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HANKEL OPERATORS, INVARIANT SUBSPACES, AND CYCLIC VECTORS IN THE DRURY-ARVESON SPACE

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Abstract. We show that every nonzero invariant subspace of the Drury-Arveson space H^2_d of the unit ball of $\mathbb{C}^d$ is an intersection of kernels of little Hankel operators. We use this result to show that if f and $1/f \in H^2_d$, then f is cyclic in H^2_d.

1. Introduction

It follows from Beurling's characterization of the invariant subspaces of the unilateral shift acting on $H^2 = H^2(\partial\mathbb{D})$ of the unit circle that every such invariant subspace equals the null space of a Hankel operator. Since the symbols of bounded Hankel operators on H^2 are given by BMOA functions, this observation ties the operator theory surrounding the unilateral shift to the H^1-BMO duality and its connection with Carleson measures. In this paper we will exhibit an analogy of this for the d-shift and the Drury-Arveson space of the unit ball $\mathbb{B}_d = \{z \in \mathbb{C}^d : |z| < 1\}$, $d \geq 1$.

For some information about the significance of the Drury-Arveson space, we direct the reader's attention to the survey article [15]. The Drury-Arveson space H^2_d is the space of analytic functions on $\mathbb{B}_d$ with reproducing kernel $k_w(z) = \frac{1}{1-\langle z,w\rangle}$, where $\langle z, w\rangle = \sum_{i=1}^d z_i \overline{w_i}$. The operator tuple $M_z = (M_{z_1}, \ldots, M_{z_d})$ acting on H^2_d is called the d-shift.

Let $Rf = \sum_{i=1}^d z_i \frac{\partial f}{\partial z_i}$ denote the radial derivative of a holomorphic function f. It is well known and easy to check that an analytic function f on $\mathbb{B}_d$ is in H^2_d if and only if $\int_{\mathbb{B}_d} |R^n f|^2 (1-|z|^2)^{2n-d} dV < \infty$ for some (or equivalently for all) $n > (d-1)/2$. Here we have used dV to denote normalized Lebesgue measure on $\mathbb{B}_d$. In Section 2 of this paper, we provide more detail about the norm on H^2_d and its connection to the norms of weighted Bergman spaces.

We will write

$$\mathcal{M}_d = \{\varphi \in \mathrm{Hol}(\mathbb{B}_d) : \varphi f \in H^2_d \text{ for all } f \in H^2_d\}$$

for the multiplier algebra of H^2_d. It was shown in [12] that

$$\mathcal{M}_d = H^\infty(\mathbb{B}_d) \cap \mathcal{C}H^2_d,$$

where $H^\infty(\mathbb{B}_d)$ denotes the algebra of bounded analytic functions on $\mathbb{B}_d$ and $\mathcal{C}H^2_d$ is the space of functions $b \in H^2_d$ such that $|R^n b|^2 (1-|z|^2)^{2n-d} dV$ is a Carleson measure for H^2_d for some (or equivalently for all) $n > (d-1)/2$. Recall that a

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positive measure μ on $\mathbb{B}_d$ is called a Carleson measure for a Hilbert function space $\mathcal{H}$ if there is a $C > 0$ such that $\int_{\mathbb{B}_d} |f|^2 d\mu \le C\|f\|^2$ for all $f \in \mathcal{H}$; see Section 3 and Lemmas 3.1 and 3.2 for more information about the space $\mathcal{C}H^2_d$.

The results of this paper are based on the theory of Hankel operators which map $\mathcal{H}$ into $\overline{\mathcal{H}}$. Here $\mathcal{H}$ denotes a Hilbert space of analytic functions on $\mathbb{B}_d$ such that $\mathrm{Hol}(\overline{\mathbb{B}_d})$ is densely contained in $\mathcal{H}$, and we have written $\overline{\mathcal{H}} = \{\overline{f} : f \in \mathcal{H}\}$ for the space of complex conjugates of $\mathcal{H}$. The space $\overline{\mathcal{H}}$ is a Hilbert space with inner product $\langle \overline{f}, \overline{g}\rangle_{\overline{\mathcal{H}}} = \langle g, f\rangle_{\mathcal{H}}$, $f, g \in \mathcal{H}$.

As in [2] or [14] we define the space of Hankel symbols

$$\mathcal{X}(\mathcal{H}) = \{b \in \mathcal{H} :\ \exists\, C > 0\ |\langle \varphi\psi, b\rangle| \le C\|\varphi\|\|\psi\|\ \forall \varphi, \psi \in \mathrm{Hol}(\overline{\mathbb{B}_d})\}.$$

Note that for every $b \in \mathcal{X}(\mathcal{H})$ the map $(\varphi, \overline{\psi}) \to \langle \varphi\psi, b\rangle$ extends to be a bounded sesquilinear form on $\mathcal{H} \times \overline{\mathcal{H}}$. Thus with each $b \in \mathcal{X}(\mathcal{H})$ we may associate the Hankel operator $H_b \in B(\mathcal{H}, \overline{\mathcal{H}})$,

$$\langle H_b\varphi, \overline{\psi}\rangle_{\overline{\mathcal{H}}} = \langle \varphi\psi, b\rangle_{\mathcal{H}}, \quad \varphi, \psi \in \mathrm{Hol}(\overline{\mathbb{B}_d}).$$

If $\mathcal{H} = H^2(\partial\mathbb{D})$, then this definition of Hankel operator differs by a rank 1 operator from the common definition as an operator $H^2(\partial\mathbb{D}) \to H^2(\partial\mathbb{D})^\perp \subseteq L^2(\partial\mathbb{D})$. For the Bergman space $L^2_a(\mathbb{B}_d) = \{f \in \mathrm{Hol}(\mathbb{B}_d) : \int_{\mathbb{B}_d} |f|^2 dV < \infty\}$, this definition coincides with what is typically referred to as the little Hankel operator.

It is known in many cases that Carleson measures can be used to describe $\mathcal{X}(\mathcal{H})$. In particular, this is known to be true for $H^2(\partial\mathbb{B}_d)$, $L^2_a(\mathbb{B}_d)$, and the Dirichlet space of one variable, $D = \{f \in \mathrm{Hol}(\mathbb{D}) : f' \in L^2_a(\mathbb{D})\}$. In fact, $\mathcal{X}(H^2(\partial\mathbb{B}_d)) = BMOA(\mathbb{B}_d)$ and this equals

$$\{b \in H^2(\partial\mathbb{B}_d) : |Rb|^2(1 - |z|^2)dV \text{ is a Carleson measure for } H^2(\partial\mathbb{B}_d)\}$$

(see [6] or [19], Theorem 5.14). Similarly, $\mathcal{X}(L^2_a(\mathbb{B}_d))$ is the Bloch space

$$\mathcal{B} = \left\{ b \in \mathrm{Hol}(\mathbb{B}_d) \mid \sup_{z \in \mathbb{B}_d} |Rb(z)|(1 - |z|^2) < \infty \right\}$$

(see, e.g., Corollary 1 of [17] together with Theorem 3.5 of [19]), and one easily checks that this coincides with

$$\{b \in L^2_a(\mathbb{B}_d) : |Rb|^2(1 - |z|^2)^2 dV \text{ is a Carleson measure for } L^2_a(\mathbb{B}_d)\}.$$

Furthermore, in [2] it was shown that for the Dirichlet space

$$\mathcal{X}(D) = \{b \in D : |b'|^2 dA \text{ is a Carleson measure for } D\}.$$

For the Drury-Arveson space we will prove the following:

Theorem 1.1. *$\mathcal{C}H^2_d \subseteq \mathcal{X}(H^2_d) \subseteq \mathcal{B}$.*

This theorem implies that $\mathcal{M}_d \subseteq \mathcal{X}(H^2_d)$, and it is this observation that will be important for the rest of the paper.

We use $\mathrm{Lat}(M_z, H^2_d)$ to denote the lattice of invariant subspaces of the d-shift. One checks that for $i = 1, \ldots, d$ one has

$$\langle H_b(z_i f), \overline{\psi}\rangle_{\overline{\mathcal{H}}} = \langle H_b f, \overline{z_i\psi}\rangle_{\overline{\mathcal{H}}} \text{ for all } f \in \mathcal{H} \text{ and } \psi \in \mathrm{Hol}(\overline{\mathbb{B}_d}).$$

This implies for each $b \in \mathcal{X}(H^2_d)$ that $\ker H_b \in \mathrm{Lat}(M_z, H^2_d)$. Moreover, we have the following theorem.

Theorem 1.2. *If* $(0) \neq \mathcal{M} \in \mathrm{Lat}(M_z, H^2_d)$*, then there are* $\{b_n\}_{n\geq 0} \subseteq \mathcal{X}(H^2_d)$ *such that*

$$\mathcal{M} = \bigcap_{n\geq 0} \ker H_{b_n}.$$

We mentioned before that it follows from Beurling's theorem that every invariant subspace of the unilateral shift (M_z, H^2) equals the kernel of a bounded Hankel operator. Similarly, it was shown in [10] that every invariant subspace $\mathcal{M}$ of the Dirichlet shift (M_z, D) satisfies $\mathcal{M} = \ker H_b$ for some $b \in \mathcal{X}(D)$. For the Bergman space no direct analog of such a theorem can hold (see [18]). We will present an example showing that if $d > 1$ one cannot expect to represent every invariant subspace as the null space of a single Hankel operator (see Example 4.3).

In the later parts of our paper, we will apply Theorem 1.2 together with the insight from [6] that $\mathcal{X}(\mathcal{H})$ is the dual of the space of weak products of $\mathcal{H}$. For a space $\mathcal{H}$ of analytic functions on $\mathbb{B}_d$, the weak product is defined as

$$\mathcal{H} \odot \mathcal{H} = \left\{ \sum_{i=1}^{\infty} f_i g_i : f_i, g_i \in \mathcal{H}, \sum_{i=1}^{\infty} \|f_i\|\|g_i\| < \infty \right\}.$$

A norm on $\mathcal{H} \odot \mathcal{H}$ is given by

$$\|h\|_* = \inf \left\{ \sum_{i=1}^{\infty} \|f_i\|\|g_i\| : h = \sum_{i=1}^{\infty} f_i g_i \right\}.$$

It is known, for example, that $H^2(\partial\mathbb{B}_d) \odot H^2(\partial\mathbb{B}_d) = H^1(\partial\mathbb{B}_d)$ and $L^2_a(\mathbb{B}_d) \odot L^2_a(\mathbb{B}_d) = L^1_a(\mathbb{B}_d)$ (see [6]). We refer the reader to [6] and [2] for details and further motivation for weak products.

For the Drury-Arveson space we don't have a natural candidate for what the weak product should be, thus we just define

$$H^1_d = H^2_d \odot H^2_d.$$

Note that since $1 \in H^2_d$ we have $H^2_d \subseteq H^1_d$. Furthermore, we have that $(H^1_d)^* = \mathcal{X}(H^2_d)$, where the duality is given by $L_b(h) = \langle h, b\rangle_{H^2_d}$ for $b \in \mathcal{X}(H^2_d)$ and $h = \sum_{i=1}^n \varphi_i \psi_i \in \mathrm{Hol}(\overline{\mathbb{B}_d}) \subseteq H^1_d$ (see [14], Theorem 1.3).

If $\mathcal{Y}$ is a Banach space of analytic functions, then a function f is called cyclic in $\mathcal{Y}$ if the polynomial multiples of f are dense in $\mathcal{Y}$. If the polynomials are dense in $\mathcal{Y}$, then a function is cyclic if and only if there is a sequence $\{p_n\}$ of polynomials such that $p_n f \to 1$. It is thus obvious from the continuous inclusion $H^2_d \subseteq H^1_d$ that any $f \in H^2_d$ which is cyclic in H^2_d must also be cyclic in H^1_d. The following corollary implies that $f \in H^2_d$ is cyclic in H^2_d if and only if it is cyclic in H^1_d. For clarity, if $S \subseteq H^2_d$, then we write $\mathrm{clos}_{H^1_d} S$ for the closure of S in H^1_d.

Corollary 1.3. *Let* $\mathcal{M} \in \mathrm{Lat}(M_z, H^2_d)$*. Then*

$$\mathcal{M} = H^2_d \cap \mathrm{clos}_{H^1_d} \mathcal{M}.$$

We note that this is analogous to Theorem 1.2 of [10], where the corollary is proved for the Dirichlet space.

For $H^p(\partial\mathbb{D})$ with $1 \leq p < \infty$, the cyclic functions are the outer functions. For other spaces, such as the Bergman space $L^2_a(\mathbb{D})$ or the Dirichlet space D, it is an open problem to characterize cyclicity. For the Drury-Arveson space, we prove the following theorem.

Theorem 1.4. *If f, g and $fg \in H^2_d$, then fg is cyclic in H^2_d if and only if both f and g are cyclic in H^2_d.*

In particular, it follows that if f and $1/f \in H^2_d$, then f is cyclic in H^2_d. It was shown by Borichev and Hedenmalm in [4] that there is $f \in L^2_a(\mathbb{D})$ such that $1/f \in L^2_a(\mathbb{D})$, but f is not cyclic in $L^2_a(\mathbb{D})$. We will provide details in Section 4, but it is easy to see that this result implies Theorem 1.4 cannot hold for $H^2(\partial\mathbb{B}_2)$. For the Dirichlet space, the analogue of Theorem 1.4 was proved in [13] by use of cut-off functions and a formula of Carleson for the Dirichlet integral of an outer function. Our current proof also reproves the Dirichlet space result and it avoids all those technicalities.

By considering the first radial derivative, it is fairly easy to see that for $d \leq 3$, whenever $f \in H^2_d$ such that $1/f$ is bounded in $\mathbb{B}_d$, then $1/f \in H^2_d$. Thus such an f must be cyclic in H^2_d. For $d \geq 4$ we were able to show such a result only under the extra hypothesis that f be in the Bloch space $\mathcal{B}$.

Theorem 1.5. *If $f \in H^2_d \cap \mathcal{B}$, and if there is $c > 0$ such that $|f(z)| \geq c$ for all $z \in \mathbb{B}_d$, then $1/f \in H^2_d$ and f is cyclic in H^2_d.*

The Corona theorem is known to hold for $\mathcal{M}_d$ (see [7]), thus any multiplier of H^2_d that is bounded below must be cyclic. The inclusions $\mathcal{M}_d \subseteq H^\infty(\mathbb{B}_d) \subseteq \mathcal{B}$ show that Theorem 1.5 generalizes this fact. In Theorem 5.4, we prove another result that can be considered a generalization of the one function Corona theorem (and the result is verified for spaces that include cases for which the Corona theorem is not known to hold).

In order to prove our results it is convenient to consider a one-parameter family of Hilbert spaces H_γ for real $\gamma > 0$. We define $\mathcal{H}_\gamma$ to be the Hilbert space of analytic functions in $\mathbb{B}_d$ with reproducing kernel $k^\gamma_\lambda(z) = \frac{1}{(1-\langle z,\lambda\rangle)^\gamma}$. As we will explain (and is well known), this family includes certain weighted Bergman and Besov spaces. We have chosen the somewhat nonstandard notation for the parameter because of our use of reproducing kernel techniques. In this paper our emphasis is on the Drury-Arveson space, but we note that for $0 < \gamma \leq 1$ the reproducing kernels are complete Nevanlinna-Pick kernels and all of our results hold in that setting as well.

2. Equivalence of bilinear forms

The family of spaces H_γ defined above includes several well-studied spaces such as $\mathcal{H}_1 = H^2_d$, $\mathcal{H}_d = H^2(\partial\mathbb{B}_d)$, and $\mathcal{H}_{d+1} = L^2_a(\mathbb{B}_d)$. Furthermore, for $\gamma > d$, the norm on $\mathcal{H}_\gamma$ can be expressed as

$$\|f\|^2_\gamma = c_\gamma \int_{|z|<1} |f(z)|^2(1-|z|^2)^{\gamma-d-1} dV(z), c_\gamma = \frac{(\gamma-1)\cdots(\gamma-d)}{d!}$$

(see [19]). In particular, we see that for $\gamma > d$, the space H_γ is a weighted Bergman space.

We will need to make some precise calculations with the inner product on $\mathcal{H}_\gamma$, and we will make use of multiindex notation. Let $\gamma > 0$, set $a_{0,\gamma} = 1$, and for $k \geq 1$ let

$$a_{k,\gamma} = \binom{\gamma+k-1}{k} = \frac{\gamma(\gamma+1)\cdots(\gamma+k-1)}{k!}. \tag{2.1}$$

Then

$$k_\lambda^\gamma(z) = \sum_{k\geq 0} a_{k,\gamma}\langle z,\lambda\rangle^k = \sum_{k\geq 0} a_{k,\gamma} \sum_{|\alpha|=k} \frac{|\alpha|!}{\alpha!} z^\alpha \overline{\lambda}^\alpha.$$

Hence for $f(z) = \sum_\alpha \hat{f}(\alpha) z^\alpha$ we have

$$\|f\|_\gamma^2 = \sum_{k=0}^\infty \frac{1}{a_{k,\gamma}} \sum_{|\alpha|=k} \frac{\alpha!}{|\alpha|!} |\hat{f}(\alpha)|^2.$$

Note that if $f = \sum_{k\geq 0} f_k$, where f_k is a homogeneous polynomial of degree k, then

$$\|f\|_\gamma^2 = \sum_{k\geq 0} \|f_k\|_\gamma^2 = \sum_{k\geq 0} \frac{1}{a_{k,\gamma}} \|f_k\|_1^2. \tag{2.2}$$

It follows from the definition of the radial derivative $R = \sum_{i=1}^d z_i \frac{\partial}{\partial z_i}$ that for any multiindex α we have that $Rz^\alpha = |\alpha| z^\alpha$. Hence $Rf = \sum_{k\geq 0} k f_k$ and

$$\langle Rf, g\rangle_\gamma = \langle f, Rg\rangle_\gamma = \sum_{k\geq 0} \frac{k}{a_{k,\gamma}} \langle f_k, g_k\rangle_1,$$

whenever the series converges.

We further note that (2.1) implies that an analytic function f is in $\mathcal{H}_\gamma$ if and only if $Rf \in \mathcal{H}_{\gamma+2}$. Thus if $n \geq 1$, then $f \in H_d^2$ if and only if $R^n f \in \mathcal{H}_{1+2n}$.

Lemma 2.1. *Let $b \in H_d^2$. Then the following are equivalent:*

(a) $b \in \mathcal{X}(H_d^2)$,

(b) *there is an integer $n \geq 0$ and a $C > 0$ such that*

$$|\langle \varphi\psi, R^n b\rangle_{n+1}| \leq C\|\varphi\|_1\|\psi\|_1 \text{ for all } \varphi,\psi \in \mathrm{Hol}(\overline{B}_d),$$

(c) *for all integers $n \geq 0$ there is a $C > 0$ such that*

$$|\langle \varphi\psi, R^n b\rangle_{n+1}| \leq C\|\varphi\|_1\|\psi\|_1 \text{ for all } \varphi,\psi \in \mathrm{Hol}(\overline{B}_d).$$

Proof. It is trivial that (c) implies (a) and that (a) implies (b), hence we only need to show the implication (b) $\Rightarrow$ (c). This will follow if we show that for each integer $n \geq 0$ there is a $c > 0$ such that

$$\left|\langle \varphi\psi, R^n b\rangle_{n+1} - \frac{1}{n+1}\langle \varphi\psi, R^{n+1} b\rangle_{n+2}\right| \leq c\|\varphi\|_1\|\psi\|_1\|b\|_1. \tag{2.3}$$

Let $f \in \mathrm{Hol}(\overline{\mathbb{B}_d})$, and let $b = \sum_{k\geq 0} b_k$ and $f = \sum_{k\geq 0} f_k$ be the homogeneous expansions of b and f. We know that $R^n b \in \mathcal{H}^{1+2n}$ for each $n \geq 0$, thus the series $\langle f, R^n b\rangle_{n+1} = \sum_{k\geq 0} \frac{k^n}{a_{k,n+1}} \langle f_k, b_k\rangle_1$ converges absolutely.

Since $\frac{a_{k,n+2}}{a_{k,n+1}} = 1 + \frac{k}{n+1}$ one easily proves from (2.2) that

$$\langle f, R^n b\rangle_{n+1} = \langle f, R^n b\rangle_{n+2} + \frac{1}{n+1}\langle f, R^{n+1} b\rangle_{n+2}.$$

Thus

$$\begin{aligned}\left|\langle f, R^n b\rangle_{n+1} - \frac{1}{n+1}\langle f, R^{n+1}b\rangle_{n+2}\right| &\le \sum_{k\ge 0}\frac{k^n}{a_{k,n+2}}\|f_k\|_1\|b_k\|_1\\ &\le \|f\|_2\left(\sum_{k\ge 0}\frac{a_{k,2}k^{2n}}{a^2_{k,n+2}}\|b_k\|_1^2\right)^{1/2}\\ &\le c\|f\|_2\|b\|_2\\ &\le c\|f\|_2\|b\|_1\end{aligned}$$

for some $c>0$. The second to last inequality follows since for each n we have $a_{k,n+1}\sim(k+1)^n$ as $k\to\infty$ (see e.g. [16], p. 58). Thus there is a $c>0$ such that for all $k\ge 0$ one has $\frac{a_{k,2}k^{2n}}{a^2_{k,n+2}}\le\frac{c}{k+1}$.

In [14] (see Theorem 1.4), it was shown that for any reproducing kernel Hilbert space $\mathcal{H}(k)$ with reproducing kernel k one has a contractive inclusion $\mathcal{H}(k)\odot\mathcal{H}(k)\subseteq\mathcal{H}(k^2)$. We apply this with $k=k^1$, the Drury-Arveson kernel, to obtain $\|\varphi\psi\|_2\le\|\varphi\|_1\|\psi\|_1$ for all $\varphi,\psi\in\operatorname{Hol}(\overline{\mathbb{B}_d})$. Inequality (2.3) then follows by substituting $f=\varphi\psi$ in the earlier estimate. ☐

3. Hankel operators and Carleson embeddings

We start by stating a special case of a theorem from [5]. Note that if $n\in\mathbb{N}, n>(d-1)/2$ and $b\in\operatorname{Hol}(\mathbb{B}_d)$, then the Carleson measure condition

$$\int_{\mathbb{B}_d}|f|^2|R^nb|^2(1-|z|^2)^{2n-d}dV\le C\|f\|^2_{H^2_d}$$

says that the function R^nb defines a bounded multiplication operator from H^2_d to H_{2n+1}. It was shown in [5] that this condition on b is independent of n, that one may even take any $n\in\mathbb{N}$, and that the analogous statement holds for all H_γ, $\gamma>0$.

Lemma 3.1 (Special case of Corollary 3.12 of [5]). *Let $\gamma>0$ and $b\in H_\gamma$. The following are equivalent:*

(a) *There are $n\ge 1$ and a constant $c>0$ such that*

$$\|fR^nb\|_{\gamma+2n}\le c\|f\|_\gamma \text{ for all } f\in H_\gamma.$$

(b) *For every $n\ge 1$ there is a constant $c>0$ such that*

$$\|fR^nb\|_{\gamma+2n}\le c\|f\|_\gamma \text{ for all } f\in H_\gamma.$$

We define the space $\mathcal{CH}_\gamma$ to be the collection of functions that satisfy (a) and (b) of the lemma. They are the functions that satisfy a Carleson embedding condition that is appropriate for the space H_γ. Also note that the space $\mathcal{C}H^2_d$ equals $\mathcal{CH}_1$. Furthermore, in [5], the authors prove the following inequalities regarding $\mathcal{CH}_\gamma$ that will be useful for us.

Lemma 3.2 (Special case of Proposition 3.6 of [5]). *Let $\gamma>0$. If $b\in\mathcal{CH}_\gamma$, then b is in the Bloch space $\mathcal{B}$ and for each $n\ge 1$ and $0\le k<n$ there is $c>0$ such that*

$$\|(R^kf)(R^{n-k}b)\|_{\gamma+2n}\le c\|f\|_\gamma$$

for all $f\in H_\gamma$.

Proof of Theorem 1.1. We start by proving the second inclusion. Let k_w be the reproducing kernel for the Drury-Arveson space and let $R_{\overline{w}}$ be the radial derivative with respect to the $\overline{w}$ variables. Then $R_{\overline{w}}k_w = (k_w - 1)k_w$, hence for $b \in \mathcal{X}(H^2_d)$ we have

$$|Rb(w)| = |\langle (k_w - 1)k_w, b\rangle| \le c\|k_w\|^2 = \frac{c}{1-|w|^2}.$$

It follows that $b \in \mathcal{B}$.

We will show now that for $b \in \mathcal{C}H^2_d = \mathcal{CH}_1$ and for sufficiently large n one has

$$|\langle R^{2n}(\varphi\psi), R^n b\rangle_{3n+1}| \le C\|\varphi\|_1\|\psi\|_1.$$

Then the first inclusion of the theorem will follow from Lemma 2.1 and the identity $\langle \varphi\psi, R^{3n}b\rangle_{3n+1} = \langle R^{2n}(\varphi\psi), R^n b\rangle_{3n+1}$.

Let n be a natural number such that $n > \frac{d-1}{2}$. Then $\langle R^{2n}(\varphi\psi), R^n b\rangle_{3n+1}$ is an inner product for a weighted Bergman space and

$$\langle R^{2n}(fg), R^n b\rangle_{3n+1} = \sum_{k=0}^{2n} \binom{2n}{k} \langle R^k f R^{2n-k} g, R^n b\rangle_{3n+1}.$$

The theorem will follow, if we can bound each term of the sum. By symmetry, it suffices to consider $0 \le k \le n$. In this case we have $2(n+k) - d > -1$ and $2(2n-k) - d > -1$, hence

$$\begin{aligned}|\langle R^k f R^{2n-k} g, R^n b\rangle_{3n+1}| &\lesssim \int_{\mathbb{B}_d} |R^k f R^{2n-k} g R^n b|(1-|z|^2)^{3n-d} dV \\ &\lesssim \left(\int_{\mathbb{B}_d} |R^k f R^n b|^2 (1-|z|^2)^{2(n+k)-d} dV\right)^{1/2} \\ &\quad \left(\int_{\mathbb{B}_d} |R^{2n-k} g|^2 (1-|z|^2)^{2(2n-k)-d} dV\right)^{1/2} \\ &\approx \|R^k f R^n b\|_{2(n+k)+1} \|R^{2n-k} g\|_{2(2n-k)+1} \\ &\lesssim \|f\|_1 \|g\|_1,\end{aligned}$$

where the last inequality follows from Lemma 3.2. □

Remark. In [5] the authors characterize certain Toeplitz and Hankel operators on weighted Besov spaces. The Hankel operators are big Hankel operators, and they are different from the operators considered here. However, it is notable that for $H = H^2_d$ the characterization of the Hankel operators in [5] also involves the space $\mathcal{C}H^2_d$.

4. Invariant subspaces and Hankel operators

The following lemma is elementary; see [10], Lemma 2.2(a).

Lemma 4.1. *If $b \in \mathcal{X}(H^2_d)$, then $\ker H_b = [b]_*^{\perp}$, where we have written $[b]_*$ for the smallest M_z^*-invariant subspace containing b.*

Theorem 4.2. *Let $\mathcal{M} \in \operatorname{Lat}(M_z, H^2_d)$, $\mathcal{M} \neq (0)$. Then there are $\{b_n\}_{n\ge 0} \subseteq \mathcal{X}(H^2_d)$ such that*

$$\mathcal{M} = \bigcap_{n\ge 0} \ker H_{b_n}.$$

Proof. Let $P_{\mathcal{M}}$ be the projection onto $\mathcal{M}$. Since the reproducing kernel $k_\lambda(z) = \frac{1}{1-\langle z,\lambda\rangle}$ is a complete Nevanlinna-Pick kernel, we may use Theorem 4.1 in [11] (also see the remarks following the proof of Theorem 2.1 in [3]) to see that there are multipliers $\varphi_n \in \mathcal{M}_d$ such that $P_{\mathcal{M}} = \sum_n M_{\varphi_n} M_{\varphi_n}^*$. Applying this relationship to the reproducing kernels k_λ, we see that $l_\lambda(z) = \frac{P_{\mathcal{M}} k_\lambda(z)}{k_\lambda(z)} = \sum_n \varphi_n(z)\overline{\varphi_n(\lambda)}$ is a positive definite function. Thus, if $\{e_n\}$ is any orthonormal basis for $\mathcal{H}(l)$, the Hilbert function space with reproducing kernel $l_\lambda(z)$, then $P_{\mathcal{M}} k_\lambda(z) = \sum_n e_n(z)\overline{e_n(\lambda)} k_\lambda(z)$. For $n \in \mathbb{N}$ and $\lambda \in \mathbb{B}_d$ set $T_n k_\lambda = \overline{e_n(\lambda)} k_\lambda$ and extend linearly to the set $\mathcal{D}$ of finite linear combinations of reproducing kernels. By use of the above identity one obtains $\|P_{\mathcal{M}} f\|^2 = \sum_n \|T_n f\|^2$ for all $f \in \mathcal{D}$. Thus each T_n extends to be a bounded operator and it is easy to see that $T_n^* = M_{e_n}$. Hence every member of every orthonormal basis of $\mathcal{H}(l)$ is a multiplier. It follows that every function in $\mathcal{H}(l)$ must be a multiplier of H^2_d.

Of course, for each $\lambda \in \mathbb{B}_d$ we have $l_\lambda \in \mathcal{H}(l)$, hence $l_\lambda \in \mathcal{M}_d$. Since $k_\lambda \in \mathcal{M}_d$ this implies that

$$P_{\mathcal{M}^\perp} k_\lambda = k_\lambda - P_{\mathcal{M}} k_\lambda = k_\lambda - k_\lambda l_\lambda \in \mathcal{M}_d \subseteq \mathcal{X}(H^2_d).$$

Then $\mathcal{M}^\perp = \bigvee_{\lambda \in \mathbb{B}_d} [P_{\mathcal{M}^\perp} k_\lambda]_*$. It is now clear that there is a countable set $\{\lambda_n\} \subseteq \mathbb{B}_d$ such that

$$\mathcal{M} = \bigcap_{n \geq 0} [P_{\mathcal{M}^\perp} k_{\lambda_n}]_*^\perp = \bigcap_{n \geq 0} \ker H_{P_{\mathcal{M}^\perp} k_{\lambda_n}}.$$

□

Example 4.3. Let $d \geq 2$ and consider

$$\mathcal{M} = \left\{ f \in H^2_d : f(0) = \frac{\partial f}{\partial z_i}(0) = 0 \text{ for } i = 1, \dots, d \right\}.$$

Then $\mathcal{M} \in \operatorname{Lat}(M_z, H^2_d)$ and $\dim \mathcal{M}^\perp = d + 1 \geq 3$. The functions in $\mathcal{M}^\perp$ are the polynomials of degree less than or equal to 1. If b is any such polynomial, then for each i we have $M_{z_i}^* b$ is a constant function, hence $\ker H_b^\perp = [b]_*$ is at most two-dimensional. Hence $\mathcal{M}$ cannot be the kernel of a single Hankel operator.

Recall from the Introduction that $(H^1_d)^* = \mathcal{X}(H^2_d)$, where for each $b \in X(H^2_d)$ and $\varphi, \psi \in \operatorname{Hol}(\overline{\mathbb{B}_d})$ the duality is given by $L_b(\varphi\psi) = \langle \varphi\psi, b\rangle$.

Lemma 4.4. *Let $f \in H^2_d, b \in \mathcal{X}(H^2_d)$, and $\varphi \in \operatorname{Hol}(\overline{\mathbb{B}_d})$. Then*

$$L_b(\varphi f) = \langle \varphi f, b\rangle_{H^2_d} = \langle H_b f, \overline{\varphi}\rangle_{\overline{H^2_d}}.$$

Proof. Let ψ_n be a sequence of polynomials such that $\psi_n \to f$ in H^2_d. Then $\varphi\psi_n \to \varphi f$ in H^1_d and $H_b\psi_n \to H_b f$ in H^2_d. Hence the lemma follows by approximation since it is true by definition if f is replaced by ψ_n. □

Theorem 4.5. *Let $\mathcal{M} \in \operatorname{Lat}(M_z, H^2_d)$. Then*

$$\mathcal{M} = H^2_d \cap \operatorname{clos}_{H^1_d} \mathcal{M}.$$

Consequently, a function $f \in H^2_d$ is cyclic in H^2_d if and only if it is cyclic in H^1_d.

Proof. It is clear that $\mathcal{M} \subseteq H^2_d \cap \operatorname{clos}_{H^1_d}\mathcal{M}$. Let $f \notin \mathcal{M}$. Then by Theorem 4.2 there is a $b \in \mathcal{X}(H^2_d)$ such that $\mathcal{M} \subseteq \ker H_b$ and $H_b f \neq 0$. Thus there is a multiindex α such that $\langle H_b f, \overline{z^\alpha}\rangle \neq 0$. If $f \in \operatorname{clos}_{H^1_d}\mathcal{M}$, then there are $f_n \in \mathcal{M}$ such that $f_n \to f$ in H^1_d. This implies that $z^\alpha f_n \to z^\alpha f$ in H^1_d. Hence $0 = \langle H_b f_n, \overline{z^\alpha}\rangle = L_b(z^\alpha f_n) \to L_b(z^\alpha f) = \langle H_b f, \overline{z^\alpha}\rangle$. This contradiction shows that $f \notin \operatorname{clos}_{H^1_d}\mathcal{M}$. □

Theorem 4.6. *Let $f, g \in H^2_d$.*

(a) *If $fg \in H^2_d$, then $fg \in [f] \cap [g]$.*

(b) *If $fg \in H^2_d$ and if f is cyclic in H^2_d, then $[fg] = [g]$.*

(c) *If $fg \in H^2_d$, then fg is cyclic in H^2_d, if and only if both f and g are cyclic in H^2_d.*

(d) *If $f, 1/f \in H^2_d$, then f is cyclic in H^2_d.*

Note that if $f, g \in H^\infty \cap H^2_d$, then $fg \in H^2_d$. One can check this by verifying that for $n > (d-1)/2$

$$R^{2n} fg = \sum_{k=0}^{2n} \binom{2n}{k} R^k f R^{2n-k} g \in H_{4n+1}.$$

Indeed, it is enough to verify the inclusion for each summand individually, and by symmetry, one only needs to consider the cases $0 \leq k \leq n$. In those cases one can use $\|R^{2n-k} g\|_{4n-2k+1} \approx \|g\|_1$ and $H^\infty \subseteq \mathcal{B}$ and hence $(1-|z|^2)^k |R^k f(z)| \leq c$ (see [19]).

Similarly, we note that if $f, g \in \mathcal{CH}_1$, then $fg \in H^2_d$. In this case we have $\|R(fg)\|_3 \leq \|gRf\|_3 + \|fRg\|_3 < \infty$, hence $R(fg) \in \mathcal{H}_3$.

Proof. (a) Suppose that f, g and $fg \in H^2_d$. Since the polynomials are dense in H^2_d, there is a sequence p_n of polynomials such that $p_n \to f$ in H^2_d. Then $\|p_n g - fg\|_* \leq \|p_n - f\|_{H^2_d} \|g\|_{H^2_d}$, hence $p_n g \to fg$ in H^1_d. Thus

$$fg \in H^2_d \cap \operatorname{clos}_{H^1_d}[g] = [g].$$

Similarly $fg \in [f]$.

(b) Now additionally suppose that f is cyclic in H^2_d. By (a) we have $[fg] \subseteq [g]$ and it suffices to show that $g \in [fg]$. Since f is cyclic, there is a sequence of polynomials p_n such that $p_n f \to 1$ in H^2_d. Then as in part (a) of the proof it follows that $p_n fg \to g$ in H^1_d. Hence $g \in H^2_d \cap \operatorname{clos}_{H^1_d}[fg] = [fg]$.

(c) If fg is cyclic in H^2_d, then by (a) $H^2_d = [fg] \subseteq [f] \cap [g]$. Hence both f and g must be cyclic. Conversely, if both f and g are cyclic, then by (b) $[fg] = [g] = H^2_d$.

(d) follows from (c) by taking $g = 1/f$. □

Remark. We now show that Theorem 4.6(d) does not hold for H_2. The reproducing kernel for the single variable Bergman space L^2_a is $k_w(z) = (1-\overline{w}z)^{-2}$. By use of the kernel it is easy to see that the operator $T : L^2_a \to H_2$ given by $(Tf)(z_1, \ldots, z_d) = f(z_1)$ is isometric. Now let $f \in L^2_a$ be a function such that $1/f \in L^2_a$, but f is not cyclic in L^2_a. The existence of such functions is due to [4]. It is easy to see that then $g = Tf$ satisfies $g, 1/g \in H_2$, but g is not cyclic in H_2.

5. Functions that are bounded below in $\mathbb{B}_d$

It is clear that the case $\gamma = 1$ of the following theorem when combined with Theorem 4.6(d) implies Theorem 1.5.

Theorem 5.1. *Let $\gamma > 0$. If $f \in H_\gamma \cap \mathcal{B}$ and $\frac{1}{f} \in H^\infty$, then $\frac{1}{f} \in H_\gamma$.*

In order to verify the conclusion of this theorem we will need to work with $R^m\left(\frac{1}{f}\right)$ for sufficiently large m. For this we will need some preliminary remarks and a lemma.

Let $f \in \mathrm{Hol}(\mathbb{B}_d)$ and $m \in \mathbb{N}$. We define A_m to be the set of all m-tuples $\eta = (\eta_1, \ldots, \eta_m)$ which satisfy $\sum_{i=1}^m i\eta_i = m$, and we write

$$T_\eta(f) = \prod_{i=1}^m (R^i f)^{\eta_i}.$$

Then Faa di Bruno's formula for the higher order derivatives of a composition (see [9]) gives

$$R^m\left(\frac{1}{f}\right) = \sum_{\eta \in A_m} \frac{m!}{\eta!} \frac{(-1)^{|\eta|}|\eta|!}{f^{|\eta|+1}} \prod_{j=1}^m \left(\frac{1}{j!}\right)^{\eta_j} T_\eta(f).$$

Since $\|\frac{1}{f}\|_\gamma \sim \|R^m \frac{1}{f}\|_{\gamma+2m}$ and the assumption of the theorem implies that $\frac{1}{f}$ is a multiplier of $H_{\gamma+2m}$ whenever $\gamma + 2m > d$, it follows that the next lemma implies Theorem 5.1.

Lemma 5.2. *Let $\gamma > 0$ and $m \in \mathbb{N}$. If $f \in \mathcal{H}_\gamma \cap \mathcal{B}$, then*

$$\|T_\eta(f)\|_{\gamma+2m} < \infty$$

for any m-tuple $\eta = (\eta_1, \ldots, \eta_m) \in A_m$.

Proof. Fix $\eta = (\eta_1, \ldots, \eta_m) \in A_m$. For $j \in \mathbb{N}$, let B_j be those $|\eta|$-tuples $\beta = (\beta_1, \ldots, \beta_{|\eta|})$ which satisfy $|\beta| = j$. Then by writing powers as products of single terms, one sees that there is a function $g : \{1, \ldots, |\eta|\} \to \{1, \ldots, m\}$ which satisfies $T_\eta(f) = \prod_{i=1}^{|\eta|} R^{g(i)} f$. Note that this implies that $\sum_{i=1}^{|\eta|} g(i) = \sum_{i=1}^m i\eta_i = m$.

Now choose $j \geq \frac{d|\eta|}{2}$, and for each $\beta \in B_j$ choose an index i_β such that $\beta_{i_\beta} \geq \frac{j}{|\eta|} \geq d/2$. This is possible since $|\beta| = j$. Then $2(\beta_{i_\beta} + g(i_\beta)) + \gamma > d$. Since $f \in \mathcal{B}$, for any $n \in \mathbb{N}$ we have that $(1-|z|^2)^n |R^n f(z)|$ is bounded in $\mathbb{B}_d$ (see [19], Theorem 3.5), hence there is a $C > 0$ such that

$$(1-|z|^2)^{(j+m)} \prod_{i=1}^{|\eta|} |R^{\beta_i + g(i)} f(z)| \leq C(1-|z|^2)^{\beta_{i_\beta} + g(i_\beta)} |R^{\beta_{i_\beta} + g(i_\beta)} f(z)|.$$

Here we also used that $\sum_{i=1}^{|\eta|} \beta_i + g(i) = j + m$.

Finally, by the Leibniz rule, we have that

$$\begin{aligned}
\|T_\eta(f)\|_{\gamma+2m} &\approx \|R^j T_\eta(f)\|_{\gamma+2(m+j)} \\
&= \left\| \sum_{\beta \in B_j} \frac{j!}{\beta!} \prod_{i=1}^{|\eta|} R^{\beta_i + g(i)} f \right\|_{\gamma+2(m+j)} \\
&\lesssim \sum_{\beta \in \beta_j} \left\| \prod_{i=1}^{|\eta|} R^{\beta_i + g(i)} f \right\|_{\gamma+2(m+j)} \\
&\lesssim \sum_{\beta \in \beta_j} \left\| R^{\beta_{t_\beta} + g(t_\beta)} f \right\|_{\gamma+2(\beta_{t_\beta} + g(t_\beta))}
\end{aligned}$$

and the result follows since $f \in \mathcal{H}_\gamma$. ☐

We mentioned in the Introduction that in the special case of the previous theorem where $f \in \mathcal{M}_d$ and $1/f \in H^\infty$, then by the one function case of the Corona theorem for $\mathcal{M}_d$ one has $1/f \in \mathcal{M}_d$ (see [7]). That result was also proved by Fang and Xia in [8]. The next theorem establishes the same conclusion in the context of H_γ, $\gamma > 0$, and without assuming that f be bounded. The proof is also significantly shorter than [8].

Lemma 5.3. *If $0 < \gamma < \beta$, then $\mathcal{CH}_\gamma \subseteq \mathcal{CH}_\beta$.*

Proof. Set $\varepsilon = \beta - \gamma > 0$ and let $b \in \mathcal{CH}_\gamma$. By Lemma 3.1 with $n = 1$ this implies that $M_{Rb} : H_\gamma \to H_{\gamma+2}$ is bounded. Using adjoints and reproducing kernels, we see that this is equivalent to the existence of $C > 0$ such that $Ck_w^{\gamma+2}(z) - Rb(z)\overline{Rb(w)}k_w^\gamma(z)$ is positive definite. We multiply this by $k_w^\varepsilon(z)$ and apply the Schur product theorem (see [1], Theorem A.1) and obtain that $Ck_w^{\beta+2}(z) - Rb(z)\overline{Rb(w)}k_w^\beta(z)$ is positive definite. This implies that $b \in \mathcal{CH}_\beta$. ☐

Theorem 5.4. *If $f \in \mathcal{CH}_\gamma$, and if there is a constant $c > 0$ such that $|f(z)| \geq c$ for all $z \in \mathbb{B}_d$, then $\frac{1}{f}$ is a multiplier for $\mathcal{H}_\gamma$.*

Proof. If $f \in \mathcal{CH}_\gamma$, then by Lemma 5.3 we have that $f \in \mathcal{CH}_{\gamma+n}$ for all $n \in \mathbb{N}_0$. Since for large n the space $H_{\gamma+n}$ is a weighted Bergman space, the hypothesis implies that $1/f$ is a multiplier of $H_{\gamma+n+2}$ for sufficiently large n. Thus the theorem will follow inductively from the claim:

Claim: If $f \in \mathcal{CH}_{\gamma+n}$ and if $1/f$ is a multiplier of $H_{\gamma+n+2}$, then $1/f$ is a multiplier of $H_{\gamma+n}$.

Let $g \in H_{\gamma+n}$. Then

$$\left\|\frac{g}{f}\right\|_{\gamma+n} \approx \left\|R\left(\frac{g}{f}\right)\right\|_{\gamma+n+2} \leq \left\|\frac{Rg}{f}\right\|_{\gamma+n+2} + \left\|\frac{gRf}{f^2}\right\|_{\gamma+n+2}$$
$$\lesssim \|Rg\|_{\gamma+n+2} + \|gRf\|_{\gamma+n+2} \lesssim \|g\|_{\gamma+n}.$$

Here the last inequality follows from Lemma 3.1. ☐

References

[1] Jim Agler and John E. McCarthy, *Pick interpolation and Hilbert function spaces*, Graduate Studies in Mathematics, vol. 44, American Mathematical Society, Providence, RI, 2002. MR1882259 (2003b:47001)

[2] Nicola Arcozzi, Richard Rochberg, Eric Sawyer, and Brett D. Wick, *Bilinear forms on the Dirichlet space*, Anal. PDE **3** (2010), no. 1, 21–47, DOI 10.2140/apde.2010.3.21. MR2663410 (2011f:31011)

[3] William Arveson, *The curvature invariant of a Hilbert module over* $\mathbf{C}[z_1, \cdots, z_d]$, J. Reine Angew. Math. **522** (2000), 173–236, DOI 10.1515/crll.2000.037. MR1758582 (2003a:47013)

[4] Alexander Borichev and Håkan Hedenmalm, *Harmonic functions of maximal growth: invertibility and cyclicity in Bergman spaces*, J. Amer. Math. Soc. **10** (1997), no. 4, 761–796, DOI 10.1090/S0894-0347-97-00244-0. MR1446365 (98a:30066)

[5] Carme Cascante, Joan Fàbrega, and Joaquín M. Ortega, *On weighted Toeplitz, big Hankel operators and Carleson measures*, Integral Equations Operator Theory **66** (2010), no. 4, 495–528, DOI 10.1007/s00020-010-1760-4. MR2609237 (2011c:47053)

[6] R. R. Coifman, R. Rochberg, and Guido Weiss, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math. (2) **103** (1976), no. 3, 611–635. MR0412721 (54 #843)

[7] Şerban Costea, Eric T. Sawyer, and Brett D. Wick, *The corona theorem for the Drury-Arveson Hardy space and other holomorphic Besov-Sobolev spaces on the unit ball in* $\mathbb{C}^n$, Anal. PDE **4** (2011), no. 4, 499–550. MR3077143

[8] Quanlei Fang and Jingbo Xia, *Corrigendum to "Multipliers and essential norm on the Drury-Arveson space" [MR2784815]*, Proc. Amer. Math. Soc. **141** (2013), no. 1, 363–368, DOI 10.1090/S0002-9939-2012-11363-2. MR2988737

[9] Warren P. Johnson, *The curious history of Faà di Bruno's formula*, Amer. Math. Monthly **109** (2002), no. 3, 217–234, DOI 10.2307/2695352. MR1903577 (2003d:01019)

[10] Shuaibing Luo and Stefan Richter, *Hankel operators and invariant subspaces of the Dirichlet space*, J. Lond. Math. Soc. (2) **91** (2015), no. 2, 423–438, DOI 10.1112/jlms/jdv001. MR3355109

[11] Scott McCullough and Tavan T. Trent, *Invariant subspaces and Nevanlinna-Pick kernels*, J. Funct. Anal. **178** (2000), no. 1, 226–249, DOI 10.1006/jfan.2000.3664. MR1800795 (2002b:47006)

[12] Joaquín M. Ortega and Joan Fàbrega, *Pointwise multipliers and decomposition theorems in analytic Besov spaces*, Math. Z. **235** (2000), no. 1, 53–81, DOI 10.1007/s002090000123. MR1785071 (2001i:46033)

[13] Stefan Richter and Carl Sundberg, *A formula for the local Dirichlet integral*, Michigan Math. J. **38** (1991), no. 3, 355–379, DOI 10.1307/mmj/1029004388. MR1116495 (92i:47035)

[14] Stefan Richter and Carl Sundberg, *Weak products of Dirichlet functions*, J. Funct. Anal. **266** (2014), no. 8, 5270–5299, DOI 10.1016/j.jfa.2014.02.005. MR3177336

[15] Orr Shalit, *Operator theory and function theory in Drury-Arveson space and its quotients*, in Operator Theory, Daniel Alpay (ed.), pp 1-50, Springer Basel, (2014) DOI 10.1007/978-3-0348-0692-3_60-1

[16] E. C. Titchmarsh, *The theory of functions*, Oxford University Press, Oxford, 1958. Reprint of the second (1939) edition. MR3155290

[17] Ke He Zhu, *Hankel operators on the Bergman space of bounded symmetric domains*, Trans. Amer. Math. Soc. **324** (1991), no. 2, 707–730, DOI 10.2307/2001738. MR1093426 (92f:47023)

[18] Kehe Zhu, *Maximal inner spaces and Hankel operators on the Bergman space*, Integral Equations Operator Theory **31** (1998), no. 3, 371–387, DOI 10.1007/BF01195126. MR1627913 (2000f:47052)

[19] Kehe Zhu, *Spaces of holomorphic functions in the unit ball*, Graduate Texts in Mathematics, vol. 226, Springer-Verlag, New York, 2005. MR2115155 (2006d:46035)

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