Corollary 5.2 of [9], we immediately know that $(\widetilde{B}_t^H)_{0\le t\le T}$ is an $\mathcal{F}_t^{B^H}$-fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2},1)$ under the new probability $Q(d\omega) = R_\varepsilon(T)P(d\omega)$ if $(\widetilde{B}_t^{1-H})_{0\le t\le T}$ is an $\mathcal{F}_t^{B^{1-H}}$-fractional Brownian motion with Hurst parameter $1-H$ under the new probability $Q(d\omega) = R_\varepsilon(T)P(d\omega)$.

Next, we want to show $(\widetilde{B}_t^{1-H})_{0\le t\le T}$ is an $\mathcal{F}_t^{B^{1-H}}$-fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2},1)$ under the new probability $Q(d\omega) = R_\varepsilon(T)P(d\omega)$. Due to [11], it suffices to show that $E^P R_\varepsilon(T) = 1$. Since $\int_0^\cdot \tilde{c}_H^{-1}(T-v)^{1-2H}h_\varepsilon(v)dv$ is absolutely continuous, for any $s \in [0,T]$ we have by (1) that

$$\Big(K_{1-H}^{-1}\int_0^\cdot \tilde{c}_H^{-1}(T-v)^{1-2H}h_\varepsilon(v)dv\Big)(s) = s^{\frac{1}{2}-H}I_{0+}^{H-\frac{1}{2}}s^{H-\frac{1}{2}}\tilde{c}_H^{-1}(T-s)^{1-2H}h_\varepsilon(s).$$

Hence, we have further that for $s \in [0,T]$,

(14)

$$\begin{aligned}&\Big|\Big(K_{1-H}^{-1}\int_0^\cdot \tilde{c}_H^{-1}(T-u)^{1-2H}h_\varepsilon(u)du\Big)(s)\Big|\\ =&\Big|\frac{1}{\Gamma(H-\frac{1}{2})}\tilde{c}_H^{-1}s^{\frac{1}{2}-H}\int_0^s u^{H-\frac{1}{2}}(T-u)^{1-2H}h_\varepsilon(u)(s-u)^{-\frac{3}{2}+H}du\Big|\\ \le&\frac{1}{\Gamma(H-\frac{1}{2})}\tilde{c}_H^{-1}s^{\frac{1}{2}-H}\Big\{\int_0^s \Big(u^{H-\frac{1}{2}}(s-u)^{-\frac{3}{2}+H}h_\varepsilon(u)\Big)^{\frac{\sigma}{\sigma-1}}du\Big\}^{\frac{\sigma-1}{\sigma}}\\ &\cdot\Big\{\int_0^s (T-u)^{\sigma(1-2H)}du\Big\}^{\frac{1}{\sigma}}\\ \le&\frac{\tilde{c}_H^{-1}}{\Gamma(H-\frac{1}{2})}s^{\frac{1}{2}-H}\frac{T^{1-2H+\frac{1}{\sigma}}+(T-s)^{1-2H+\frac{1}{\sigma}}}{|\sigma(1-2H)+1|^{\frac{1}{\sigma}}}\Big\{\int_0^s \Big(u^{H-\frac{1}{2}}(s-u)^{-\frac{3}{2}+H}\Big)^{\sigma'}du\Big\}^{\frac{1}{\sigma'}}\\ &\cdot\Big\{\int_0^s |h_\varepsilon(u)|^q du\Big\}^{\frac{1}{q}}\\ \le&C^{\frac{1}{2}}(H,\sigma,\sigma')s^{-\frac{3}{2}+H+\frac{1}{\sigma'}}\Big[T^{1-2H+\frac{1}{\sigma}}+(T-s)^{1-2H+\frac{1}{\sigma}}\Big]\Big\{\int_0^s |h_\varepsilon(u)|^q du\Big\}^{\frac{1}{q}},\end{aligned}$$

where

$$C(H,\sigma,\sigma') = \Big[\frac{B(-\sigma'/2 + H\sigma' + 1, -3\sigma'/2 + H\sigma' + 1)^{1/\sigma'}}{\tilde{c}_H \Gamma(H-\frac{1}{2})|\sigma(1-2H)+1|^{\frac{1}{\sigma}}}\Big]^2.$$

Substituting (12) into (14), we can further obtain that for the fixed $T > r$,

$$\begin{aligned}
&\frac{1}{C(H,\sigma,\sigma')}\int_0^T \Big|\Big(K^{-1}_{1-H}\int_0^{\cdot} \tilde{c}_H^{-1}(T-u)^{1-2H}h_\varepsilon(u)du\Big)(s)\Big|^2 ds \\
&\le \int_0^T s^{2H+\frac{2}{\sigma'}-3}\Big[T^{1-2H+\frac{1}{\sigma}} + (T-s)^{1-2H+\frac{1}{\sigma}}\Big]^2\Big\{\int_0^s |h_\varepsilon(u)|^q du\Big\}^{\frac{2}{q}} ds \\
&\le T^{\frac{2}{\sigma}+\frac{2}{\sigma'}-2H}\Big[\frac{1}{H+\frac{1}{\sigma'}-1} + 2B\Big(2H+\frac{2}{\sigma'}-2, 3-4H+\frac{2}{\sigma}\Big)\Big] \qquad (15)\\
&\quad\cdot\Big\{(1+(TK_1)^q)[(\frac{2}{T-r})^{q-1}\varepsilon^q|\eta(-r)|^q \\
&\qquad + 2^{q-1}\varepsilon^q\|\eta\|^q_{\mathcal{H}}] + 2^{q-1}(\varepsilon K_2)^q(T-r)|\eta(-r)|^q\Big\}^{2/q} \\
&=: 2\rho(T,r,H,\eta,\varepsilon).
\end{aligned}$$

Using the well-known Novikov criterion, one can have $E^P R_\varepsilon(T) = 1$. Now let us put $\varepsilon = 1$. Note that

$$P_T f(\xi) = E^Q\{f(X_T^1)\} = E^P\{R_1(T)f(X_T^\varepsilon)\} = E^P\{R_1(T)f(\eta(T)+X_T)\}$$

and so we can get the shift Harnack inequality

$$\begin{aligned}
(P_T f(\xi))^p &\le P_T f^p(\eta+\cdot)(\xi)(E^P(R_1^{\frac{p}{p-1}}(T)))^{p-1} \\
&\le P_T f^p(\eta+\cdot)(\xi)\exp\Big[\frac{p}{p-1}C(H,\sigma,\sigma')\rho(T,r,H,\eta)\Big].
\end{aligned}$$

The proof is now complete. □

Remark 3.1. By virtue of $\frac{1}{2} < H < 1$, we have $2H - \frac{3}{2} < H - \frac{1}{2}$. On the other hand, we have $\frac{\sigma}{\sigma-1} = \frac{1}{1-\frac{1}{\sigma}} < \frac{1}{1-(H-\frac{1}{2})} = \frac{2}{3-2H}$ since $\frac{1}{\sigma} < H - \frac{1}{2}$. Thus, σ, σ' in Theorem 3.2 are reasonable.

Remark 3.2. In [8], Fan established the shift Harnack inequality and the integration by parts formula for equation (2) without delay by directly using Girsanov transformations for fractional Brownian motion with Hurst parameter $\frac{1}{2} < H < 1$. However, the condition that b is Fréchet differentiable such that ∇b is bounded and Hölder continuous of order $1 - \frac{1}{2H} < \rho < 1$ is required.

Theorem 3.3. *Assume the same conditions as in Theorem* 3.2. *Moreover, if we assume that b, F is differential, and $|\nabla b| \le K_3$, $|\nabla F| \le K_4$, where K_3 and K_4 are positive constants and ∇ is the gradient operator, then we have that for any $T > r$ and $f \in C_b^1(\mathcal{L})$,*

$$\begin{aligned}
(P_T\nabla_\eta f)(\xi) = E\Big\{f(X_T^\xi)\frac{\tilde{c}_H^{-1}}{\Gamma(H-\frac{1}{2})}\int_0^T \Big\langle s^{\frac{1}{2}-H}(T-s)^{1-2H}\int_0^s u^{H-\frac{1}{2}}(s-u)^{-\frac{3}{2}+H} \\
\Big[\Lambda(u) + \nabla_{\Theta(u)}b(X^\xi(u)) + \nabla_{\Theta_u}F(X_u^\xi)\Big]du, dB_s\Big\rangle.
\end{aligned}$$

Proof. Use the same notation as in the proof of Theorem 3.2. Since ∇b and ∇F are bounded, we have

$$\frac{d}{d\varepsilon}R_\varepsilon(T)\Big|_{\varepsilon=0} = \frac{\tilde{c}_H^{-1}}{\Gamma(H-\frac{1}{2})}\int_0^T \Big\langle s^{\frac{1}{2}-H}(T-s)^{1-2H}\int_0^s u^{H-\frac{1}{2}}(s-u)^{-\frac{3}{2}+H} \Big[\Lambda(u)+\nabla_{\Theta(u)}b(X^\xi(u))+\nabla_{\Theta_u}F(X_u^\xi)\Big]du, dB_s\Big\rangle$$

which immediately implies that

(16)

$$\begin{aligned}(P_T\nabla_\eta f)(\xi) &= \lim_{\varepsilon\to 0^+}\Big[\frac{P_Tf(\cdot-\varepsilon\eta)(\xi)-E^Pf(X_T^\xi)}{-\varepsilon}\Big]\\ &= \lim_{\varepsilon\to 0^+}\Big[\frac{E^PR_\varepsilon(T)f(X_T^\varepsilon)-\varepsilon\eta)-E^Pf(X_T^\xi)}{-\varepsilon}\Big]\\ &= \lim_{\varepsilon\to 0^+}\Big[\frac{E^PR_\varepsilon(T)f(X_T^\varepsilon)-E^Pf(X_T^\xi)}{-\varepsilon}\Big]\\ &= E\Big\{f(X_T^\xi)\frac{\tilde{c}_H^{-1}}{\Gamma(H-\frac{1}{2})}\int_0^T\Big\langle s^{\frac{1}{2}-H}(T-s)^{1-2H}\int_0^s u^{H-\frac{1}{2}}(s-u)^{-\frac{3}{2}+H}\\ &\quad \Big[\Lambda(u)+\nabla_{\Theta(u)}b(X^\xi(u))+\nabla_{\Theta_u}F(X_u^\xi)\Big]du, dB_s\Big\rangle.\end{aligned}$$

The proof is now complete. □

Remark 3.3. There are many interesting applications of the shift Harnack inequality and the integration by parts formula, e.g., to estimate the density with respect to the Lebesgue measure for distributions and Markov operators, to the log shift Harnack inequality and distribution properties of the underlying transition probability.

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