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THE SUBPROJECTIVITY OF THE PROJECTIVE TENSOR PRODUCT OF TWO $C(K)$ SPACES WITH $|K| = \aleph_0$

ELÓI MEDINA GALEGO AND CHRISTIAN SAMUEL

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Abstract. We prove that the projective tensor product of two $C(K)$ spaces, where K is an infinite countable metric compact space, is c_0-saturated and is therefore a subprojective space. This completes some recent work on subprojectivity of projective tensor products involving $C(K)$ spaces by T. Oikhberg and E. Spinu.

1. Introduction and the main theorem

We use standard terminology and notation on Banach space theory; see e.g. [4]. Subspace means a closed linear infinite dimensional subspace. Operator means a bounded linear operator.

This paper is motivated by a recent and systematic study of subprojectivity on Banach spaces by T. Oikhberg and E. Spinu [7]. Recall that a Banach space X is called subprojective if any of its subspaces contain a further subspace complemented in X.

The simplest examples of subprojective Banach spaces are ℓ_p spaces $(1 \leq p < \infty)$, c_0 [9, Lemma 2] and the spaces $C(\alpha)$ of all scalar continuous functions defined on the interval of countable ordinals $[1, \alpha]$, endowed with the order topology [10, Main Theorem]. On the other hand, by Milutin's theorem, [11, Theorem 21.5.10] and [9, Corollary 2], the spaces $C(K)$ of all scalar countinuous functions defined on an uncountable compact metric space are not subprojective.

In their paper, T. Oikhberg and E. Spinu examine, among other things, the stability of the subprojectivity under projective and injective tensor products. Here we turn our attention to the subprojectivity of the projective tensor product $X\widehat{\otimes}Y$ of two subprojective Banach spaces X and Y [7, Question 3.12].

It is known that $\ell_p\widehat{\otimes}\ell_q$, $\ell_p\widehat{\otimes}c_0$ and $c_0\widehat{\otimes}c_0$ $(1 \leq p, q < \infty)$ are subprojective ([7, Corollary 3.3] and [8, Theorem 5]).

In order to state our main result we need to recall that a Banach space X is said to be c_0-saturated if any of its subspaces contain a subspace isomorphic to c_0. The spaces $C(\alpha)$ are also c_0-saturated for every countable ordinal α [10, Main Theorem].

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In this paper we use a method which is different from that used by Oikhberg-Spinu [7] and Oja [8]. Here we use Tong's results on diagonal block operators [14] and we show:

Theorem 1.1. *Let K_1 and K_2 be two countable compact metric spaces. Then $C(K_1)\widehat{\otimes}C(K_2)$ is c_0-saturated.*

So, we deduce easily from the theorem of Sobczyk [13] the "sufficient" part of the following theorem.

Theorem 1.2. *Let K_1 and K_2 be two compact metric spaces. Then $C(K_1)\widehat{\otimes}C(K_2)$ is subprojective if and only if K_1 and K_2 are countable.*

Denote by ω the first infinite ordinal and by ω_1 the first uncountable ordinal. By the classical theorem of Mazurkiewicz and Sierpiński [6] we know that an infinite countable compact metric space is homeomorphic to an interval $[1,\alpha]$, where $\omega \leq \alpha < \omega_1$.

Thus, we will establish Theorem 1.1 by proving:

Theorem 1.3. *Suppose that $\omega \leq \alpha, \beta < \omega_1$. Then $C(\alpha)\widehat{\otimes}C(\beta)$ is c_0-saturated.*

2. Some preliminary results

We start by recalling some basic facts on projective tensor products of Banach spaces [3], [12]. Let E, F be two Banach spaces and denote by $\mathcal{B}(E,F)$ the space of bounded bilinear functionals on $E \times F$. The projective tensor norm of $u = \sum_{i=1}^{n} a_i \otimes b_i \in E \otimes F$ is defined by

$$\|u\| = \sup\left\{ \left|\sum_{i=1}^{n} \varphi(a_i, b_i)\right| \,:\, \varphi \in \mathcal{B}(E,F),\ \|\varphi\| \leq 1 \right\}.$$

As usual, $E\widehat{\otimes}F$ denotes the completion of $E \otimes F$ with respect to the projective tensor norm. If necessary we denote by $\| \ \|_{\pi(E\otimes F)}$ the projective tensor norm on $E\widehat{\otimes}F$.

We recall that if M is a subspace of F, the norm induced by $\| \ \|_{\pi(E\otimes F)}$ on $E \otimes M$ is not necessarily equivalent to the norm $\| \ \|_{\pi(E\otimes M)}$. Nevertheless, we have the following result of Grothendieck ([3, Corollary 1, p. 40], [12, Proposition 2.4]) which will play an important role in this work.

Theorem 2.1. *Let E and F be Banach spaces. Suppose that M is a complemented subspace of F and P is a bounded linear projection from F onto M. Then for every $u \in E\widehat{\otimes}M$, we have $\|u\|_{\pi(E\otimes F)} \leq \|u\|_{\pi(E\otimes M)} \leq \|P\|\,\|u\|_{\pi(E\otimes F)}$.*

We need to introduce the definition of a diagonal block sequence used by Tong in his paper [14].

Throughout this paper we denote by $(P_m)_{m\geq 1}$ the sequence of natural projections associated to the unit vector basis of c_0. It is convenient to define $P_0 = 0$.

We say that a sequence $(u_k)_{k\geq 1}$ of $c_0\widehat{\otimes}c_0$ is a diagonal block sequence if there exist a strictly increasing $(n_k)_{k\geq 1}$ of integers such that $u_1 = (P_{n_1} \otimes P_{n_1})(u_1)$ and $u_k = ((P_{n_k} - P_{n_{k-1}}) \otimes (P_{n_k} - P_{n_{k-1}}))(u_k)$ for every integer $k \geq 2$. We prove Theorem 1.3 by a transfinite induction, and the first step is the following lemma.

Lemma 2.2. *$c_0\widehat{\otimes}c_0$ is c_0-saturated.*

Proof. For every integer n we denote $R_n = I_{c_0} \otimes P_n + P_n \otimes I_{c_0} - P_n \otimes P_n$.

We suppose that $c_0\widehat{\otimes}c_0$ is not c_0-saturated, so there exists a subspace E of $c_0\widehat{\otimes}c_0$ which does not contain a subspace isomorphic to c_0. We shall show that this leads to a contradiction.

Let $0 < \varepsilon < 1$. We construct by induction a normalized sequence $(x_k)_{k\geq 1}$ of E and a strictly increasing sequence $(n_k)_{k\geq 0}$ of integers such that $n_0 = 0$ which satisfy, for every integer $k \geq 1$,

$$\big\|((P_{n_k} - P_{n_{k-1}}) \otimes (P_{n_k} - P_{n_{k-1}}))(x_k) - x_k\big\| \leq \frac{\varepsilon}{2^k}. \tag{2.1}$$

To begin we choose $x_1 \in E$, $\|x_1\| = 1$. We have $x_1 = \lim_{n\to\infty}(P_n \otimes P_n)(x_1)$ so we can fix an integer n_1 such that

$$\|x_1 - (P_{n_1} \otimes P_{n_1})(x_1)\| \leq \frac{\varepsilon}{2}.$$

Now let $i \geq 1$ and suppose that we have a normalized finite sequence $(x_k)_{1\leq k\leq i}$ of E and a finite sequence of integers $n_1 < \cdots < n_i$ such that (2.1) is satisfied for $1 \leq k \leq i$.

The subspace $\operatorname{Im} R_{n_i}$ of $c_0\widehat{\otimes}c_0$ is isomorphic to c_0, and so it is c_0-saturated. It follows that R_{n_i} is not an isomorphism from E onto $R_{n_i}(E)$, so there exists $x_{i+1} \in E$ which satisfies $\|x_{i+1}\| = 1$ and $\|R_{n_i}(x_{i+1})\| \leq \varepsilon/2^{i+2}$. There exists an integer $n_{i+1} > n_i$ such that

$$\|x_{i+1} - (P_{n_{i+1}} \otimes P_{n_{i+1}})(x_{i+1})\| \leq \frac{\varepsilon}{2^{i+2}}.$$

It is easy to infer from the last inequalities that x_{i+1} satisfies (2.1). Let $z_k = (P_{n_k} - P_{n_{k-1}}) \otimes (P_{n_k} - P_{n_{k-1}})(x_k)$. The sequence $(z_k)_k$ is a seminormalized diagonal block sequence of $c_0\widehat{\otimes}c_0$, so by the result of Tong, [14, Theorem 4.6], it is equivalent to the unit basis of c_0. For $\varepsilon > 0$ small enough, $(x_k)_k$ is a basic sequence of E equivalent to the unit basis of c_0; hence we have a contradiction.

For the continuation it is convenient to introduce new notation. We denote by $C_0(\alpha)$ the subspace of $C(\alpha)$ given by $\{ f \in C(\alpha)\,;\, f(\alpha) = 0 \}$. According to [1, Lemma 1] $C_0(\alpha)$ is isomorphic to $C(\alpha)$ for $\omega \leq \alpha$. Let $0 \leq \beta < \gamma < \alpha$. We denote this by

$$C([\beta + 1, \gamma]) = \{ f \in C_0(\alpha)\,;\, f = f 1_{[\beta+1,\gamma]} \},$$

where $1_{[\beta+1,\gamma]}$ is the characteristic function of the interval $[\beta+1, \gamma]$. Let $1 \leq \gamma \leq \alpha$ be an ordinal. We denote by S_γ the operator from $C_0(\alpha)$ to $C_0(\alpha)$ defined by $S_\gamma(f) = f 1_{[1,\gamma]}$.

For every $(m, \gamma) \in [1, \omega) \times [1, \alpha)$, we denote by $T_{m,\gamma}$ the operator of $c_0\widehat{\otimes}C_0(\alpha)$ given by

$$P_m \otimes I_{C_0(\alpha)} + I_{c_0} \otimes S_\gamma - P_m \otimes S_\gamma.$$

We know [2, Lemma 5.1] that $\operatorname{Im} T_{m,\gamma}$ is isomorphic to $C_0(\alpha) \times c_0\widehat{\otimes}C(\gamma)$.

The previous lemma is the first step $\alpha = \omega$ of the proof by a transfinite induction of the following lemma.

Lemma 2.3. *For every ordinal number $\omega \leq \alpha < \omega_1$, the space $c_0\widehat{\otimes}C_0(\alpha)$ is c_0-saturated.*

Proof. Let α be a countable ordinal such that, for every $\omega \leq \gamma < \alpha$, the space $c_0\widehat{\otimes}C_0(\gamma)$ is c_0-saturated. It is obvious that $c_0\widehat{\otimes}C_0(\alpha)$ is c_0-saturated if α is a successor. Suppose now that α is a limit ordinal and that $c_0\widehat{\otimes}C_0(\alpha)$ is not c_0-saturated. This means that there exists a subspace E of $c_0\widehat{\otimes}C_0(\alpha)$ which does not contain a subspace isomorphic to c_0. Let $0 < \varepsilon < 1$. We construct by induction a normalized sequence $(x_i)_{i\geq 1}$ of E, a strictly increasing sequence $(m_i)_{i\geq 1}$ of integers and a strictly increasing sequence $(\gamma_i)_{i\geq 1}$ of $[1,\alpha)$ such that

$$\|x_1 - (P_{m_1} \otimes S_{\gamma_1})(x_1)\| \leq \frac{\varepsilon}{2} \tag{2.2}$$

and, for every integer $i \geq 2$,

$$\|x_i - ((P_{m_i} - P_{m_{i-1}}) \otimes (S_{\gamma_i} - S_{\gamma_{i-1}}))(x_i)\| \leq \frac{\varepsilon}{2^i}\,. \tag{2.3}$$

We choose $x_1 \in E$, $\|x_1\| = 1$. We know that $x_1 = \lim_{m\to\infty,\gamma\to\alpha}(P_m \otimes S_\gamma)(x_1)$, so we may fix an integer m_1 and an ordinal $\gamma_1 < \alpha$ such that (2.2) holds. Let i be an integer ≥ 1 and suppose we have a finite normalized sequence $(x_k)_{1\leq k\leq i}$ of E, a finite sequence of integers $m_1 < \cdots < m_i$ and a finite sequence $\gamma_1 < \cdots < \gamma_i$ of $[1,\alpha)$ such that (2.2) and (2.3) are satisfied for $k = 1,\ldots,i$. It follows from our assumption and from [5, Lemma 1] that $C_0(\alpha) \times c_0\widehat{\otimes}C(\gamma)$ is c_0-saturated. Hence T_{m_i,γ_i} is not an isomorphism from E onto its image so there exists $x_{i+1} \in E$ such that $\|x_{i+1}\| = 1$ and

$$\|T_{m_i,\gamma_i}(x_{i+1})\| \leq \varepsilon/2^{i+2}. \tag{2.4}$$

There exist an integer $m_{i+1} > m_i$ and an ordinal $\gamma_i < \gamma_{i+1} < \alpha$ such that

$$\|x_{i+1} - (P_{m_{i+1}} \otimes S_{\gamma_{i+1}})(x_{i+1})\| \leq \varepsilon/2^{i+2}. \tag{2.5}$$

We deduce easily from (2.4) and (2.5) that (2.3) holds for $i+1$. Let $z_1 = (P_{m_1} \otimes S_{\gamma_1})(x_1)$ and, for $i \geq 2$, $z_i = (P_{m_i} - P_{m_{i-1}}) \otimes (S_{\gamma_i} - S_{\gamma i_1})(x_i)$. The sequence $(z_i)_{i\geq 1}$ is a seminormalized diagonal block sequence of $c_0\widehat{\otimes}C_0(\alpha)$, and thus, by [2, Theorem 4.2], it is equivalent to the unit basis of c_0. For $\varepsilon > 0$ small enough the sequence $(x_i)_{i\geq 1}$ is a basic sequence of E equivalent to $(z_i)_{i\geq 1}$; hence we have a contradiction. ☐

3. Diagonal block sequences in $C_0(\alpha)\widehat{\otimes}C_0(\beta)$ spaces

In order to prove the result announced we introduce diagonal block sequences in $C_0(\alpha)\widehat{\otimes}C_0(\beta)$. Let $(\gamma_i)_{i\geq 1}$ be a strictly increasing sequence of $[1,\alpha)$ and $(\theta_i)_{i\geq 1}$ a strictly increasing sequence of $[1,\beta)$. We denote by R_{θ_i} the operator of $C_0(\beta)$ defined by $R_{\theta_i}(g) = g1_{[1,\theta_i]}$ for every $g \in C_0(\beta)$.

Definition 3.1. For every $1 \leq k < \omega$, we denote by Π_k the operator of $C_0(\alpha)\widehat{\otimes}C_0(\beta)$ given by

$$\Pi_k(u) = \begin{cases} (S_{\gamma_1} \otimes R_{\theta_1})(u) & \text{if } k = 1, \\ ((S_{\gamma_k} - S_{\gamma_{k-1}}) \otimes (R_{\theta_k} - R_{\theta_{k-1}}))(u) & \text{if } k \geq 2. \end{cases}$$

We say that a sequence $(f_k)_{k\geq 1}$ of $C_0(\alpha)\widehat{\otimes}C_0(\beta)$ is a diagonal block sequence associated to the sequences $(\theta_k)_{k\geq 1}$ and $(\gamma_k)_{k\geq 1}$ if $\Pi_k(f_k) = f_k$ for every integer k.

Lemma 3.2. *Let $\omega \leq \alpha,\beta < \omega_1$. Every normalized diagonal block sequence of $C_0(\alpha)\widehat{\otimes}C_0(\beta)$ is equivalent to the unit basis of c_0.*

Proof. Let $(f_k)_{k\geq 1}$ be a normalized diagonal block sequence of $C_0(\alpha)\widehat{\otimes}C_0(\beta)$ associated to the sequences $(\theta_k)_{k\geq 1}$ and $(\gamma_k)_{k\geq 1}$. Let $0<\varepsilon<1$. For every integer $k\geq 1$ we fix $g_k\in C_0(\alpha)\otimes C_0(\beta)$ such that $\|f_k-g_k\|\leq \varepsilon/2^k$. We denote $h_k=\Pi_k(g_k)$; it is obvious that $\|f_k-h_k\|\leq\varepsilon/2^k$. For every integer k there exist an integer n_k and $u_1^k,\ldots,u_{n_k}^k\in C_0(\alpha), v_1^k,\ldots,v_{n_k}^k\in C_0(\beta)$ such that $h_k=\sum_{n=1}^{n_k}u_n^k\otimes v_n^k$. We may suppose that $(S_{\gamma_k}-S_{\gamma_{k-1}})(u_n^k)=u_n^k$ and $(R_{\theta_k}-R_{\theta_{k-1}})(v_n^k)=v_n^k$ for every integer $1\leq n\leq n_k$. $\square$

The subspace

$$E_k=\{\,u\in C_0(\alpha)\,;\,u=(S_{\gamma_k}-S_{\gamma_{k-1}})(u)\,\}$$

is a $\mathcal{L}_{\infty,1+\varepsilon}$ space for every $\varepsilon>0$, so there exists a finite dimensional subspace X_k of E_k such that $u_1^k,\ldots,u_{n_k}^k\in X_k$, $d(X_k,\ell_\infty^{d_k})\leq 2$ and a projection π_k of E_k onto X_k, $\|\pi_k\|\leq 2$, where d_k is the dimension of X_k.

In the same way the subspace

$$F_k=\{\,v\in C_0(\beta)\,;\,v=(R_{\theta_k}-R_{\theta_{k-1}})(v)\,\}$$

is a $\mathcal{L}_{\infty,1+\varepsilon}$ space for every $\varepsilon>0$, so there exists a finite dimensional subspace Y_k of F_k such that $v_1^k,\ldots,v_{n_k}^k\in Y_k$, $d(Y_k,\ell_\infty^{d_k'})\leq 2$ and a projection π_k' of F_k onto Y_k, $\|\pi_k'\|\leq 2$, where d_k' is the dimension of Y_k.

Let N be an integer. We denote $X=X_1+\cdots+X_N,\ Y=Y_1+\cdots+Y_N,\ d=d_1+\cdots+d_N,\ d'=d_1'+\cdots+d_N'$. It is clear that $d(X,\ell_\infty^{d})\leq 2$ and $d(Y,\ell_\infty^{d'})\leq 2$.

Let π be the operator from $C_0(\alpha)$ onto X defined for every u by

$$\pi(u)(\gamma)=\begin{cases}\pi_1S_{\gamma_1}(u)(\gamma) & \text{if } 1\leq\gamma\leq\gamma_1,\\ \pi_k(S_{\gamma_k}-S_{\gamma_{k-1}})(u)(\gamma) & \text{if } \gamma_{k-1}+1\leq\gamma\leq\gamma_k \text{ and } 2\leq k\leq N,\end{cases}$$

where π is a projection and $\|\pi\|\leq 2$.

In the same way the operator π' from $C_0(\beta)$ onto Y defined for every v by

$$\pi'(v)(\gamma)=\begin{cases}\pi_1'R_{\theta_1}(v)(\gamma) & \text{if } 1\leq\gamma\leq\gamma_1\\ \pi_k'(R_{\theta_k}-R_{\theta_{k-1}})(v)(\gamma) & \text{if } \gamma_{k-1}+1\leq\gamma\leq\gamma_k \text{ and } 2\leq k\leq N\end{cases}$$

is a projection and $\|\pi'\|\leq 2$.

The operator $\pi\otimes\pi'$ is a projection onto $X\otimes Y$ of norm ≤ 4.

Let $\lambda_1,\ldots,\lambda_N$ be scalars; by [3, Corollaire 1, p. 40] we have

$$\Big\|\sum_{i=1}^N\lambda_ih_i\Big\|_{\pi(C_0(\alpha)\otimes C_0(\beta))}\leq\Big\|\sum_{i=1}^N\lambda_ih_i\Big\|_{\pi(X\otimes Y)}\leq 4\Big\|\sum_{i=1}^N\lambda_ih_i\Big\|_{\pi(C_0(\alpha)\otimes C_0(\beta))}.$$

Now we use Tong's result to compute $\Big\|\sum_{i=1}^N\lambda_ih_i\Big\|_{\pi(X\otimes Y)}$.

For every integer $1\leq i\leq N$, let $T_i:X_i\to\ell_\infty^{d_i}$ (resp. $T_i':Y_i\to\ell_\infty^{d_i'}$) be an isomorphism such that $\|T_i\|=1$ and $\|T_i^{-1}\|\leq 2$ (resp. $\|T_i'\|=1$ and $\|T_i'^{-1}\|\leq 2$). The operator $T\otimes T':X\widehat{\otimes}Y\to\ell_\infty^d\widehat{\otimes}\ell_\infty^{d'}$ is an isomorphism such that $\|T\otimes T'\|=1$ and $\|T^{-1}\otimes T'^{-1}\|\leq 4$. We have

$$\frac14\Big\|\sum_{i=1}^N\lambda_ih_i\Big\|_{\pi(X\otimes Y)}\leq\Big\|\sum_{i=1}^N\lambda_i(T\otimes T')(h_i)\Big\|_{\pi(\ell_\infty^d\otimes\ell_\infty^{d'})}\leq\Big\|\sum_{i=1}^N\lambda_ih_i\Big\|_{\pi(X\otimes Y)}.$$

The sequence $((T \otimes T')(h_i))_{1\le i\le N}$ is a diagonal block sequence of $\ell_\infty^d \widehat{\otimes} \ell_\infty^{d'}$ and so, by Tong's result,

$$\left\| \sum_{i=1}^{N} \lambda_i (T \otimes T')(h_i) \right\|_{\pi(\ell_\infty^d \otimes \ell_\infty^{d'})} = \max_{1\le i\le N} |\lambda_i| \, \|(T \otimes T')(h_i)\|.$$

Hence $(h_i)_{i\geq 1}$ is a seminormalized basic sequence equivalent to the unit sequence of c_0.

For $\varepsilon > 0$ small enough the sequence $(f_i)_{i\geq 1}$ is equivalent to $(h_i)_{i\geq 1}$ and thus equivalent to the unit basis of c_0.

4. Proof of the main theorem

We are now ready to prove Theorem 1.3 and the "necessary" part of Theorem 1.2.

Proof of Theorem 1.3. For every ordinal $\omega \le \beta < \omega_1$ we consider the property

$(\mathcal{P}_\beta)$ for every $\omega \le \alpha < \omega_1$ the space $C(\alpha)\widehat{\otimes}C(\beta)$ is c_0-saturated.

We show by a transfinite induction that $(\mathcal{P}_\beta)$ is true for every β. The property $(\mathcal{P}_\omega)$ is true by Lemma 2.2.

We suppose that $\omega < \beta < \omega_1$ is a limit ordinal and that $(\mathcal{P}_\theta)$ is true for every $\omega \le \theta < \beta$.

Now we show, by a transfinite induction on the ordinal $\alpha \in \langle \omega, \omega_1)$, that $C_0(\alpha)\widehat{\otimes} C_0(\beta)$ is c_0-saturated. The result is true if $\alpha = \omega$.

We suppose that α is a limit ordinal and that $C_0(\gamma)\widehat{\otimes}C_0(\beta)$ is c_0-saturated for every $\omega \le \gamma < \alpha$.

We suppose also that $C_0(\alpha)\widehat{\otimes}C_0(\beta)$ is not c_0-saturated and show that this assumption leads to a contradiction.

So there exists a subspace E of $C_0(\alpha)\widehat{\otimes}C_0(\beta)$ which does not contain a subspace isomorphic to c_0. Let $0 < \varepsilon < 1$. It is easy to construct by induction a normalized sequence $(f_i)_{i\geq 1}$ of E, a strictly increasing sequence $(\gamma_i)_{i\geq 1}$ of $\langle 1, \alpha\rangle$ and a strictly increasing sequence $(\theta_i)_{i\geq 1}$ of $\langle 1, \beta\rangle$ such that

- $\|f_1 - (S_{\gamma_1} \otimes R_{\theta_1})(f_1)\| \le \varepsilon/2$,
- for every integer $i \geq 2$, $\|f_i - ((S_{\gamma_i} - S_{\gamma_{i-1}}) \otimes (R_{\theta_i} - R_{\theta_{i-1}}))(f_i)\| \le \varepsilon/2^i$.

It follows by Lemma 3.2 that for $\varepsilon > 0$ small enough the sequence $(f_i)_{i\geq 1}$ is equivalent to the unit basis of c_0 in contradiction with our assumption.

Proof of the "necessary" part of Theorem 1.2. Let K_1 and K_2 be two infinite compact metric spaces and suppose that $C(K_1)\widehat{\otimes}C(K_2)$ is subprojective. It follows that $C(K_1)\widehat{\otimes}c_0$ is subprojective, so by [7, Proposition 4.2.] the compact K_1 is scattered.

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Department of Mathematics, University of São Paulo, São Paulo, Brazil 05508-090
E-mail address: `eloi@ime.usp.br`

I2M, Aix Marseille Université, CNRS, UMR 7353, 13453 Marseille Cedex 20, France
E-mail address: `christian.samuel@univ-amu.fr`