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A POSITIVE GRASSMANNIAN ANALOGUE OF THE PERMUTOHEDRON

LAUREN K. WILLIAMS

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ABSTRACT. The classical permutohedron Perm_n is the convex hull of the points $(w(1),\ldots,w(n)) \in \mathbb{R}^n$ where w ranges over all permutations in the symmetric group S_n. This polytope has many beautiful properties – for example it provides a way to visualize the weak Bruhat order: if we orient the permutohedron so that the longest permutation w_0 is at the "top" and the identity e is at the "bottom", then the one-skeleton of Perm_n is the Hasse diagram of the weak Bruhat order. Equivalently, the paths from e to w_0 along the edges of Perm_n are in bijection with the reduced decompositions of w_0. Moreover, the two-dimensional faces of the permutohedron correspond to braid and commuting moves, which by the Tits Lemma, connect any two reduced expressions of w_0.

In this note we introduce some polytopes $\mathrm{Br}_{k,n}$ (which we call *bridge polytopes*) which provide a positive Grassmannian analogue of the permutohedron. In this setting, *BCFW-bridge decompositions* of *reduced plabic graphs* play the role of reduced decompositions. We define $\mathrm{Br}_{k,n}$ and explain how paths along its edges encode BCFW-bridge decompositions of the longest element $\pi_{k,n}$ in the *circular Bruhat order*. We also show that two-dimensional faces of $\mathrm{Br}_{k,n}$ correspond to certain local moves for plabic graphs, which by a result of Postnikov, connect any two reduced plabic graphs associated to $\pi_{k,n}$. All of these results can be generalized to the positive parts of Schubert cells. A useful tool in our proofs is the fact that our polytopes are isomorphic to certain *Bruhat interval polytopes*. Conversely, our results on bridge polytopes allow us to deduce some corollaries about the structure of Bruhat interval polytopes.

1. Introduction

The *totally nonnegative part* $(Gr_{k,n})_{\geq 0}$ of the real Grassmannian $Gr_{k,n}$ is the locus where all Plücker coordinates are nonnegative [Lus94,Rie99,Pos06]. Postnikov initiated the combinatorial study of $(Gr_{k,n})_{\geq 0}$: he showed that if one stratifies the space based on which Plücker coordinates are positive and which are zero, one gets a cell decomposition. The cells are in bijection with several families of combinatorial objects, including decorated permutations, and *equivalence classes of reduced plabic graphs*. Reduced plabic graphs are a certain family of planar bicolored graphs embedded in a disk. These graphs are very interesting: they can be used to parameterize cells in $(Gr_{k,n})_{\geq 0}$ [Pos06] and to understand the *cluster algebra structure* on the Grassmannian [Sco06]; they also arise as *soliton graphs*

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associated to the KP hierarchy [KW11]. Most recently, reduced plabic graphs and the totally nonnegative Grassmannian $(Gr_{k,n})_{\geq 0}$ have been studied in connection with *scattering amplitudes* in $N = 4$ super Yang-Mills [AHBC[12]]. Motivated by physical considerations, the authors gave a systematic way of building up a reduced plabic graph for a given cell of $(Gr_{k,n})_{\geq 0}$, which they called the *BCFW-bridge construction*.

In many ways, reduced plabic graphs behave like reduced decompositions of the symmetric group: for example, just as any two reduced decompositions of the same permutation can be related by braid and commuting moves, any two reduced plabic graphs associated to the same decorated permutation can be related via certain local moves [Pos06]. The goal of this note is to highlight another way in which reduced plabic graphs behave like reduced decompositions. We will introduce a polytope called the *bridge polytope*, which is a positive Grassmannian analogue of the permutohedron in the following sense: just as the one-skeleton of the permutohedron encodes reduced decompositions of permutations, the one-skeleton of the bridge polytope encodes BCFW-bridge decompositions of reduced plabic graphs. Moreover, just as the two-dimensional faces of permutohedra encode braid and commuting moves for reduced decompositions, the two-dimensional faces of bridge polytopes encode local moves for reduced plabic graphs.

2. Background

2.1. **Reduced decompositions, Bruhat order, and the permutohedron.**

Recall that the *symmetric group* S_n is the group of all permutations on n letters. If we let s_i denote the simple transposition exchanging i and $i+1$, then S_n is a Coxeter group generated by the s_i (for $1 \leq i < n$), subject to the relations $s_i^2 = 1$, as well as the *commuting relations* $s_i s_j = s_j s_i$ for $|i - j| \geq 2$ and *braid relations* $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$.

Given $w \in S_n$, there are many ways to write w as a product of simple transpositions. If $w = s_{i_1} \dots s_{i_\ell}$ is a minimal-length such expression, we call $s_{i_1} \dots s_{i_\ell}$ a *reduced expression* of w, and ℓ the *length* $\ell(w)$ of w. The well-known Tits Lemma asserts that any two reduced expressions can be related by a sequence of braid and commuting moves if and only if they are reduced expressions for the same permutation.

There are two important partial orders on S_n, both of which are graded, with rank function given by the length of the permutation. The *strong Bruhat order* is the transitive closure of the cover relation $u \prec v$, where $u \prec v$ means that $(ij)u = v$ for $i < j$ and $\ell(v) = \ell(u) + 1$. Here (ij) is the transposition (not necessarily simple) exchanging i and j. We use the symbol $\preceq$ to indicate the strong Bruhat order. The strong Bruhat order has many nice combinatorial properties: it is thin and shellable, and hence is a CW poset [Bjö84, Ede81, Pro82, BW82]. Moreover, this partial order is closely connected to the geometry of the complete flag variety Fl_n: it encodes when one Schubert cell is contained in the closure of another. Somewhat less well-known is its connection to total positivity [Lus94, FS00, Her14]. The totally nonnegative part $U^+_{\geq 0}$ of the unipotent part of GL_n has a decomposition into cells, indexed by permutations, and the strong Bruhat order describes when one cell is contained in the closure of another [Lus94].

The other partial order associated to S_n is the *(left) weak Bruhat order*. This is the transitive closure of the cover relation $u \lessdot v$, where $u \lessdot v$ means that $s_i u = v$

and $\ell(v) = \ell(u) + 1$. We use the symbol $\leq$ to indicate the weak Bruhat order. The weak Bruhat order is related to reduced decompositions in the following way: given any $w \in S_n$, the saturated chains from the identity e to w in the weak Bruhat order are in bijection with reduced decompositions of w. Moreover, the weak order has the advantage of being conveniently visualized in terms of a polytope called the permutohedron, as we now describe.

Definition 2.1. The *permutohedron* Perm_n in $\mathbb{R}^n$ is the convex hull of the points $\{(z(1), \ldots, z(n)) \mid z \in S_n\}$.

See Figure 1 for the permutohedron Perm_4. We remark that the permutohedron is also connected to the geometry of the complete flag variety Fl_n: it is the image of Fl_n under the moment map.

Given a permutation $z \in S_n$, we will often refer to it using the notation $(z(1), \ldots, z(n))$, or $(z_1, \ldots, z_n)$, where z_i denotes $z(i)$.

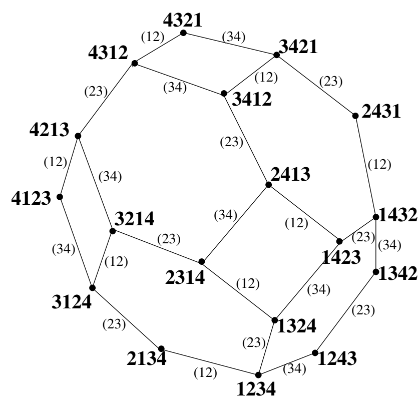

FIGURE 1. The permutohedron Perm_4. Edges are labeled by the values swapped when we go between the permutations labeling the two vertices.

The following proposition is well known; see e.g. [BLVS[99]].

Proposition 2.2. *Let u and v be permutations in S_n with $\ell(u) \leq \ell(v)$. There is an edge between vertices u and v in the permutohedron* Perm_n *if and only if $v = s_i u$ for some i and $\ell(v) = \ell(u) \pm 1$, in which case we label the corresponding edge of the permutohedron by s_i. Moreover, the two-dimensional faces of* Perm_n *are either squares (with edges labeled in an alternating fashion by s_i and s_j, where $|i-j| \geq 2$) or hexagons (with edges labeled in an alternating fashion by s_i and s_{i+1} for some i).*

See Figure 1 for the example of Perm_4. Proposition 2.2 implies that edges of the permutohedron correspond to cover relations in the weak Bruhat order, and the two-dimensional faces correspond to braid and commuting relations for reduced words.

Let w_0 be the longest permutation in S_n. This permutation is unique, and can be defined by $w_0(i) = n + 1 - i$. Proposition 2.2 can be equivalently restated as follows.

Proposition 2.3. *The shortest paths from e to w_0 along the one-skeleton of the permutohedron* Perm_n *are in bijection with the reduced decompositions of w_0.*

For example, we can see from Figure 1 that there is a path

$$e \lessdot (2,1,3,4) \lessdot (3,1,2,4) \lessdot (4,1,2,3) \lessdot (4,2,1,3) \lessdot (4,3,1,2) \lessdot (4,3,2,1) = w_0$$

in the permutohedron Perm_4. Reading off the edge labels of this path gives rise to the reduced decomposition $s_1 s_2 s_1 s_3 s_2 s_1$ of w_0.

2.2. **The positive Grassmannian, permutations, circular Bruhat order, and plabic graphs.** In this section we introduced the positive Grassmannian, and some combinatorial objects associated to it, including permutations and plabic graphs. As we will see, *reduced plabic graphs* are in many ways analogous to reduced decompositions of permutations.

The *real Grassmannian $Gr_{k,n}$* is the space of all k-dimensional subspaces of $\mathbb{R}^n$. An element of $Gr_{k,n}$ can be viewed as a full-rank $k \times n$ matrix modulo left multiplication by nonsingular $k \times k$ matrices. In other words, two $k \times n$ matrices represent the same point in $Gr_{k,n}$ if and only if they can be obtained from each other by row operations. Let $\binom{[n]}{k}$ be the set of all k-element subsets of $[n] := \{1, \ldots, n\}$. For $I \in \binom{[n]}{k}$, let $\Delta_I(A)$ be the *Plücker coordinate*, that is, the maximal minor of the $k \times n$ matrix A located in the column set I. The map $A \mapsto (\Delta_I(A))$, where I ranges over $\binom{[n]}{k}$, induces the *Plücker embedding* $Gr_{k,n} \hookrightarrow \mathbb{RP}^{\binom{n}{k}-1}$. The *totally nonnegative part of the Grassmannian* (sometimes called the *positive Grassmannian*) $(Gr_{k,n})_{\geq 0}$ is the subset of $Gr_{k,n}$ such that all Plücker coordinates are nonnegative.

If we partition $(Gr_{k,n})_{\geq 0}$ into strata based on which Plücker coordinates are positive and which are zero, we obtain a decomposition into *positroid cells* [Pos06]. Postnikov showed that the cells are in bijection with, and naturally labeled by, several families of combinatorial objects, including *Grassmann necklaces*, *decorated permutations* and *equivalence classes of reduced plabic graphs*. He also introduced the *circular Bruhat order* on decorated permutations, which describes when one cell is contained in another. Like the strong Bruhat order, the circular Bruhat order is graded, thin, shellable, and hence is a CW poset [Wil07].

Definition 2.4. A *Grassmann necklace* is a sequence $\mathcal{I} = (I_1, \ldots, I_n)$ of subsets $I_r \subset [n]$ such that, for $i \in [n]$, if $i \in I_i$, then $I_{i+1} = (I_i \setminus \{i\}) \cup \{j\}$, for some $j \in [n]$; and if $i \notin I_i$ then $I_{i+1} = I_i$. (Here indices i are taken modulo n.) In particular, we have $|I_1| = \cdots = |I_n|$, which is equal to some $k \in [n]$. We then say that $\mathcal{I}$ is a Grassmann necklace of *type* (k, n).

To construct the Grassmann necklace associated to a positroid cell, one uses the following construction.

Lemma 2.5. *Define the shifted linear order $<_i$ by $i <_i i+1 <_i \cdots <_i n <_i 1 <_i \cdots <_i i-1$. Given $A \in Gr_{k,n}$, let $\mathcal{I}(A) = (I_1, \ldots, I_n)$ be the sequence of subsets in $[n]$ such that, for $i \in [n]$, I_i is the lexicographically minimal subset of $\binom{[n]}{k}$ with respect to the shifted linear order such that $\Delta_{I_i}(A) \neq 0$. Then $\mathcal{I}(A)$ is a Grassmann necklace of type (k, n).*

Definition 2.6. A *decorated permutation* π on n letters is a permutation on n letters in which fixed points have one of two colors, "clockwise" and "counterclockwise". A position i of π is called a *weak excedance* if $\pi(i) > i$ or if $\pi(i) = i$ for i a counterclockwise fixed point.

Definition 2.7. Given a Grassmann necklace $\mathcal{I}$, define a decorated permutation $\pi = \pi(\mathcal{I})$ by requiring that

(1) if $I_{i+1} = (I_i \setminus \{i\}) \cup \{j\}$, for $j \neq i$, then $\pi(j) = i$,
(2) if $I_{i+1} = I_i$ and $i \in I_i$, then $\pi(i) = i$ is a counterclockwise fixed point,
(3) if $I_{i+1} = I_i$ and $i \notin I_i$, then $\pi(i) = i$ is a clockwise fixed point.

As before, indices are taken modulo n.

Using the above constructions, one may show the following.

Theorem 2.8 ([Pos06]). *The cells of $(Gr_{k,n})_{\geq 0}$ are in bijection with Grassmann necklaces of type (k, n) and with decorated permutations on n letters with exactly k weak excedances.*

Definition 2.9. The *totally positive Grassmannian* $(Gr_{k,n})_{>0}$ is the subset of $(Gr_{k,n})_{\geq 0}$ consisting of elements where all Plücker coordinates are strictly positive. It is the unique top-dimensional cell of $(Gr_{k,n})_{\geq 0}$, and it is labeled by the decorated permutation $\pi_{k,n} := (n-k+1, n-k+2, \dots, n, 1, 2, \dots, n-k)$.

More generally, there is a nice class of positroid cells called the *TP* or *totally positive Schubert cells.*

Definition 2.10. A *TP Schubert cell* is the unique positroid cell of greatest dimension which lies in the intersection of $(Gr_{k,n})_{\geq 0}$ with a usual Schubert cell for $Gr_{k,n}$. The TP Schubert cells in $(Gr_{k,n})_{\geq 0}$ are in bijection with k-element subsets J of $[n] = \{1, 2, \dots, n\}$. To calculate the decorated permutation associated to the TP Schubert cell indexed by J, write $J = \{j_1 < \cdots < j_k\}$, and the complement of J as $J^c = \{h_1 < \cdots < h_{n-k}\}$. Then the corresponding decorated permutation $\pi = \pi(J)$ is defined by $\pi(h_i) = i$ for all $1 \leq i \leq n-k$, and $\pi(j_i) = n-k+i$ for all $1 \leq i \leq k$. (Any fixed points that arise are considered to be weak excedances if and only if they are in positions labeled by J.)

A *Grassmannian permutation* is a permutation $\pi = (\pi(1), \dots, \pi(n))$ which has at most one descent, i.e., at most one position i such that $\pi(i) > \pi(i+1)$. Note that the permutations of the form $\pi(J)$ defined above are precisely the inverses of the Grassmannian permutations, since $\pi(J)^{-1} = (h_1, h_2, \dots, h_{n-k}, j_1, j_2, \dots, j_k)$.

In the case that $J = \{1, 2, \dots, k\}$, the corresponding TP Schubert cell is $(Gr_{k,n})_{>0}$ and $\pi(J) = \pi_{k,n}$.

To describe the partial order on positroid cells, it is convenient to represent each decorated permutation by a related affine permutation, as was done in [KLS13]. Then, the circular Bruhat order for $(Gr_{k,n})_{\geq 0}$ can be viewed as the restriction of the affine Bruhat order to the corresponding affine permutations, together with a new bottom element $\hat{0}$ added to the poset.

Definition 2.11. Given a decorated permutation π on n letters, we construct its affinization $\tilde{\pi}$ as follows. If $\pi(i) < i$, set $\tilde{\pi}(i) := \pi(i) + n$. If $\pi(i) = i$ and i is a clockwise fixed point, set $\tilde{\pi}(i) := i$. If $\pi(i) = i$ and i is a counterclockwise fixed point, set $\tilde{\pi}(i) := i + n$. Finally, if $\pi(i) > i$, set $\tilde{\pi}(i) := \pi(i)$.

Clearly the affine permutation constructed above is a map from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, 2n\}$ such that $i \leq \tilde{\pi}(i) \leq i + n$. And modulo n, it agrees with the underlying permutation π.

We now discuss plabic graphs, which e.g. are useful for constructing parameterizations of positroid cells [Pos06], and for understanding the cluster structure on the Grassmannian [Sco06].

Definition 2.12. A *planar bicolored graph* (or *plabic graph*) is an undirected graph G drawn inside a disk (and considered modulo homotopy) plus n vertices on the boundary of the disk (which we call *boundary vertices*) labeled $1, \ldots, n$ in clockwise order. The remaining *internal vertices* are strictly inside the disk and are colored in black and white. We require that each boundary vertex is incident to a single edge.

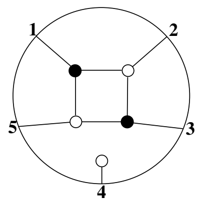

FIGURE 2. A plabic graph.

We will always assume that a plabic graph G has no isolated components, and that any degree 1 vertex of G must be adjacent to the boundary. The following map gives a connection between plabic graphs and decorated permutations.

Definition 2.13. Given a plabic graph G, the *trip* T_i is the directed path which starts at the boundary vertex i, and follows the "rules of the road", namely it turns right at each black vertex that it encounters and left at each white vertex. Note that T_i will also end at a boundary vertex. The *decorated trip permutation* π_G is the permutation such that $\pi_G(i) = j$ whenever T_i ends at j. Moreover, if there is a white (respectively, black) boundary leaf at boundary vertex i, then $\pi_G(i) = i$ is a counterclockwise (respectively, clockwise) fixed point.

We now define some local transformations of plabic graphs, which are analogous in some ways to braid and commuting moves for reduced expressions in the symmetric group.

(M1) SQUARE MOVE. If a plabic graph has a square formed by four trivalent vertices whose colors alternate, then we can switch the colors of these four vertices.

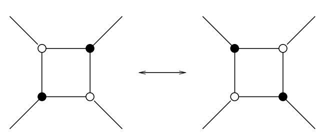

FIGURE 3. Square move.

(M2) UNICOLORED EDGE CONTRACTION/UNCONTRACTION. If a plabic graph contains an edge with two vertices of the same color, then we can contract this edge into a single vertex with the same color. We can also uncontract a vertex into an edge with vertices of the same color.

FIGURE 4. Unicolored edge contraction.

(M3) MIDDLE VERTEX INSERTION/REMOVAL. If a plabic graph contains a vertex of degree 2, then we can remove this vertex and glue the incident edges together; on the other hand, we can always insert a vertex (of any color) in the middle of any edge.

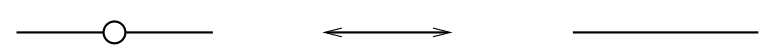

FIGURE 5. Middle vertex insertion/ removal.

(R1) PARALLEL EDGE REDUCTION. If a network contains two trivalent vertices of different colors connected by a pair of parallel edges, then we can remove these vertices and edges, and glue the remaining pair of edges together.

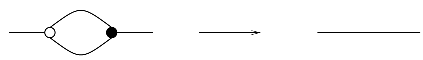

FIGURE 6. Parallel edge reduction.

Definition 2.14 ([Pos06]). Two plabic graphs are called *move-equivalent* if they can be obtained from each other by moves (M1)-(M3). The *move-equivalence class* of a given plabic graph G is the set of all plabic graphs which are move-equivalent to G. A leafless plabic graph without isolated components is called *reduced* if there is no graph in its move-equivalence class to which we can apply (R1).

The following result is analogous to the Tits Lemma for reduced decompositions of permutations.

Theorem 2.15 ([Pos06, Theorem 13.4]). *Two reduced plabic graphs are move-equivalent if and only if they have the same decorated trip permutation.*

From the definition above, it is not so easy to detect whether a given plabic graph is reduced. One characterization was given in [Pos06, Theorem 13.2]. Another very simple characterization was given in [KW11].

Definition 2.16. Given a plabic graph G with n boundary vertices, start at each boundary vertex i and label every edge along trip T_i with i. After doing this for each boundary vertex, each edge will be labeled by up to two numbers (between 1 and n). If an edge is labeled by two numbers $i < j$, write $[i, j]$ on that edge.

We say that a plabic graph has the *resonance property*, if after labeling edges as above, the set E of edges incident to a given vertex has the following property:

- there exist numbers $i_1 < i_2 < \cdots < i_m$ such that when we read the labels of E, we see the labels $[i_1, i_2], [i_2, i_3], \ldots, [i_{m-1}, i_m], [i_1, i_m]$ appear in clockwise order.

This property and the following characterization of reduced plabic graphs were given in [KW11].

Theorem 2.17 ([KW11, Theorem 10.5])**.** *A plabic graph is reduced if and only if it has the resonance property.*

2.3. **BCFW-bridge decompositions.** Given a reduced plabic graph, it is easy to construct its corresponding trip permutation, as explained in Definition 2.13. But how can we go backwards, i.e., given the permutation, how can we construct a corresponding reduced plabic graph? One procedure to do this was given in [Pos06, Section 20]. Another elegant solution – the *BCFW-bridge construction* – was given in [AHBC^{12}, Section 3.2].

To explain the BCFW-bridge construction, we use the representation of each decorated permutation π as an affine permutation $\tilde{\pi}$; see Definition 2.11. We say that position i of $\tilde{\pi}$ is a *fixed point* if $\tilde{\pi}(i) = i$ or $\tilde{\pi}(i) = i + n$, and we say that $\tilde{\pi}$ is a *decoration of the identity* if it has a fixed point in each position.

Definition 2.18 (The BCFW-bridge construction)**.** Given a decorated permutation π on n letters, which is not simply a decoration of the identity, we start by choosing a pair of adjacent positions $i < j$ such that $\tilde{\pi}(i) < \tilde{\pi}(j)$, and i and j are not fixed points. Here *adjacent* means that either $j = i + 1$, or $i < j$ and every position h such that $i < h < j$ is a fixed point. We record the transposition $\tau = (ij)$ and swap the entries in positions i and j in $\tilde{\pi}$. Any entries in the resulting permutation which are fixed points are designated as *frozen*, and henceforth ignored. We continue this process on the resulting affine permutation, until the end result is a decoration of the identity. Finally we use the sequence of transpositions τ as a recipe for adding "bridges", thereby constructing a plabic graph.

See Table 1 and Figure 7 for an example of a bridge decomposition of the permutation $\pi = (4, 6, 5, 1, 2, 3)$ (with corresponding affine permutation $\tilde{\pi} = (4, 6, 5, 7, 8, 9)$). Here a *bridge* is the subgraph shown at the left of Figure 7, and a bridge decomposition is a graph built by attaching successive bridges. The sequence of transpositions τ gives a recipe for where to place successive bridges.

TABLE 1. A BCFW-bridge decomposition of $\tilde{\pi} = (4, 6, 5, 7, 8, 9)$. The frozen entries are boxed.

τ	1 $\downarrow$	2 $\downarrow$	3 $\downarrow$	4 $\downarrow$	5 $\downarrow$	6 $\downarrow$
(34)	4	6	5	7	8	9
(23)	4	6	7	5	8	9
(12)	4	7	6	5	8	9
(56)	$\boxed{7}$	4	6	5	8	9
(45)	$\boxed{7}$	4	6	5	9	8
(34)	$\boxed{7}$	4	6	9	$\boxed{5}$	8
(46)	$\boxed{7}$	4	$\boxed{9}$	6	$\boxed{5}$	8
(24)	$\boxed{7}$	4	$\boxed{9}$	8	$\boxed{5}$	$\boxed{6}$
	7	8	9	4	5	6

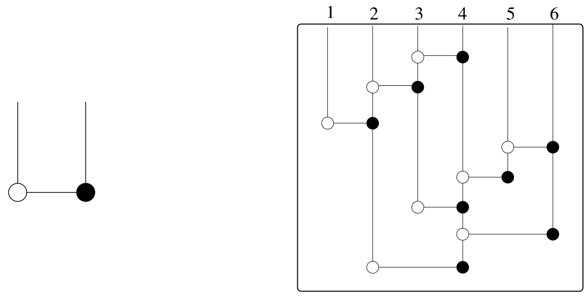

FIGURE 7. A single bridge, and the plabic graph G associated to the bridge decomposition from Table 1. Note that the trip permutation of G equals the product (24)(46)(34)(45)(56)(12)(23)(34) of transpositions τ (reading right to left) in the bridge decomposition, which equals the corresponding decorated permutation $\pi = (4, 6, 5, 1, 2, 3)$.

Proposition 2.19 ([AHBC[12], Section 3.2])**.** *Given a decorated permutation π, the BCFW-bridge construction constructs a reduced plabic graph whose trip permutation is π.*

3. Bridge polytopes and their one-skeleta

The goal of this section is to introduce some polytopes that we will call *bridge polytopes*, because their one-skeleta encode BCFW-bridge decompositions of reduced plabic graphs. This statement is analogous to the fact that the one-skeleton of the permutohedron encodes reduced decompositions of w_0. More specifically, for each k-element subset J of $[n]$, we will introduce a bridge polytope Br_J which encodes bridge decompositions of reduced plabic graphs for the permutation $\pi(J)$ labeling the corresponding TP Schubert cell. In the case that $J = \{1, 2, \ldots, k\}$, we will also denote Br_J by $\mathrm{Br}_{k,n}$. $\mathrm{Br}_{k,n}$ will be the polytope encoding bridge decompositions of $\pi_{k,n}$, the decorated permutation labeling the totally positive Grassmannian.

Before giving the general construction, we present an example in Figure 8, which encodes the bridge decompositions of plabic graphs with trip permutation $\pi = (3, 4, 1, 2)$. The figure at the left shows the polytope obtained by taking the convex hull of the permutations

$$\{(z_1, z_2, z_3, z_4) \mid z_1 \geq 1, z_2 \geq 2, z_3 \leq 3, z_4 \leq 4\}.$$

The edge between two permutations is labeled by the *values* of the swapped entries. The figure at the right within Figure 8 shows the same polytope, but now vertices are labeled by affine permutations. A vertex which was labeled by z at the left is labeled at the right by the affinization $\widetilde{z^{-1}}$ of z^{-1}. Note that we took the inverse in order to get a vertex-labeling of the polytope such that edges correspond to the *positions* of the swapped entries, agreeing with the sequence of transpositions defined in Definition 2.18.

The main result of this section will imply that the shortest paths in the one-skeleton of the bridge polytope from "top" to "bottom" (from $(3, 4, 1, 2)$ to $(1, 2, 3, 4)$ using the vertex-labeling at the left, and from $(3, 4, 5, 6)$ to $(5, 6, 3, 4)$ using the

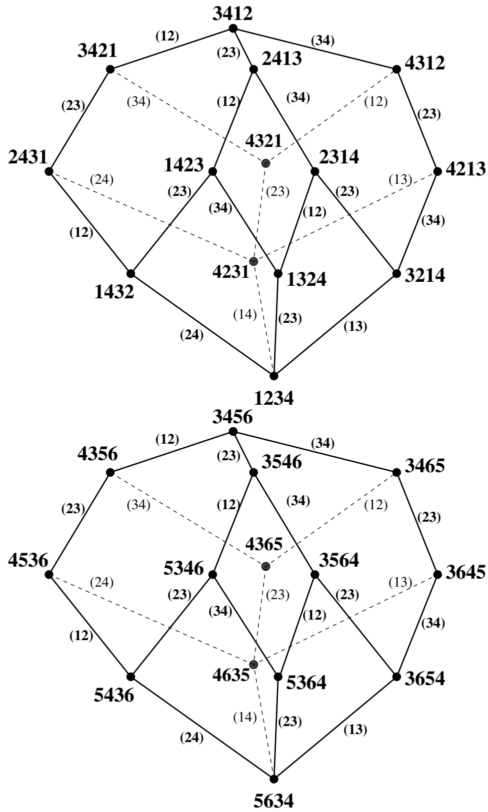

Figure 8. Two labelings of a bridge polytope. At the left, vertices are labeled by ordinary permutations, and edges are labeled by the pair of *values* which are swapped. At the right, vertices are labeled by affine permutations, and edges are labeled by the pair of *positions* which are swapped. The map from vertex-labels at the left to vertex-labels at the right is $z \mapsto \widetilde{z^{-1}}$. In both cases, the shortest paths along the one-skeleton from top to bottom encode bridge decompositions of the permutation $\pi_{2,4} = (3,4,1,2)$.

vertex-labeling at the right) are in bijection with the bridge decompositions of the reduced plabic graphs with trip permutation $\pi_{2,4} = (3,4,1,2)$. For example, the leftmost path from top to bottom is labeled by the sequence of transpositions (12), (23), (12), (24); the corresponding plabic graph is shown in Figure 9.

We now turn to the general construction.

Definition 3.1. Let $J \subset \{1,2,\dots,n\}$ and set

$$S_J = \{\pi \in S_n \mid \pi(j) \geq j \text{ for } j \in J \text{ and } \pi(j) \leq j \text{ for } j \notin J\}. \tag{1}$$

In other words, any permutation in S_J is required to have a *weak excedance* in position j for each $j \in J$, and a *weak nonexcedance* in position j for each $j \notin J$.

We define a polytope Br_J by

$$\mathrm{Br}_J = \mathrm{Conv}\{(\pi(1),\pi(2),\dots,\pi(n)) \mid \pi \in S_J\} \subset \mathbb{R}^n. \tag{2}$$

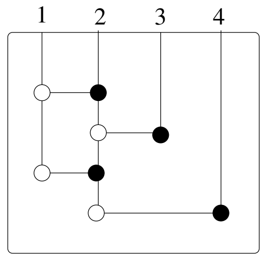

FIGURE 9. The plabic graph coming from the sequence of bridges (12), (23), (12), (24). Note that its trip permutation is $(3, 4, 1, 2) = (24)(12)(23)(12)$.

In the special case that $J = \{1, 2, \dots, k\}$, we write

$$\mathrm{Br}_{k,n} = \mathrm{Br}_J = \{\pi \in S_n \mid \pi(j) \geq j \text{ for } 1 \leq j \leq k \text{ and } \pi(j) \leq j \text{ for } k+1 \leq j \leq n\}.$$

Recall the definition of $\pi(J)$ from Definition 2.10, the decorated permutation labeling the corresponding TP Schubert cell. Let $e(J)$ be the decoration of the identity which has counterclockwise fixed points precisely in positions J.

Theorem 3.2 below is our main result; we will defer its proof to Section 4. We state Theorem 3.2 using the vertex-labeling of the bridge polytope with ordinary permutations, as in Definition 3.1; of course the result can also be stated using the vertex-labeling by affine permutations.

Theorem 3.2. *Let J be a k-element subset of $[n]$. The shortest paths from $\pi(J)$ to the identity permutation e along the one-skeleton of the bridge polytope Br_J are in bijection with the BCFW-bridge decompositions of the permutation $\pi(J)^{-1}$, where a sequence of edge-labels in a path is interpreted as a sequence of transpositions τ in the bridge decomposition. Equivalently, there is an edge between two vertices π and $\hat{\pi}$ of the bridge polytope Br_J if and only if there exists some pair $i < \ell$ such that $(i\ell)\pi = \hat{\pi}$, and $\pi(j) = \hat{\pi}(j) = j$ for $i < j < \ell$. In other words, π and $\hat{\pi}$ differ by swapping the values i and ℓ, and $i+1, i+2, \dots, \ell-1$ are fixed points of both π and $\hat{\pi}$.*

Because of Corollary 3.3 below, we think of the bridge polytope $\mathrm{Br}_{k,n}$ as a positive Grassmannian analogue of the permutohedron. (The notation $\pi_{k,n}$ comes from Definition 2.9.)

Corollary 3.3. *The shortest paths in the bridge polytope $\mathrm{Br}_{k,n}$ from $\pi_{k,n}$ down to the identity e are in bijection with the BCFW-bridge decompositions of the permutation $\pi_{n-k,n}$. In other words, we can read off all BCFW-bridge decompositions of $\pi_{n-k,n}$ by recording the edge-labels on all shortest paths from $\pi_{k,n}$ to e in the one-skeleton of $\mathrm{Br}_{k,n}$.*

4. Bruhat interval polytopes and the proof of Theorem 3.2

Bruhat interval polytopes are a class of polytopes which were recently studied by Kodama and the author in [KW13], in connection with the full Kostant-Toda lattice on the flag variety. The combinatorial properties of Bruhat interval polytopes were further investigated by Tsukerman and the author in [TW14]. In this section our main goal is to prove Theorem 3.2. We will also show that bridge polytopes are isomorphic to certain Bruhat interval polytopes, and we will deduce a characterization of the one-skeleta of a large class of Bruhat interval polytopes.

Definition 4.1. Let $u, v \in S_n$ such that $u \leq v$ in (strong) Bruhat order. The *Bruhat interval polytope* $\mathsf{Q}_{u,v}$ is defined as the convex hull

$$\mathsf{Q}_{u,v} = \mathrm{Conv}\{(z(1), \ldots, z(n)) \mid \text{ for } z \in S_n \text{ such that } u \leq z \leq v\}.$$

Note that when u is the identity and $v = w_0$, the longest permutation in S_n, the Bruhat interval polytope $\mathsf{Q}_{u,v}$ equals the permutohedron Perm_n.

Theorem 4.2 is one of the main results from [TW14]. It allows us to define a partial order in Definition 4.3 which, in the case that $u = e$ and $v = w_0$, generalizes the weak Bruhat order on permutations.

Theorem 4.2 ([TW14, Theorem 4.1])**.** *Every face of a Bruhat interval polytope is itself a Bruhat interval polytope. In particular, every edge of a Bruhat interval polytope corresponds to a cover relation in the strong Bruhat order.*

Definition 4.3. The *weak Bruhat interval order* on the permutations $[u, v] = \{z \mid u \leq z \leq v\}$ is the partial order determined by the following cover relations: $z \lessdot z'$ if and only if $\ell(z') = \ell(z) + 1$ and there is an edge between z and z' in the Bruhat interval polytope $\mathsf{Q}_{u,v}$. A *chain* between u and v in $\mathsf{Q}_{u,v}$ is a path in the one-skeleton of the polytope whose vertices correspond to a saturated chain in the weak Bruhat interval order on $[u, v]$.

Remark 4.4. Note that the weak Bruhat interval order inherits a grading from the length function. We will also refer to chains from u to v in $\mathsf{Q}_{u,v}$ as *shortest paths* from u to v in the one-skeleton of $\mathsf{Q}_{u,v}$, since they correspond to the paths that go through the minimal number of vertices.

Lemma 4.5. *The bridge polytope* Br_J *is affinely isomorphic to the Bruhat interval polytope* $\mathsf{Q}_{e,\pi(J)^{-1}}$ *by a map which simply permutes the coordinates of* $\mathbb{R}^n$*. More specifically, recall that for* J *a* k*-element subset of* $[n]$*, we write* $J = \{j_1 < \cdots < j_k\}$*, and* $J^c = \{h_1 < \cdots < h_{n-k}\}$*. Consider the map* $\psi : \mathbb{R}^n \to \mathbb{R}^n$ *defined by* $\psi(x_1, \ldots, x_n) = (x_{h_1}, x_{h_2}, \ldots, x_{h_{n-k}}, x_{j_1}, x_{j_2}, \ldots, x_{j_k})$*. Then* ψ *is an isomorphism from* Br_J *to* $\mathsf{Q}_{e,\pi(J)^{-1}}$*, which takes the vertices* $\pi(J)$ *and* e *of* $\mathrm{Br}_{k,n}$ *to the vertices* e *and* $\pi(J)^{-1}$ *of* $\mathsf{Q}_{e,\pi(J)^{-1}}$*, respectively.*

Proof. Since ψ simply permutes the coordinates of $\mathbb{R}^n$, it obviously acts as an affine isomorphism on any polytope. Moreover, ψ takes the vertex $(z(1), \ldots, z(n))$ labeled by the permutation z in $\mathrm{Br}_{k,n}$ to the vertex labeled by the permutation $z\pi(J)^{-1}$ of $\mathsf{Q}_{e,\pi(J)^{-1}}$. It is not hard to show that ψ maps the set of permutations S_J to the set of permutations in the interval $[e, \pi(J)^{-1}]$. □

Remark 4.6. By Remark 4.4 and Lemma 4.5, we can view the vertices of each bridge polytope Br_J as the vertices of a graded poset, with a greatest element $\pi(J)$ and a least element e.

Using Definition 4.3 and Lemma 4.5, we make the following definition.

Definition 4.7. A *chain* or *shortest path* between $\pi(J)$ and e in the bridge polytope Br_J is a path in the one-skeleton of Br_J whose vertices, after applying ψ, correspond to a chain between e and $\pi(J)^{-1}$ in $\mathsf{Q}_{e,\pi(J)^{-1}}$.

Once we have established Theorem 3.2, Lemma 4.5 immediately implies the following description of the one-skeleton of any Bruhat interval polytopes $\mathsf{Q}_{e,w}$, when w is a Grassmannian permutation.

Corollary 4.8. *Let w be a Grassmannian permutation in S_n, i.e., a permutation with at most one descent. Then there is an edge between vertices u and v in $\mathsf{Q}_{e,w}$ if and only if u and v satisfy the following properties:*

- $v = (i\ell)u$,
- *in the permutations u and v, each of the values $i+1, i+2, \dots, \ell-1$ are in precisely the same positions that they are in w.*

We now prove Theorem 3.2, by establishing Lemma 4.9, Proposition 4.10, and Theorem 4.11, below.

Lemma 4.9. *If there is an edge in Br_J between π and $\hat{\pi}$, then there exists some transposition $(i\ell)$ such that $(i\ell)\pi = \hat{\pi}$.*

Proof. By Theorem 4.2, every edge of a Bruhat interval polytope connects two vertices u and v, where $v = (i\ell)$ for some transposition $(i\ell)$. Lemma 4.9 now follows from Lemma 4.5. $\square$

Proposition 4.10. *Suppose there is an edge in Br_J between π and $\hat{\pi}$ where $(i\ell)\pi = \hat{\pi}$ for some $i < \ell$, i.e., π and $\hat{\pi}$ differ by swapping the values i and ℓ. Then $i+1, i+2, \dots, \ell-1$ must be fixed points of π and $\hat{\pi}$.*

Proof. We use a proof by contradiction. Suppose that not all of $i+1, i+2, \dots, \ell-1$ are fixed points. Let j be the smallest element of $\{i+1, i+2, \dots, \ell-1\}$ which is not a fixed point. Let a, b, and c be the positions of i, ℓ, and j in π.

Without loss of generality, $\pi_c = j > c$. So $c \in J$ and is a position of a weak excedance in any permutation in S_J. Since we have an edge in the polytope between π and $\hat{\pi}$, there exists a $\lambda \in \mathbb{R}^n$ such that $\lambda \cdot x := \sum_h \lambda_h x_h$ (applied to $x \in S_J$) is maximized precisely on π and $\hat{\pi}$. In particular, we must have $\lambda_a = \lambda_b$.

If $\lambda_a = \lambda_b < \lambda_c$, then define $\tilde{\pi}$ so that it agrees with π except in positions b and c, where we have $\tilde{\pi}_b = j$ and $\tilde{\pi}_c = \ell$. Since $\pi_b = \ell$ and $\hat{\pi}_b = i$ and $i < j < \ell$, permutations in S_J are allowed to have a bth coordinate of j. Since $j > c$ and $j < \ell$, we have $\ell > c$. So permutations in S_J are allowed to have a cth coordinate of ℓ. Therefore $\tilde{\pi} \in S_J$. But $\lambda \cdot \tilde{\pi} > \lambda \cdot \pi$, which is a contradiction.

If $\lambda_a = \lambda_b > \lambda_c$, then define $\tilde{\pi}$ so that it agrees with π except in positions a and c, where we have $\tilde{\pi}_a = j$ and $\tilde{\pi}_c = i$. Since $\pi_a = i$ and $\hat{\pi}_a = \ell$ and $i < j < \ell$, permutations in S_J are allowed to have an ath coordinate of j. Since $\pi_c = j > c$, and $i+1, i+2, \dots, j-1$ are fixed points of π, $c \notin \{i+1, i+2, \dots, j-1\}$. So $c \leq i$, and permutations in S_J are allowed to have a cth coordinate of i (recall that $c \in J$ and hence c must be a weak excedance in any element of S_J). Therefore $\tilde{\pi} \in S_J$. But $\lambda \cdot \tilde{\pi} > \lambda \cdot \pi$, which is a contradiction. $\square$

Theorem 4.11. *Consider the bridge polytope Br_J and two permutations π and $\hat{\pi}$ in $S_J \subset S_n$ such that $(i\ell)\pi = \hat{\pi}$ and $i+1, i+2, \dots, \ell-1$ are fixed points of π and $\hat{\pi}$. Choose a vector $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ whose coordinates come from the set $\{n^2, n^4, \dots, n^{2(n-1)}\}$, and such that*

$$\begin{aligned}\lambda_{\pi^{-1}(1)} < \lambda_{\pi^{-1}(2)} < \cdots < \lambda_{\pi^{-1}(i-1)} < \square < \lambda_{\pi^{-1}(i)} \\ = \lambda_{\pi^{-1}(\ell)} < \square < \lambda_{\pi^{-1}(\ell+1)} < \cdots < \lambda_{\pi^{-1}(n-1)} < \lambda_{\pi^{-1}(n)}.\end{aligned}$$

Here the $\square$ *on the first line represents the coordinates* $\lambda_{\pi^{-1}(j)}$ *for* $j \in J$ *and* $i < j < \ell$*, sorted from left to right in increasing order of* j*, and the* $\square$ *on the second line represents the coordinates* $\lambda_{\pi^{-1}(j)}$ *for* $j \notin J$ *and* $i < j < \ell$*, sorted from left to right in increasing order of* j*.*

Then as x *ranges over* S_J*, the dot product* $\lambda \cdot x$ *is maximized precisely on* π *and* $\hat{\pi}$*. In particular, there is an edge in* Br_J *between* π *and* $\hat{\pi}$*.*

Proof. To prove Theorem 4.11, we will first show that to solve for the $x \in S_J$ on which $\lambda \cdot x$ is maximized, it suffices to use the greedy algorithm. More precisely, we will show that we can construct such x by considering the values λ_j from largest to smallest: at the step where we consider λ_j, if that value λ_j is unique among $\{\lambda_1, \dots, \lambda_n\}$, then we need to maximize the coordinate x_j; and otherwise, if $\lambda_j = \lambda_{j'}$, we need to maximize the sum of the coordinates $x_j + x_{j'}$. After showing this, we will then demonstrate that when we run the greedy algorithm, we will get precisely the outcomes π and $\hat{\pi}$.

We start by showing that the greedy algorithm maximizes the dot product $\lambda \cdot x$ for $x \in S_J$. Recall that the coordinates of λ come from the set $\{n^2, \dots, n^{2(n-1)}\}$. Let $h \in [n]$ be such that $\lambda_h = n^{2(n-1)}$. We first suppose that λ_h is distinct from all other values λ_j. We claim that we need to maximize x_h subject to the condition $x = (x_1, \dots, x_n) \in S_J$. Let w be the maximum value that appears in the hth position of any permutation in S_J, and let $x \in S_J$ such that $x_h = w$. Let $y \in S_J$ be some other permutation with $y_h \le w - 1$. Then $\lambda \cdot x > \lambda_h x_h = wn^{2(n-1)}$. On the other hand, $\lambda \cdot y = \lambda_h y_h + \sum_{j \ne h} \lambda_j y_j \le (w-1)n^{2(n-1)} + (n-1) \cdot n \cdot n^{2(n-2)}$. Since $(w-1)n^{2(n-1)} + (n-1) \cdot n \cdot n^{2(n-2)} < wn^{2(n-1)}$, we have that $\lambda \cdot x > \lambda \cdot y$.

Now suppose that $\lambda_h = \lambda_{h'} = n^{2(n-1)}$ for some $h' \ne h$. We claim that we need to maximize $x_h + x_{h'}$ subject to the condition $x = (x_1, \dots, x_n) \in S_J$. Let $w = \max\{x_h + x_{h'} \mid x \in S_J\}$, and let $x \in S_J$ with $x_h + x_{h'} = w$. Let $y \in S_J$ be some other permutation with $y_h + y_{h'} \le w - 1$. Then $\lambda \cdot x > \lambda_h x_h + \lambda_{h'} x_{h'} = wn^{2(n-1)}$. On the other hand, $\lambda \cdot y = \lambda_h y_h + \lambda_{h'} y_{h'} + \sum_{j \ne h, j \ne h'} \lambda_j y_j \le (w-1)n^{2(n-1)} + n(n-2)n^{2(n-2)}$. Since $(w-1)n^{2(n-1)} + n(n-2)n^{2(n-2)} < wn^{2(n-1)}$, we are done.

Next we let $k \in [n]$ be such that $\lambda_k = n^{2(n-2)}$. In the case that there is no other $\lambda_j = n^{2(n-2)}$, we can use the same argument as above, and the inequality $(w-1)n^{2(n-2)} + (n-2) \cdot n \cdot n^{2(n-3)} < w \cdot n^{2(n-2)}$, to show that we now need to maximize the value of x_k subject to the conditions that we imposed on x in the previous paragraph. And in the case that we have $\lambda_k = \lambda_{k'} = n^{2(n-2)}$ for $k \ne k'$, we can use the same argument as above, and the inequality

$$(w-1)n^{2(n-2)} + n(n-3)n^{2(n-3)} < w \cdot n^{2(n-2)}$$

to show that we need to maximize the value of $x_k + x_{k'}$ subject to the conditions that we imposed on x in the previous paragraph.

Continuing in this fashion, we see that this greedy algorithm will compute all $x \in S_J$ such that $\lambda \cdot x$ is maximized.

Now we need to show that running our greedy algorithm produces precisely the permutations π and $\hat{\pi}$.

Step 1. Looking at the ordering of the coordinates of λ as described in Theorem 4.11, we first need to maximize $x_{\pi^{-1}(n)}$, then $x_{\pi^{-1}(n-1)}, \dots$, then $x_{\pi^{-1}(\ell+1)}$. Therefore we place n in position $\pi^{-1}(n)$, $n-1$ in position $\pi^{-1}(n-1)$, $\dots$, $\ell+1$ in position

$\pi^{-1}(\ell+1)$. (Note that there exist permutations $x \in S_J$ with these coordinates in these positions, since π and $\hat{\pi}$ are two such permutations.)

Step 2. Next we need to maximize the values that we put in positions $\pi^{-1}(j)$, for $i < j < \ell$ and $\pi^{-1}(j) = j \notin J$. But these are positions of weak nonexcedances in S_J, so the best we can do is to put fixed points there. Note that π and $\hat{\pi}$ also have fixed points in these positions, so the x we have built so far agrees with both π and $\hat{\pi}$.

Step 3. Now we want to maximize the sum of the values that we can put in positions $\pi^{-1}(i)$ and $\pi^{-1}(\ell)$. The values ℓ and i are unused, and ℓ is the greatest unused value so far, so one possibility is to put the values $\{i, \ell\}$ into positions $\{\pi^{-1}(i), \pi^{-1}(\ell)\}$. The two possible assignments of this form would agree thus far with either π or $\hat{\pi}$. We claim that these two possible assignments maximize the sum of the values that we can put in positions $\pi^{-1}(i)$ and $\pi^{-1}(\ell)$.

To prove the claim, we assume for the sake of contradiction that there is some other $x \in S_J$ that satisfies the constraints so far, and such that $x_{\pi^{-1}(i)} + x_{\pi^{-1}(\ell)} > i + \ell$. Since we have already placed the values $\{\ell+1, \ell+2, \dots, n\}$ and the values $\{j \mid j \notin J, i < j < \ell\}$ into the permutation we have built so far in Steps 1 and 2, the values in positions $\pi^{-1}(i)$ and $\pi^{-1}(\ell)$ in such an x would either be: (a) ℓ and k, for some $k \in J$ such that $i < k < \ell$, or (b) k' and k, for some $k, k' \in J$ such that $i < k' < k < \ell$. Note that by definition, any $x \in S_J$ needs to have weak excedances in positions $T := \{j \mid i < j < \ell, j \in J\}$. But in Case (a), the largest $|T|$ values remaining to place in those positions are $\{i\} \cup \{j \mid i < j < \ell, j \in J, j \neq k\}$. And in Case (b), the largest $|T|$ values remaining are $\{i\} \cup \{j \mid i < j < \ell, j \in J, j \neq k, j \neq k'\} \cup \{\ell\}$. In either case, it is impossible to find an assignment of values so that each position in T is a weak excedance. This proves the claim, showing that to maximize the sum $x_{\pi^{-1}(i)} + x_{\pi^{-1}(\ell)}$, we must put the values $\{i, \ell\}$ into positions $\pi^{-1}(i)$ and $\pi^{-1}(\ell)$.

Step 4. We now need to assign values to positions T. The largest $|T|$ values remaining are precisely the elements of T. So since $x \in S_J$ needs to have weak excedances in positions T, we need to put a fixed point in each position of T.

Step 5. Finally we want to maximize the values that we place in positions $\pi^{-1}(i-1), \dots, \pi^{-1}(1)$. The remaining unused values are $i-1, \dots, 1$. So for $i-1 \geq j \geq 1$, we place j in position $\pi^{-1}(j)$.

This greedy algorithm has constructed precisely two permutations, π and $\hat{\pi}$, and clearly $\lambda \cdot \pi = \lambda \cdot \hat{\pi}$. Therefore if we consider the values $\lambda \cdot x$ for $x \in S_J$, $\lambda \cdot x$ is maximized precisely on π and $\hat{\pi}$. It follows that there is an edge in Br_J between π and $\hat{\pi}$. □

5. The two-dimensional faces of bridge polytopes

In this section we describe the two-dimensional faces of bridge polytopes, and explain how they are related to the moves for reduced plabic graphs.

Theorem 5.1. *A two-dimensional face of a bridge polytope is either a square, a trapezoid, or a regular hexagon, with labels as in Figure* 10.

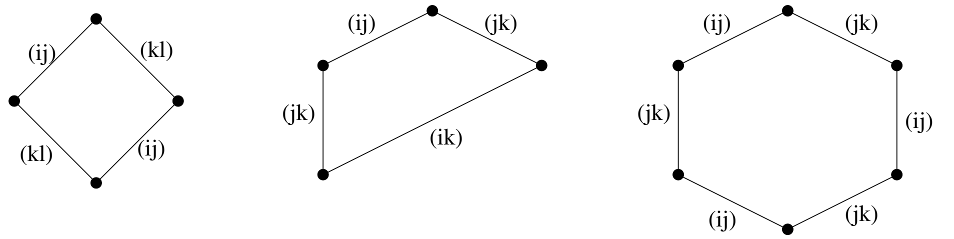

FIGURE 10. The possible two-dimensional faces of a bridge polytope. Here we have $i < j < k < l$ for the square, $i < j < k$ or $k < j < i$ for the trapezoid, and $i < j < k$ for the hexagon.

Proof. Consider a two-dimensional face of a bridge polytope. By Lemma 4.9, each edge is parallel to a vector of the form $e_i - e_\ell$, and has length an integer multiple of $\sqrt{2}$. A simple calculation with dot products shows that the possible angles among the edges are $\frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}$: more precisely, two vectors $e_i - e_j$ and $e_k - e_l$ (where i, j, k, l are distinct) have angle $\frac{\pi}{2}$; two vectors $e_i - e_j$ and $e_i - e_k$ have angle $\frac{\pi}{3}$; and two vectors $e_i - e_j$ and $-e_i + e_k$ have angle $\frac{2\pi}{3}$.

The only possibilities for such a polygon are: a square, a trapezoid (with angles $\frac{\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}$), a regular hexagon (all angles are $\frac{2\pi}{3}$), an equilateral triangle, or a parallelogram. In the first three cases, the labels on the edges of the polygons must be as in Figure 10. In the case of the square, it follows from the rules of bridge decompositions that the intervals $[i, j]$ and $[k, l]$ must be disjoint, and hence without loss of generality $i < j < k < l$. (E.g. k cannot lie in between i and j, because in order to perform the swap i and j, all elements between i and j must be fixed points. And we are not allowed to swap fixed points.) A similar argument explains the ordering on i, j, k for the trapezoid and the hexagon.

We now argue that it is impossible for a two-dimensional face to be a triangle or a parallelogram. Note that if one traverses any cycle in the one-skeleton of a bridge polytope, the product of the corresponding edge labels must be 1. It follows that there cannot be a two-face with an odd number of sides, because the product of an odd number of transpositions is an odd permutation, and hence is never the identity. Therefore an equilateral triangle is impossible. Finally consider the case of a face which is a parallelogram. Its edge labels must have alternating labels (ij), (ik), (ij), (ik). But the product $(ij)(ik)(ij)(ik)$ is not equal to the identity permutation. □

Remark 5.2. The same proof shows that any two-dimensional face of a Bruhat interval polytope is a square, a trapezoid, or a hexagon, as shown in Figure 10.

Theorem 5.3. *The three kinds of two-dimensional faces of bridge polytopes correspond to simple applications of the local moves for plabic graphs. More specifically, consider a shortest path p from $\pi_{k,n}$ (or more generally $\pi(J)$) to e in the one-skeleton of the bridge polytope $\mathrm{Br}_{k,n}$ (or more generally Br_J). Choose a two-dimensional face F such that p traverses half the sides of F, and modify p along F, obtaining a new path p' which goes around the other sides of F. Then the reduced plabic graphs corresponding to p and p' are either the same, or are related by the local moves for plabic graphs as in Figure* 11.

Proof. The proof of Theorem 5.3 is illustrated in Figure 11. The two reduced plabic graphs related by a square face in a bridge polytope are in fact the same, as shown in the top figure within Figure 11. The two plabic graphs related by a trapezoidal face are related by moves of type (M3), as shown in the middle figure. And the two plabic graphs related by a hexagonal face are related by a combination of moves (M1) and (M3), as shown in the bottom figure. Note that in the latter case, the dashed edge in Figure 11 must be present, or else the plabic graph associated to our bridge decomposition would not be reduced. ☐

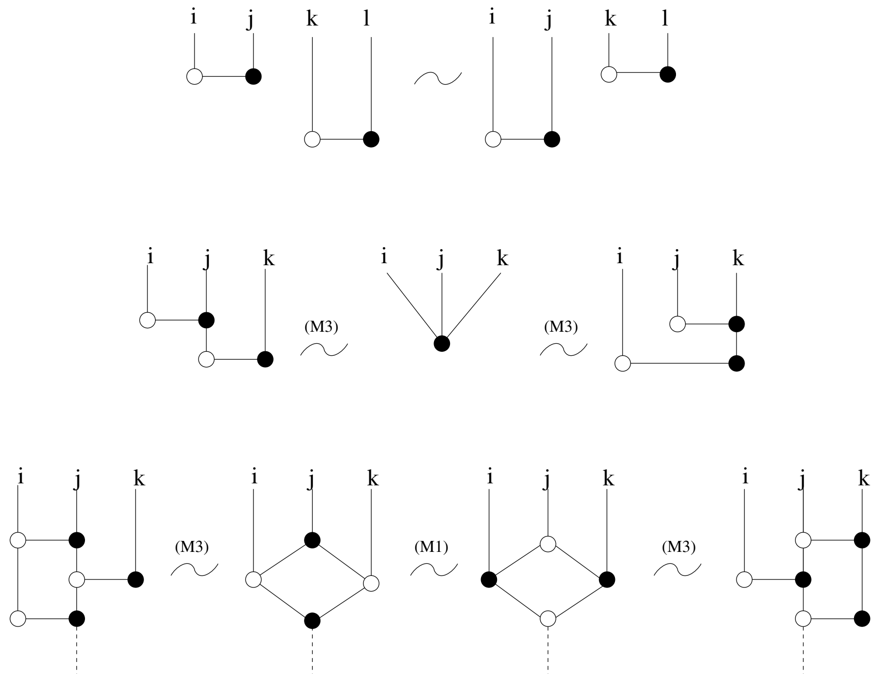

FIGURE 11. A graphical proof of Theorem 5.3.

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REFERENCES

[AHBC^{12}] Nima Arkani-Hamed, Jacob L. Bourjaily, Freddy Cachazo, Alexander B. Goncharov, Alexander Postnikov, and Jaroslav Trnka, *Scattering amplitudes and the positive Grassmannian*, 2012. Preprint, `arXiv:1212.5605`.

[Bjö84] A. Björner, *Posets, regular CW complexes and Bruhat order*, European J. Combin. **5** (1984), no. 1, 7–16, DOI 10.1016/S0195-6698(84)80012-8. MR746039 (86e:06002)

[BLVS^{99}] Anders Björner, Michel Las Vergnas, Bernd Sturmfels, Neil White, and Günter M. Ziegler, *Oriented matroids*, 2nd ed., Encyclopedia of Mathematics and its Applications, vol. 46, Cambridge University Press, Cambridge, 1999. MR1744046 (2000j:52016)

[BW82] Anders Björner and Michelle Wachs, *Bruhat order of Coxeter groups and shellability*, Adv. in Math. **43** (1982), no. 1, 87–100, DOI 10.1016/0001-8708(82)90029-9. MR644668 (83i:20043)

[Ede81] Paul H. Edelman, *The Bruhat order of the symmetric group is lexicographically shellable*, Proc. Amer. Math. Soc. **82** (1981), no. 3, 355–358, DOI 10.2307/2043939. MR612718 (82j:06004)

[FS00] Sergey Fomin and Michael Shapiro, *Stratified spaces formed by totally positive varieties*, Michigan Math. J. **48** (2000), 253–270, DOI 10.1307/mmj/1030132717. Dedicated to William Fulton on the occasion of his 60th birthday. MR1786489 (2001i:20095)

[Her14] Patricia Hersh, *Regular cell complexes in total positivity*, Invent. Math. **197** (2014), no. 1, 57–114, DOI 10.1007/s00222-013-0480-1. MR3219515

[KLS13] Allen Knutson, Thomas Lam, and David E. Speyer, *Positroid varieties: juggling and geometry*, Compos. Math. **149** (2013), no. 10, 1710–1752, DOI 10.1112/S0010437X13007240. MR3123307

[KW11] Yuji Kodama and Lauren K. Williams, *KP solitons, total positivity, and cluster algebras*, Proc. Natl. Acad. Sci. USA **108** (2011), no. 22, 8984–8989, DOI 10.1073/pnas.1102627108. MR2813307 (2012f:37142)

[KW13] Y. Kodama and L. Williams, *The full Kostant-Toda hierarchy on the positive flag variety*, 2013. Preprint, `arXiv:1308.5011`, to appear in Comm. Math. Phys.

[Lus94] G. Lusztig, *Total positivity in reductive groups*, Lie theory and geometry, Progr. Math., vol. 123, Birkhäuser Boston, Boston, MA, 1994, pp. 531–568. MR1327548 (96m:20071)

[Pos06] Alexander Postnikov, *Total positivity, Grassmannians, and networks*, 2006. Preprint, `arXiv:math/0609764`.

[Pro82] Robert A. Proctor, *Classical Bruhat orders and lexicographic shellability*, J. Algebra **77** (1982), no. 1, 104–126, DOI 10.1016/0021-8693(82)90280-0. MR665167 (84j:20044)

[Rie99] Konstanze Rietsch, *An algebraic cell decomposition of the nonnegative part of a flag variety*, J. Algebra **213** (1999), no. 1, 144–154, DOI 10.1006/jabr.1998.7665. MR1674668 (2000e:14086)

[Sco06] Joshua S. Scott, *Grassmannians and cluster algebras*, Proc. London Math. Soc. (3) **92** (2006), no. 2, 345–380, DOI 10.1112/S0024611505015571. MR2205721 (2007e:14078)

[TW14] Emmanuel Tsukerman and Lauren Williams, *Bruhat interval polytopes*, 2014. Preprint, `arXiv:1406.5202`.

[Wil07] Lauren K. Williams, *Shelling totally nonnegative flag varieties*, J. Reine Angew. Math. **609** (2007), 1–21, DOI 10.1515/CRELLE.2007.059. MR2350779 (2008m:14100)

Department of Mathematics, University of California, Berkeley, California 94720-3840

E-mail address: `williams@math.berkeley.edu`