

ON COPIES OF THE ABSOLUTE GALOIS GROUP IN $\text{Out } \hat{F}_2$

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ABSTRACT. In this article we consider outer Galois actions on a free profinite group of rank two, induced by the étale fundamental group of a projective line minus three points or of a pointed elliptic curve over a number field. Under mild technical assumptions their respective images uniquely determine the curves and the number fields.

1. INTRODUCTION

In 1979 G.V. Belyĭ famously proved the following result; see [3, Theorem 4]:

Theorem (Belyĭ). *Let X be a smooth projective algebraic curve defined over $\overline{\mathbb{Q}}$. Then there exists a morphism $f: X \rightarrow \mathbb{P}_{\overline{\mathbb{Q}}}^1$ of algebraic curves over $\overline{\mathbb{Q}}$ which is unramified outside $\{0, 1, \infty\} \subset \mathbb{P}^1$.*

Here we think of $\overline{\mathbb{Q}}$ as the set of all algebraic numbers in \mathbb{C} .

One of the most important consequences of this theorem is the existence of a continuous injective homomorphism

$$(1) \quad \varrho_{01\infty}: G_{\mathbb{Q}} \hookrightarrow \text{Out } \hat{F}_2,$$

where $G_{\mathbb{Q}} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is the absolute Galois group of the rational numbers, \hat{F}_2 is the profinite completion of a free group on two letters and Out denotes the outer isomorphism group. The map (1) is obtained from the short exact sequence of étale fundamental groups

$$(2) \quad 1 \rightarrow \pi_1(\mathbb{P}_{\overline{\mathbb{Q}}}^1 \setminus \{0, 1, \infty\}, *) \rightarrow \pi_1(\mathbb{P}_{\mathbb{Q}}^1 \setminus \{0, 1, \infty\}, *) \rightarrow G_{\mathbb{Q}} \rightarrow 1$$

in which the kernel can be identified with the profinite completion of

$$\pi_1^{\text{top}}(\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}, *) \simeq F_2.$$

From Belyĭ's theorem it is easy to deduce:

Corollary (Belyĭ). *The map (1) is injective.*

Note that since $G_{\mathbb{Q}}$ is compact (1) is therefore a homeomorphism onto its image.

The proof, which can be found in [3, §4], is based on the following observations: by Belyĭ's theorem every algebraic curve over $\overline{\mathbb{Q}}$ is birational to some finite étale covering of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, but such coverings correspond to conjugacy classes of open

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subgroups of \hat{F}_2 . Since $G_{\mathbb{Q}}$ operates faithfully on birationality classes of algebraic curves over $\overline{\mathbb{Q}}$, the corollary follows.

Choosing a base point $*$ defined over \mathbb{Q} we obtain a splitting of the sequence (2) and hence a lift of (1) to an injection

$$(3) \quad G_{\mathbb{Q}} \hookrightarrow \text{Aut } \hat{F}_2;$$

the most popular base point is the tangential base point $*$ $= \overrightarrow{01}$ as defined in [4]. Alexander Grothendieck urged his fellow mathematicians in [6] to study the image of (1) or (3) with the hope of arriving at a purely combinatorial description of $G_{\mathbb{Q}}$. He gave a candidate for the image, known today as the (profinite) Grothendieck–Teichmüller group $\widehat{\text{GT}} \subset \text{Aut } \hat{F}_2$ (see [12] for an overview). By construction $G_{\mathbb{Q}} \hookrightarrow \widehat{\text{GT}}$, but the other inclusion remains an open conjecture.

There are, however, still other embeddings $G_K \hookrightarrow \text{Out } \hat{F}_2$ for each number field $K \subset \mathbb{C}$. For each elliptic curve E over K we set $E^* = E \setminus \{0\}$ and obtain a short exact sequence

$$1 \rightarrow \pi_1(E_{\overline{\mathbb{Q}}}^*) \rightarrow \pi_1(E^*) \rightarrow G_K \rightarrow 1$$

analogous to (2). Choosing a basis \mathcal{B} of $H_1(E(\mathbb{C}), \mathbb{Z})$ we construct an identification $\pi_1(E_{\overline{\mathbb{Q}}}^*) \simeq \hat{F}_2$ below, and hence an injection (cf. Theorem 2 below)

$$\varrho_E = \varrho_{E, \mathcal{B}}: G_K \hookrightarrow \text{Out } \hat{F}_2.$$

We will actually require this basis to be positive, i.e., positively oriented for the intersection pairing.

Theorem A. *For $j = 1, 2$ let $K_j \subset \mathbb{C}$ be a number field, E_j an elliptic curve over K_j and \mathcal{B}_j a positive basis of $H_1(E_j(\mathbb{C}), \mathbb{Z})$. Assume that*

$$\varrho_{E_1, \mathcal{B}_1}(G_{K_1}) = \varrho_{E_2, \mathcal{B}_2}(G_{K_2})$$

as subgroups of $\text{Out } \hat{F}_2$. Then $K_1 = K_2$ and there exists an isomorphism $E_1 \simeq E_2$ over K_1 sending \mathcal{B}_1 to \mathcal{B}_2 .

It is necessary to assume that the bases are positive: let τ denote complex conjugation, let K be a non-real number field with $\tau(K) = K$ and let E be an elliptic curve over K with E not isomorphic to $\tau(E)$. Then complex conjugation defines a real diffeomorphism $E(\mathbb{C}) \rightarrow \tau(E)(\mathbb{C})$ sending each positive basis \mathcal{B} of $H_1(E(\mathbb{C}), \mathbb{Z})$ to a negative basis $\tau(\mathcal{B})$ of $H_1(\tau(E)(\mathbb{C}), \mathbb{Z})$, and $\varrho_{E, \mathcal{B}}$ and $\varrho_{\tau(E), \tau(\mathcal{B})}$ have the same image.

Outline. After recalling some preparatory material in section 2 we will prove Theorem A in section 3 and finally draw some easy consequences in section 4.

2. SOME ANABELIAN GEOMETRY

We recall some facts about étale fundamental groups of hyperbolic curves over number fields.

Definition 1. Let k be a field and Y a smooth curve over k . Let X be the smooth projective completion of Y and $S = X(\overline{k}) \setminus Y(\overline{k})$; let g be the genus of X and n the cardinality of S . Then Y is called *hyperbolic* if $\chi(Y) = 2 - 2g - n < 0$.

If $k \subseteq \mathbb{C}$, then Y is hyperbolic if and only if the universal covering space of $Y(\mathbb{C})$ is biholomorphic to the unit disc. Both \mathbb{P}^1 minus three points and an elliptic curve minus its origin are hyperbolic.

Now assume that $k = K \subset \mathbb{C}$ is a number field. By [1, XIII 4.3] the sequence

$$(4) \quad 1 \rightarrow \pi_1(Y_{\overline{\mathbb{Q}}}, *) \rightarrow \pi_1(Y, *) \rightarrow G_K \rightarrow 1$$

induced by the ‘fibration’ $Y_{\overline{\mathbb{Q}}} \rightarrow Y \rightarrow \text{Spec } K$ is exact. By the usual group-theoretic constructions this sequence defines a homomorphism

$$(5) \quad G_K \rightarrow \text{Out } \pi_1(Y_{\overline{\mathbb{Q}}}, *),$$

and the group $\pi_1(Y_{\overline{\mathbb{Q}}}, *)$ is the profinite completion of $\pi_1^{\text{top}}(Y(\mathbb{C}), *)$, which is either a free group (in the affine case) or can be presented as

$$\langle a_1, \dots, a_g, b_1, \dots, b_g \mid [a_1, b_1] \cdots [a_g, b_g] = 1 \rangle.$$

Theorem 2. *The homomorphism (5) is injective.*

Proof. This is [7, Theorem C]. □

We also note for later use that the sequence (4) can be reconstructed from (5):

Lemma 3. *Let G be a profinite group, and let π be a profinite group which is isomorphic to the étale fundamental group of a hyperbolic curve over \mathbb{C} . Let $\varphi: G \rightarrow \text{Out } \pi$ be a continuous group homomorphism. Then there exists a short exact sequence of profinite groups*

$$1 \rightarrow \pi \rightarrow H \rightarrow G \rightarrow 1$$

inducing φ , and it is unique in the sense that if another such sequence is given with H' in the middle, then there exists an isomorphism $H' \rightarrow H$ such that the diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & \pi & \longrightarrow & H & \longrightarrow & G \longrightarrow 1 \\ & & \parallel & & \downarrow \simeq & & \parallel \\ 1 & \longrightarrow & \pi & \longrightarrow & H' & \longrightarrow & G \longrightarrow 1 \end{array}$$

commutes.

Proof. Let $\mathcal{Z}(\pi)$ denote the centre of π . The obstruction to the existence of such a sequence is a class in $H^3(G, \mathcal{Z}(\pi))$ by [9, Chapter IV, Theorem 8.7], but since $\mathcal{Z}(\pi)$ is trivial by [2, Proposition 18], the obstruction is automatically zero. Given the existence of one such sequence, the isomorphism classes of all such sequences are in bijection with $H^2(G, \mathcal{Z}(\pi)) = 0$ by [9, Chapter IV, Theorem 8.8]. □

We may safely ignore basepoints for the following reason: if $y, y' \in Y(\overline{\mathbb{Q}})$, then there exists an isomorphism $\pi_1(Y_{\overline{\mathbb{Q}}}, y) \simeq \pi_1(Y_{\overline{\mathbb{Q}}}, y')$, canonical up to inner automorphisms. Hence the outer automorphism groups of both are canonically identified. Furthermore, since both basepoints map to the same tautological base point of $\text{Spec } k$, the whole sequence (4) is changed only by inner automorphisms of the kernel when basepoints are changed within $Y(\overline{\mathbb{Q}})$. So we drop basepoints from the notation in the sequel.

This has the more general consequence that we sometimes need to deal with *outer homomorphism classes* rather than homomorphisms in the usual sense. If G and H are groups (discrete or profinite), then an outer homomorphism class $f: G \dashrightarrow H$ is an H -conjugacy class of group homomorphisms $G \rightarrow H$. The étale

fundamental group is then a functor from the category of connected schemes to the category of profinite groups with outer homomorphism classes.

If X and Y are hyperbolic curves over a number field K and $f: X \rightarrow Y$ is an isomorphism, we obtain a commutative diagram

$$(6) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \pi_1(X_{\overline{\mathbb{Q}}}) & \longrightarrow & \pi_1(X) & \longrightarrow & G_K \longrightarrow 1 \\ & & \downarrow \simeq & & \downarrow \simeq & & \parallel \\ 1 & \longrightarrow & \pi_1(Y_{\overline{\mathbb{Q}}}) & \longrightarrow & \pi_1(Y) & \longrightarrow & G_K \longrightarrow 1. \end{array}$$

Theorem 4. *Let K be a number field and X, Y hyperbolic curves over K . Let $f: \pi_1(X) \rightarrow \pi_1(Y)$ be an isomorphism of fundamental groups commuting with the projections to G_K . Then f is induced by a unique isomorphism of K -varieties $X \rightarrow Y$, and can be inserted into a commutative diagram of the form (6).*

Proof. This holds more generally for K finitely generated over \mathbb{Q} . It was conjectured by Grothendieck in [5] and proved in the affine case by Tamagawa in [13, Theorem 0.3] and in the projective case by Mochizuki in [11]. \square

3. THE GALOIS ACTIONS

Let F_2 be the free group on two letters and let \hat{F}_2 be its profinite completion. Consider the following objects:

- (i) a number field $K \subset \mathbb{C}$,
- (ii) an elliptic curve E over K and
- (iii) a basis \mathcal{B} of the homology group $H_1(E(\mathbb{C}), \mathbb{Z})$.

Let $E^* = E \setminus \{0\}$; then $\pi_1^{\text{top}}(E^*(\mathbb{C}))$ is a free group of rank two whose maximal abelian quotient can be identified with $H_1(E(\mathbb{C}), \mathbb{Z})$. By the following lemma, the group isomorphism

$$\mathbb{Z}^2 \rightarrow H_1(E(\mathbb{C}), \mathbb{Z}), \quad (m, n) \mapsto mx + ny \text{ where } \mathcal{B} = (x, y),$$

can be lifted uniquely to an outer isomorphism class

$$(7) \quad F_2 \dashrightarrow \pi_1^{\text{top}}(E^*(\mathbb{C})).$$

Lemma 5. *Let F and G be free groups of rank two, and let $f: F^{\text{ab}} \rightarrow G^{\text{ab}}$ be an isomorphism between their maximal abelian quotients. Then there exists an isomorphism $\tilde{f}: F \rightarrow G$ inducing f ; it is uniquely determined by f up to inner automorphisms of F .*

Proof. It is enough to prove this lemma in the case where $F = G = F_2$; but this is a reformulation of the well-known result that the natural map

$$\text{Out } F_2 \rightarrow \text{Aut}(\mathbb{Z}^2) = \text{GL}(2, \mathbb{Z})$$

is an isomorphism. \square

Since the profinite completion of $\pi_1^{\text{top}}(E^*(\mathbb{C}))$ can be identified with $\pi_1(E_{\overline{\mathbb{Q}}}^*)$ we obtain an outer isomorphism class

$$(8) \quad \iota_{\mathcal{B}}: \hat{F}_2 \dashrightarrow \pi_1(E_{\overline{\mathbb{Q}}}^*).$$

Hence pulling back the Galois action on $\pi_1(E_{\overline{\mathbb{Q}}}^*)$ along (8) defines an injective homomorphism

$$(9) \quad \varrho_{E, \mathcal{B}}: G_K \hookrightarrow \text{Out } \hat{F}_2.$$

Lemma 6. *Let E be an elliptic curve over $\overline{\mathbb{Q}}$ and let $\sigma \in G_{\overline{\mathbb{Q}}}$. Let*

$$f: \pi_1(E^*) \dashrightarrow \pi_1(\sigma(E^*))$$

be an outer isomorphism class of profinite groups which can be obtained in each of the following ways:

- (i) *it is the map of étale fundamental groups induced via functoriality by the tautological isomorphism of schemes $t: E^* \rightarrow \sigma(E^*)$;*
- (ii) *it is the profinite completion of an outer isomorphism class*

$$\pi_1^{\text{top}}(E^*(\mathbb{C})) \dashrightarrow \pi_1^{\text{top}}(\sigma(E^*)(\mathbb{C}))$$

induced by an orientation-preserving isomorphism of real Lie groups

$$h: E(\mathbb{C}) \rightarrow \sigma(E)(\mathbb{C}).$$

Then σ is the identity, and so is the fundamental group isomorphism in (ii).

Proof. Let $E[2] \subset E$ be the 2-torsion subgroup and set $E^\dagger = E \setminus E[2]$. The multiplication-by-2 map is a normal étale covering $E^\dagger \rightarrow E^*$, therefore $\pi_1(E^\dagger)$ is a normal open subgroup of $\pi_1(E^*)$. From assumption (i) we see that f maps $\pi_1(E^\dagger)$ isomorphically to $\pi_1(\sigma(E^\dagger))$. Similarly h maps $E[2]$ to $\sigma(E)[2]$, hence (i) and (ii) hold with every $*$ replaced by \dagger .

The quotient of E^\dagger by the identification $x \sim -x$ is isomorphic over $\overline{\mathbb{Q}}$ to a scheme of the form $\mathbb{P}^1 \setminus \{0, 1, \infty, \lambda\}$, and we obtain a commutative diagram of schemes

$$(10) \quad \begin{array}{ccccc} E^\dagger & \xrightarrow{\wp} & \mathbb{P}_{\overline{\mathbb{Q}}}^1 \setminus \{0, 1, \infty, \lambda\} & \xrightarrow{\iota} & \mathbb{P}_{\overline{\mathbb{Q}}}^1 \setminus \{0, 1, \infty\} \\ \downarrow t & & & & \downarrow t \\ \sigma(E^\dagger) & \xrightarrow{\wp} & \mathbb{P}_{\overline{\mathbb{Q}}}^1 \setminus \{0, 1, \infty, \sigma(\lambda)\} & \xrightarrow{\iota} & \mathbb{P}_{\overline{\mathbb{Q}}}^1 \setminus \{0, 1, \infty\} \end{array}$$

where the horizontal maps are morphisms of $\overline{\mathbb{Q}}$ -schemes and all vertical maps are base change morphisms along $\sigma: \overline{\mathbb{Q}} \rightarrow \overline{\mathbb{Q}}$. (The maps \wp are not necessarily Weierstraß \wp -functions, but up to Möbius transformations on \mathbb{P}^1 they are, whence our notation.)

There is a very similar commutative diagram of topological spaces:

$$(11) \quad \begin{array}{ccccc} E^\dagger(\mathbb{C}) & \xrightarrow{\wp} & \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty, \lambda\} & \xrightarrow{\iota} & \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\} \\ \downarrow h & & & & \downarrow H \\ \sigma(E^\dagger)(\mathbb{C}) & \xrightarrow{\wp} & \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty, \sigma(\lambda)\} & \xrightarrow{\iota} & \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\} \end{array}$$

where the horizontal maps are holomorphic maps between Riemann surfaces and the vertical maps are orientation-preserving homeomorphisms.

We claim that the horizontal compositions in (10) and (11), i.e., the composite maps of the form

$$(12) \quad E^\dagger \xrightarrow{\wp} \mathbb{P}^1 \setminus \{0, 1, \infty, \lambda\} \xrightarrow{\iota} \mathbb{P}^1 \setminus \{0, 1, \infty\},$$

induce surjections on fundamental groups. This can be checked in the topological case; the maps induced by (12) on topological fundamental groups have the form

$$(13) \quad \ker s \rightarrow \langle a_1, a_2, a_3, a_4 \mid a_1 a_2 a_3 a_4 = 1 \rangle \rightarrow \langle a_1, a_2, a_3 \mid a_1 a_2 a_3 = 1 \rangle$$

where the second map is given by $a_4 \mapsto 1$, and

$$s: \langle a_1, a_2, a_3, a_4 \mid a_1 a_2 a_3 a_4 = 1 \rangle \rightarrow \mathbb{Z}/2\mathbb{Z}, \quad a_i \mapsto 1 \bmod 2.$$

Hence $a_j a_4 \mapsto a_j$ for $j = 1, 2, 3$ under (13), and therefore (13) is surjective.

From this we deduce that the two diagrams (10) and (11) induce the same commutative diagrams of outer homomorphism classes between the étale fundamental groups: the groups are clearly the same, and so are the homomorphisms induced by the horizontal maps and by the vertical maps on the left. But since the maps $\iota \circ \wp$ induce surjections on fundamental groups, the vertical maps on the right also have to induce the same homomorphisms.

In particular the base change map t induced by $\sigma: \overline{\mathbb{Q}} \rightarrow \overline{\mathbb{Q}}$ and the orientation-preserving homeomorphism H define the same element in $\text{Out } \pi_1(\mathbb{P}_{\overline{\mathbb{Q}}}^1 \setminus \{0, 1, \infty\})$. But H is homotopic to the identity, hence this element has to be trivial; and by Theorem 2 for $Y = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ the automorphism σ has to be trivial, too. \square

We note a result closely related to Lemma 6; see [10]:

Theorem 7 (Matsumoto–Tamagawa). *Let E be an elliptic curve defined over a number field $K \subset \mathbb{C}$. Then the images of the outer Galois representation*

$$\text{Gal}(\overline{\mathbb{Q}}/K) \rightarrow \text{Out } \pi_1(E_{\overline{\mathbb{Q}}}^*)$$

and the profinite closure of the topological monodromy

$$\widehat{\text{SL}(2, \mathbb{Z})} \rightarrow \text{Out } \pi_1(E_{\mathbb{C}}^*) = \text{Out } \pi_1(E_{\overline{\mathbb{Q}}}^*)$$

intersect trivially.

We also need one more result on isomorphisms, this time between Galois groups. Let K and L be number fields in \mathbb{C} , and assume that $\sigma \in G_{\mathbb{Q}}$ satisfies $\sigma(K) = L$. Then we can define a group isomorphism

$$\Phi_{\sigma}: G_K \rightarrow G_L, \quad \tau \mapsto \sigma \tau \sigma^{-1}.$$

Theorem 8 (Neukirch–Uchida). *Let $K, L \subset \mathbb{C}$ be number fields and let $\Phi: G_K \rightarrow G_L$ be a continuous group isomorphism. Then there exists a unique $\sigma \in G_{\mathbb{Q}}$ with $\sigma(K) = L$ and $\Phi = \Phi_{\sigma}$.*

For the proof see [14].

Proof of Theorem A. The bases \mathcal{B}_j of $H_1(E_j(\mathbb{C}), \mathbb{Z})$ define an orientation-preserving isomorphism between these two cohomology groups, hence an orientation-preserving isomorphism of real Lie groups $h: E_1(\mathbb{C}) \rightarrow E_2(\mathbb{C})$ and an isomorphism of profinite fundamental groups

$$h_* = \iota_{\mathcal{B}_2}^{-1} \circ \iota_{\mathcal{B}_1}: \pi_1(E_{1, \overline{\mathbb{Q}}}^*) \rightarrow \pi_1(E_{2, \overline{\mathbb{Q}}}^*).$$

Since the representations ϱ_{E_j} are injective there is a unique isomorphism of profinite groups $\Phi: G_{K_1} \rightarrow G_{K_2}$ such that $\varrho_{E_1} = \varrho_{E_2} \circ \Phi$. By Theorem 8 this has to be of the form Φ_{σ} for a unique isomorphism $\sigma \in G_{\mathbb{Q}}$ with $\sigma(K_1) = K_2$. We shall construct an isomorphism $\sigma(E_1) \rightarrow E_2$ of elliptic curves over K_2 .

Consider the short exact homotopy sequences for the three varieties $\sigma(E_1)$, E_1 , E_2 over their respective base fields; they can be completed to the following commutative diagram:

$$\begin{array}{ccccccc}
 1 & \longrightarrow & \pi_1(\sigma(\overline{E}_1^*)) & \longrightarrow & \pi_1(\sigma(E_1^*)) & \longrightarrow & G_{K_2} \longrightarrow 1 \\
 & & m_* \downarrow \simeq & & m_* \downarrow \simeq & & \Phi_\sigma^{-1} \downarrow \\
 1 & \longrightarrow & \pi_1(\overline{E}_1^*) & \longrightarrow & \pi_1(E_1^*) & \longrightarrow & G_{K_1} \longrightarrow 1 \\
 & & h_* \downarrow \simeq & & & & \Phi_\sigma \downarrow \\
 1 & \longrightarrow & \pi_1(\overline{E}_2^*) & \longrightarrow & \pi_1(E_2^*) & \longrightarrow & G_{K_2} \longrightarrow 1.
 \end{array}$$

Here the lower rectangle commutes trivially by exactness of the rows, and the upper two squares commute by functoriality of the fundamental group. From Lemma 3 we obtain an isomorphism $F: \pi_1(E_1^*) \rightarrow \pi_1(E_2^*)$ that makes the resulting diagram commute:

$$\begin{array}{ccccccc}
 1 & \longrightarrow & \pi_1(\sigma(E_{1,\overline{\mathbb{Q}}}^*)) & \longrightarrow & \pi_1(\sigma(E_1^*)) & \longrightarrow & G_{K_2} \longrightarrow 1 \\
 & & m_* \downarrow \simeq & & m_* \downarrow \simeq & & \Phi_\sigma^{-1} \downarrow \\
 1 & \longrightarrow & \pi_1(E_{1,\overline{\mathbb{Q}}}^*) & \longrightarrow & \pi_1(E_1^*) & \longrightarrow & G_{K_1} \longrightarrow 1 \\
 & & h_* \downarrow \simeq & & F \downarrow & & \Phi_\sigma \downarrow \\
 1 & \longrightarrow & \pi_1(E_{2,\overline{\mathbb{Q}}}^*) & \longrightarrow & \pi_1(E_2^*) & \longrightarrow & G_{K_2} \longrightarrow 1.
 \end{array}$$

By Theorem 4 the group isomorphism $F \circ m_*: \pi_1(\sigma(E_1^*)) \rightarrow \pi_1(E_2^*)$ must be induced by a unique isomorphism $g: \sigma(E_1) \rightarrow E_2$ of K_2 -schemes. But this means that

$$m_*: \pi_1(\sigma(E_{1,\overline{\mathbb{Q}}}^*)) \rightarrow \pi_1(E_{1,\overline{\mathbb{Q}}}^*)$$

is induced by the orientation-preserving homeomorphism

$$h^{-1} \circ g^{\text{an}};$$

by Lemma 6 we find that σ must be the identity, so $K_1 = K_2$ and g is the desired isomorphism. \square

4. CONCLUDING REMARKS

From Theorem A we can easily deduce several analogous statements. Recall that two subgroups H', H'' of a group G are called directly commensurable if $H' \cap H''$ has finite index both in H' and in H'' ; they are called widely commensurable if $gH'g^{-1}$ and H'' are directly commensurable for some $g \in G$.

Corollary 9. *For $j = 1, 2$ let $K_j \subset \mathbb{C}$ be a number field, E_j an elliptic curve over K_j and \mathcal{B}_j a positive basis of $H_1(E_j(\mathbb{C}), \mathbb{Z})$. Let I_j be the image of $\varrho_{E_j, \mathcal{B}_j}: G_{K_j} \rightarrow \text{Out } \hat{F}_2$.*

- (i) $I_1 = I_2$ if and only if $K_1 = K_2$ and there exists an isomorphism $E_1 \simeq E_2$ over K_1 sending \mathcal{B}_1 to \mathcal{B}_2 .
- (ii) I_1 and I_2 are conjugate in $\text{Out } \hat{F}_2$ if and only if there exists a field isomorphism $\sigma: K_1 \rightarrow K_2$ such that $\sigma(E_1) \simeq E_2$ as elliptic curves over K_2 .

- (iii) I_1 and I_2 are directly commensurable if and only if there exists an isomorphism $E_{1,\mathbb{C}} \rightarrow E_{2,\mathbb{C}}$ sending \mathcal{B}_1 to \mathcal{B}_2 .
- (iv) I_1 and I_2 are widely commensurable if and only if $j(E_1)$ and $j(E_2)$ lie in the same $G_{\mathbb{Q}}$ -orbit in $\overline{\mathbb{Q}}$, where j denotes the j -invariant from the classical theory of elliptic curves.

Proof. (i) is Theorem A and (ii) follows easily from Theorems 4 and 8. For (iii) we can find an open subgroup G_{L_j} of each G_{K_j} such that these two subgroups have the same image; we can then apply (i) to $E_i \otimes_{K_i} L_i$. Vice versa any isomorphism between two elliptic curves over \mathbb{C} that admit models over number fields must already be defined over some number field. (iv) follows similarly from (ii). \square

Corollary 10. *Let K be a number field, E an elliptic curve over K and \mathcal{B} a basis of $H_1(E_j(\mathbb{C}), \mathbb{Z})$. Then $\varrho_{E,\mathcal{B}}(G_K)$ and $\varrho_{01\infty}(G_{\mathbb{Q}})$ are not widely commensurable in $\text{Out } \hat{F}_2$.*

Proof. Assume they were widely commensurable; after enlarging the fields of definition K and \mathbb{Q} to some suitable number fields L_1, L_2 the two Galois images would actually be conjugate in $\text{Out } \hat{F}_2$. As in the proof of Theorem A we would obtain an isomorphism $\sigma \in G_{\mathbb{Q}}$ with $\sigma(L_1) = L_2$ and a commutative diagram

$$\begin{array}{ccccccc}
 1 & \longrightarrow & \pi_1(\sigma(E_{\mathbb{Q}}^*)) & \longrightarrow & \pi_1(\sigma(E)_{L_2}) & \longrightarrow & G_{L_2} \longrightarrow 1 \\
 & & \downarrow \simeq & & \downarrow \simeq & & \parallel \\
 1 & \longrightarrow & \pi_1(\mathbb{P}_{\mathbb{Q}}^1 \setminus \{0, 1, \infty\}) & \longrightarrow & \pi_1(\mathbb{P}_{L_2}^1 \setminus \{0, 1, \infty\}) & \longrightarrow & G_{L_2} \longrightarrow 1,
 \end{array}$$

hence by Theorem 4 an isomorphism $\sigma(E_{\mathbb{Q}}^*) \rightarrow \mathbb{P}_{\mathbb{Q}}^1 \setminus \{0, 1, \infty\}$, which is absurd. \square

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