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QUASICONFORMAL EXTENSION OF MEROMORPHIC FUNCTIONS WITH NONZERO POLE

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ABSTRACT. In this note, we consider meromorphic univalent functions $f(z)$ in the unit disc with a simple pole at $z = p \in (0, 1)$ which have a k-quasiconformal extension to the extended complex plane $\widehat{\mathbb{C}}$, where $0 \le k < 1$. We denote the class of such functions by $\Sigma_k(p)$. We first prove an area theorem for functions in this class. Next, we derive a sufficient condition for meromorphic functions in the unit disc with a simple pole at $z = p \in (0, 1)$ to belong to the class $\Sigma_k(p)$. Finally, we give a convolution property for functions in the class $\Sigma_k(p)$.

1. INTRODUCTION

Let $\mathbb{C}$ denote the complex plane and $\widehat{\mathbb{C}}$ denote the extended complex plane $\mathbb{C} \cup \{\infty\}$. We shall use the following notation: $\mathbb{D} = \{z : |z| < 1\}$, $\partial\mathbb{D} = \{z : |z| = 1\}$, $\overline{\mathbb{D}} = \{z : |z| \le 1\}$, $\mathbb{D}^* = \{z : |z| > 1\}$, $\overline{\mathbb{D}^*} = \{z : |z| \ge 1\}$. Let f be a meromorphic and univalent function in the unit disc $\mathbb{D}$ with a simple pole at $z = p \in [0, 1)$ of residue 1. Since $f(z) - 1/(z - p)$ is analytic in $\mathbb{D}$, one has an expression of the form

$$f(z) = \frac{1}{z - p} + \sum_{n=0}^{\infty} a_n z^n, \quad z \in \mathbb{D}. \tag{1.1}$$

We denote the class of such functions by $\Sigma(p)$. Let $\Sigma^0(p)$ be the subclass of $\Sigma(p)$ consisting of those functions f for which $a_0 = 0$ in the above expansion. Note that if $f, g \in \Sigma^0(p)$ are related by $g = M \circ f$ for a Möbius transformation M, then $f = g$.

For a given number $0 \le k < 1$, $\Sigma_k(p)$ stands for the class of those functions in $\Sigma(p)$ which admit k-quasiconformal extension to the extended plane $\widehat{\mathbb{C}}$. Here, a mapping $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is called k-quasiconformal if F is a homeomorphism and has locally L^2-derivatives on $\mathbb{C} \setminus \{F^{-1}(\infty)\}$ (in the sense of distribution) satisfying $|\bar{\partial}F| \le k|\partial F|$ a.e., where $\partial F = \partial F/\partial z$ and $\bar{\partial}F = \partial F/\partial \bar{z}$. Note that such an F is called K-quasiconformal more often, where $K = (1 + k)/(1 - k) \ge 1$, in the literature. The quantity $\mu = \bar{\partial}F/\partial F$ is called the complex dilatation of F. See the standard textbook [5] by Lehto and Virtanen for basic properties of quasiconformal mappings. Set $\Sigma_k^0(p) = \Sigma^0(p) \cap \Sigma_k(p)$.

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O. Lehto [3] refined the Bieberbach-Gronwall area theorem to the functions in $\Sigma_k(0)$ in the following form.

Theorem A. *Let $0 \le k < 1$. Suppose that $f(z) = z^{-1} + a_0 + a_1 z + a_2 z^2 + \dots$ is a function in $\Sigma_k(0)$. Then*

$$\sum_{n=1}^{\infty} n|a_n|^2 \le k^2.$$

Here, equality holds if and only if

$$f(z) = \frac{1}{z} + a_0 + a_1 z, \quad z \in \mathbb{D},$$

with $|a_1| = k$. Moreover, its k-quasiconformal extension is given by setting

$$f(z) = \frac{1}{z} + a_0 + \frac{a_1}{\bar{z}} \qquad \text{for } z \in \overline{\mathbb{D}^*}.$$

On the other hand, the area theorem was extended by P. N. Chichra [1] to functions in $\Sigma(p)$ as follows.

Theorem B. *Let $f \in \Sigma(p)$ have the expansion in* (1.1). *Then*

$$\sum_{n=1}^{\infty} n|a_n|^2 \le \frac{1}{(1-p^2)^2}. \tag{1.2}$$

Equality holds for the function

$$f_p(z) = \frac{1}{z-p} + a_0 + \frac{z}{1-p^2}.$$

Our first result establishes an area theorem for the class $\Sigma_k(p)$. Interestingly, the form of extremal functions is different from that of the function f_p in Theorem B.

Theorem 1. *Let $0 \le k < 1$ and $0 \le p < 1$. Suppose that $f \in \Sigma_k(p)$ is expressed in the form of* (1.1). *Then*

$$\sum_{n=1}^{\infty} n|a_n|^2 \le \frac{k^2}{(1-p^2)^2}. \tag{1.3}$$

Here, equality holds if and only if f is of the form

$$f(z) = \frac{1}{z-p} + a_0 + \frac{a_1 z}{1-pz} \qquad \text{for } z \in \mathbb{D}, \tag{1.4}$$

where a_0 and a_1 are constants with $|a_1| = k$. Moreover, a k-quasiconformal extension of this f is given by setting

$$f(z) = \frac{1}{z-p} + a_0 + \frac{a_1}{\bar{z}-p} \qquad \text{for } z \in \overline{\mathbb{D}^*}. \tag{1.5}$$

Observe that this is a natural extension of Theorem A. We remark that the function in (1.4) belongs to $\Sigma(p)$ as long as $|a_1| \le 1$ (see the latter part of the proof of Theorem 1 below). This function with $|a_1| = 1$ provides another extremal case in (1.2). As an immediate corollary of the theorem, we obtain the following.

Corollary 1. *Let $0 < p < 1$ and $0 < k < 1$. For $f \in \Sigma_k(p)$ with the expansion* (1.1)*, the following inequality holds:*

$$|a_1| < \frac{k}{1-p^2}.$$

Note that the inequality $|a_1| \le 1/(1-p^2)$ for $f \in \Sigma(p)$ is sharp in view of Theorem B. We have no exact value of the best upper bound, say, $M(p,k)$ of $|a_1|$ for $f \in \Sigma_k(p)$. The extremal function in Theorem 1 and compactness of the class $\Sigma_k^0(p)$ yield, at least, the estimates $k \le M(p,k) < k/(1-p^2)$ for $p, k \in (0,1)$.

Secondly, we provide a sufficient condition for functions of the form (1.1) to belong to the class $\Sigma_k(p)$.

Theorem 2. *Let $0 \le k < 1$ and $0 \le p < 1$. Suppose that ω is an analytic function in $\mathbb{D}$ such that $|\omega'(z)| \le k(1+p)^{-2}$ for $z \in \mathbb{D}$. Then the function f given by*

$$f(z) = \frac{1}{z-p} + \omega(z), \quad z \in \mathbb{D},$$

is a member of $\Sigma_k(p)$. A k-quasiconformal extension is given by setting

$$f(z) = \frac{1}{z-p} + \omega(1/\bar{z}), \quad \text{for } z \in \mathbb{D}^*. \tag{1.6}$$

We note that J. G. Krzyż [2] proved this theorem when $p = 0$. He also gave a convolution theorem in the same paper [2]. We can also extend it to a modified convolution. The modified Hadamard product (or the modified convolution) $f \star g$ of two functions $f, g \in \Sigma(p)$ with expansions

$$f(z) = \frac{1}{z-p} + \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \frac{1}{z-p} + \sum_{n=0}^{\infty} b_n z^n, \quad z \in \mathbb{D} \tag{1.7}$$

is defined by

$$(f \star g)(z) = \frac{1}{z-p} + \sum_{n=0}^{\infty} a_n\, b_n z^n, \quad z \in \mathbb{D}. \tag{1.8}$$

Our third result concerns this Hadamard product.

Theorem 3. *Let $f \in \Sigma_{k_1}(p)$ and $g \in \Sigma_{k_2}(p)$ for some $k_1, k_2, p \in [0,1)$. If $\alpha = k_1 k_2 (1-p)^{-2} < 1$, then the modified Hadamard product $f \star g$ belongs to $\Sigma_\alpha(p)$.*

As we mentioned above, this result reduces to a theorem due to Krzyż [2] when $p = 0$.

We prove Theorem 1 in Section 2 and Theorems 2 and 3 in Section 3. We also give another proof of a part of Theorem 1 as a concluding remark in Section 3.

2. Proof of Theorem 1

We start the section by proving the area theorem for functions in the class $\Sigma_k(p)$. We follow the idea due to Lehto [3].

Proof of Theorem 1. Let $f \in \Sigma_k(p)$ have the expansion in (1.1). We may suppose that f is already extended to a k-quasiconformal mapping of $\widehat{\mathbb{C}}$ onto itself. If $k = 0$, then the assertion clearly holds. Hence, we assume that $k > 0$ in the rest of the proof. To start with, we first make a change of variables. We define $\phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ by

$\phi(\zeta) = f(1/\zeta)$. Note that ϕ has locally L^2-derivatives on $\mathbb{C}\setminus\{\phi^{-1}(\infty)\} = \mathbb{C}\setminus\{1/p\}$. Since the function $\psi(\zeta) = \phi(\zeta) - \zeta/(1-p\zeta)$ has the expression

$$\psi(\zeta) = \phi(\zeta) - \frac{\zeta}{1-p\zeta} = \sum_{n=0}^{\infty} \frac{a_n}{\zeta^n}, \quad \zeta \in \mathbb{D}^*, \tag{2.1}$$

and, in particular, is bounded and analytic near the point $\zeta = 1/p$, the function ψ has locally L^2-derivatives on $\mathbb{C}$. Therefore, for every $r > 0$, we can apply the Cauchy-Pompeiu formula (see [5, III §7] for details) to the function ψ in the disc $|\zeta| < r$ to obtain

$$\psi(\zeta) = \frac{1}{2\pi i}\int\limits_{|w|=r} \frac{\psi(w)}{w-\zeta}\,dw - \frac{1}{\pi}\iint\limits_{|w|<r} \frac{\bar\partial\psi(w)}{w-\zeta}\,dudv,$$

where $w = u + iv$. We note that $\psi(\zeta) \to a_0$ as $\zeta \to \infty$ and $\bar\partial\psi(\zeta) = 0$ for $|\zeta| > 1$. Letting $r \to +\infty$, we thus get

$$\psi(\zeta) = a_0 - \frac{1}{\pi}\iint\limits_{|w|<1} \frac{\bar\partial\psi(w)}{w-\zeta}\,dudv, \quad \zeta \in \mathbb{C}. \tag{2.2}$$

We differentiate the above expression with respect to ζ and obtain

$$\partial\psi(\zeta) = -\frac{1}{\pi}\iint\limits_{|w|<1} \frac{\bar\partial\psi(w)}{(w-\zeta)^2}\,dudv = H[\bar\partial\psi](\zeta), \quad \zeta \in \mathbb{C}, \tag{2.3}$$

where H is the two dimensional Hilbert transformation. (Strictly speaking, the above integral should be understood as Cauchy's principal value for $|\zeta| \le 1$. See [5, III §7] for details.) Since H is a linear isometry of $L^2(\mathbb{C})$, in conjunction with (2.3), we have

$$\iint\limits_{\mathbb{D}} |\bar\partial\psi(\zeta)|^2\,d\xi d\eta = \iint\limits_{\mathbb{C}} \left|H[\bar\partial\psi](\zeta)\right|^2 d\xi d\eta = \iint\limits_{\mathbb{C}} |\partial\psi(\zeta)|^2\,d\xi d\eta, \tag{2.4}$$

where $\zeta = \xi + i\eta$. Next, we recall that Chichra indeed showed the following relation in the proof of Theorem B:

$$\mathrm{Area}\,(\mathbb{C}\setminus f(\mathbb{D})) = \pi\left[\frac{1}{(1-p^2)^2} - \sum_{n=1}^{\infty} n|a_n|^2\right].$$

We remark that $f(\partial\mathbb{D})$ is of area zero because f is quasiconformal. Noting $\phi(\mathbb{D}) = f(\mathbb{D}^*) = \mathbb{C}\setminus f(\overline{\mathbb{D}})$, we thus have the relation

$$\mathrm{Area}\,\phi(\mathbb{D}) = \pi\left[\frac{1}{(1-p^2)^2} - \sum_{n=1}^{\infty} n|a_n|^2\right]. \tag{2.5}$$

Since $|\bar\partial\phi| \le k|\partial\phi|$ a.e., the Jacobian J_ϕ of ϕ satisfies the inequality

$$J_\phi = |\partial\phi|^2 - |\bar\partial\phi|^2 \ge (1-k^2)|\partial\phi|^2 \ge (k^{-2}-1)|\bar\partial\phi|^2 = (k^{-2}-1)|\bar\partial\psi|^2.$$

Hence, we obtain

$$\mathrm{Area}\,\phi(\mathbb{D}) = \iint\limits_{\mathbb{D}} J_\phi(\zeta)d\xi d\eta \ge (k^{-2}-1)\iint\limits_{\mathbb{D}} |\bar\partial\psi(\zeta)|^2 d\xi d\eta. \tag{2.6}$$

Next, we see from (2.4) that

$$\iint\limits_{\mathbb{D}} |\bar\partial\psi(\zeta)|^2\, d\xi d\eta = \iint\limits_{\mathbb{C}} |\partial\psi(\zeta)|^2\, d\xi d\eta \geq \iint\limits_{|\zeta|>1} |\partial\psi(\zeta)|^2\, d\xi d\eta. \tag{2.7}$$

It is easy to evaluate the right-most integral above by using the expansion in (2.1) as follows:

$$\iint\limits_{|\zeta|>1} |\partial\psi(\zeta)|^2\, d\xi d\eta = \pi \sum_{n=1}^{\infty} n|a_n|^2.$$

Plugging this with (2.5), (2.6) and (2.7), we obtain

$$(k^{-2}-1)\pi \sum_{n=1}^{\infty} n|a_n|^2 \leq \pi \left[\frac{1}{(1-p^2)^2} - \sum_{n=1}^{\infty} n|a_n|^2\right],$$

which yields the desired inequality.

Finally, we analyze the equality case for (1.3). Suppose that equality holds in (1.3). Then, equalities must hold both in (2.6) and in (2.7). The equality in (2.7) implies that $\partial\psi = 0$ on $\mathbb{D}$. In other words, $h = \bar\psi$ is analytic on $\mathbb{D}$. Therefore, $\phi(\zeta) = \zeta/(1-p\zeta) + \overline{h(\zeta)}$. The equality in (2.6) means that $|\bar\partial\phi/\partial\phi|$ is the constant k a.e. on $\mathbb{D}$. Since $\bar\partial\phi(\zeta)/\partial\phi(\zeta) = \overline{h'(\zeta)}(1-p\zeta)^2$, it then implies that the analytic function $h'(\zeta)(1-p\zeta)^2$ has constant modulus k and therefore a constant α with $|\alpha| = k$. Hence, $h'(\zeta) = \alpha(1-p\zeta)^{-2}$ for $|\zeta| < 1$. Integrating it, we obtain $h(\zeta) = \alpha\zeta/(1-p\zeta) + h(0)$. Thus, we finally have the form

$$\phi(\zeta) = \frac{\zeta}{1-p\zeta} + \overline{h(0)} + \frac{\bar\alpha\bar\zeta}{1-p\bar\zeta}, \quad \zeta \in \mathbb{D}.$$

Therefore,

$$f(z) = \frac{1}{z-p} + \overline{h(0)} + \frac{\bar\alpha}{\bar z - p}, \quad z \in \mathbb{D}^*,$$

whose boundary values on $\partial\mathbb{D}$ are the same as those of the meromorphic function

$$g(z) = \frac{1}{z-p} + \overline{h(0)} + \frac{\bar\alpha z}{1-pz}.$$

Since $f(z) - g(z)$ is bounded analytic on $\mathbb{D}$, $f(z)$ is identically equal to $g(z)$ on $\mathbb{D}$ by the maximum principle. In particular, $h(0) = \overline{a_0}$ and $\alpha = \overline{a_1}$. Thus we have seen that the function f must have the form (1.4) if equality holds in (2.7). We need to show that the function f of the form (1.4) is indeed a member of $\Sigma_k(p)$. We first show that f is univalent in $\mathbb{D}$. We compute

$$f(z_1) - f(z_2) = \frac{z_2 - z_1}{(z_1-p)(z_2-p)} \left[1 - a_1 \left(\frac{z_1-p}{1-pz_1} \cdot \frac{z_2-p}{1-pz_2}\right)\right]$$

for $z_1, z_2 \in \mathbb{D}$. Since

$$\left| a_1 \left(\frac{z_1-p}{1-pz_1} \cdot \frac{z_2-p}{1-pz_2}\right)\right| < |a_1| = k \leq 1$$

for $z_1, z_2 \in \mathbb{D}$, we see that $f(z_1) \neq f(z_2)$ if $z_1 \neq z_2$. Hence, $f \in \Sigma(p)$. On the other hand, the function in (1.5) agrees with that in (1.4) on the boundary $|z| = 1$, and is a composition of the Möbius transformation $1/(z-p)$ with the k-quasiconformal affine mapping $w + a_0 + a_1\bar w$. Thus we conclude that f belongs to $\Sigma_k(p)$. $\square$

3. Proof of Theorems 2 and 3 and a concluding remark

We start with the proof of Theorem 2.

Proof of Theorem 2. We first show the theorem under the additional condition that ω is analytic on the closed unit disc $\overline{\mathbb{D}}$; in other words, ω is analytic on the disc $|z| < R$ for some $R > 1$. The function $f(z) = 1/(z-p) + \omega(z)$ now extends to $\overline{\mathbb{D}}$ analytically. Since $|\omega'| \le k(1+p)^{-2}$, we have the inequality

$$|\omega(z_1) - \omega(z_2)| \le \frac{k}{(1+p)^2}|z_1 - z_2|, \quad z_1, z_2 \in \overline{\mathbb{D}}.$$

We now see that

$$\left|\frac{1}{z_1 - p} - \frac{1}{z_2 - p}\right| = \frac{|z_1 - z_2|}{|z_1 - p||z_2 - p|} \ge \frac{|z_1 - z_2|}{(1+p)^2}, \quad z_1, z_2 \in \overline{\mathbb{D}}.$$

Hence, we have

$$|f(z_1) - f(z_2)| \ge \frac{1-k}{(1+p)^2}|z_1 - z_2|$$

for $z_1, z_2 \in \overline{\mathbb{D}}$, which proves that $f(z)$ is univalent on $\overline{\mathbb{D}}$.

For a while, we denote by $g(z)$ the function appearing in the right-hand side of (1.6) for $|z| > 1/R$. Obviously, $g(z)$ agrees with $f(z) = 1/(z-p) + \omega(z)$ on $\partial\mathbb{D}$. We show now that g is orientation-preserving and locally C^1-diffeomorphic on a neighborhood of $\overline{\mathbb{D}^*}$ and locally k-quasiconformal on $\mathbb{D}^*$. To deal with the point at infinity as an ordinary point, we use the coordinate $\zeta = 1/z$. Then

$$g(z) = \frac{\zeta}{1 - p\zeta} + \omega(\bar{\zeta}), \quad |\zeta| < R.$$

A simple computation yields

$$\frac{\partial g}{\partial \zeta} = \frac{1}{(1-p\zeta)^2} \quad \text{and} \quad \frac{\partial g}{\partial \bar{\zeta}} = \omega'(\bar{\zeta}).$$

Hence, the Jacobian of g satisfies

$$J_g(1/\zeta)|\zeta|^{-4} = \frac{1}{|1-p\zeta|^4} - |\omega'(\bar{\zeta})|^2 \ge \frac{1-k^2}{(1+p)^4} > 0, \quad \zeta \in \overline{\mathbb{D}}.$$

The inverse function theorem now implies that g is a local C^1-diffeomorphism at each point of $\overline{\mathbb{D}^*}$. Moreover, the complex dilatation $\mu = \bar{\partial} g/\partial g$ of g satisfies

$$|\mu(\zeta)| = |1-p\zeta|^2|\omega'(\bar{\zeta})| \le (1+p)^2|\omega'(\bar{\zeta})| \le k < 1, \quad \zeta \in \mathbb{D},$$

which means that g is locally k-quasiconformal on $\mathbb{D}^*$.

Define $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ by

$$F(z) = \begin{cases} f(z), & z \in \overline{\mathbb{D}}, \\ g(z), & z \in \mathbb{D}^*. \end{cases}$$

Since $f(z) = g(z)$ on $\partial\mathbb{D}$ the function F is continuous on $\widehat{\mathbb{C}}$. Furthermore, F is locally univalent on $\widehat{\mathbb{C}}$, which implies that $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a topological covering projection. Since $\widehat{\mathbb{C}}$ is simply connected, F must be globally univalent; namely, F is a homeomorphism of $\widehat{\mathbb{C}}$ onto itself.We recall that F is k-quasiconformal off the unit circle $\partial\mathbb{D}$. Since the unit circle is removable for quasiconformality (see [5, p.205]), we therefore conclude that F is a k-quasiconformal extension of f. Thus we have shown that the theorem is valid if ω is analytic on $\overline{\mathbb{D}}$.

Finally the proof of Theorem 2 will be complete, if we consider a general ω satisfying the assumptions of this theorem. To this end, for $0 < r < 1$, let us consider the function $\omega_r(z) = \omega(rz), z \in \overline{\mathbb{D}}$. Since ω_r is analytic on $\overline{\mathbb{D}}$ and satisfies $|\omega_r'(z)| \le k/(1+p)^2$, the function $f_r(z) = 1/(z-p)+\omega_r(z)$ has the k-quasiconformal extension F_r constructed above. By a normality criterion for k-quasiconformal homeomorphisms of $\widehat{\mathbb{C}}$ (see, for instance, [5, II §5, Theorem 5.3]), one can see that F_r converges to a k-quasiconformal mapping, say F, uniformly on $\widehat{\mathbb{C}}$ as $r \to 1^-$. It is evident that F gives the required quasiconformal extension of $f(z) = 1/(z-p) + \omega(z)$. □

A straightforward application of Theorem 2 yields the following sufficient condition for a function f of the form (1.1) to belong to $\Sigma_k(p)$.

Corollary 2. *Let $0 \le p < 1$ and $0 \le k < 1$. Suppose that a meromorphic function $f(z)$ on $\mathbb{D}$ has the form* (1.1). *If*

$$\sum_{n=1}^{\infty} n|a_n| \le \frac{k}{(1+p)^2},$$

then $f \in \Sigma_k(p)$.

Proof. This immediately follows from Theorem 2 because

$$|\omega'(z)| \le \sum_{n=1}^{\infty} n|a_n||z|^{n-1} \le \sum_{n=1}^{\infty} n|a_n| \le \frac{k}{(1+p)^2}, \quad z \in \mathbb{D}.$$

□

Next we prove Theorem 3.

Proof of Theorem 3. Let $f \in \Sigma_{k_1}(p)$ and $g \in \Sigma_{k_2}(p)$ be expressed as in (1.4). Then Theorem 1 gives us

$$\sum_{n=1}^{\infty} n|a_n|^2 \le \frac{{k_1}^2}{(1-p^2)^2} \quad \text{and} \quad \sum_{n=1}^{\infty} n|b_n|^2 \le \frac{{k_2}^2}{(1-p^2)^2}.$$

Now an application of Cauchy-Schwarz inequality together with the aforementioned inequalities yields

$$\begin{aligned} \sum_{n=1}^{\infty} n|a_n b_n| &= \sum_{n=1}^{\infty} (\sqrt{n}|a_n|)(\sqrt{n}|b_n|) \\ &\le \left(\sum_{n=1}^{\infty} n|a_n|^2\right)^{1/2} \left(\sum_{n=1}^{\infty} n|b_n|^2\right)^{1/2} \\ &\le \frac{k_1 k_2}{(1-p^2)^2} = \frac{\alpha}{(1+p)^2}, \end{aligned}$$

where $\alpha = k_1 k_2 (1-p)^{-2}$. Since $\alpha < 1$ by assumption, the desired result follows from Corollary 2. □

Remark. We conclude the present note with an outline of another proof of (1.3) based on Lehto's principle (cf. [4, II.3.3]) and Theorem B. First of all, we recall the definition of the complex Banach (indeed, Hilbert) space ℓ^2. This is the set of sequences $x = \{x_n\}_{n=1}^{\infty}$ of complex numbers with the norm

$$\|x\|_{\ell^2} = \left(\sum_{n=1}^{\infty} |x_n|^2 \right)^{1/2} < \infty.$$

It is enough to show (1.3) for functions in $\Sigma_k^0(p)$ only. Suppose that $f \in \Sigma_k^0(p)$ is already extended to a k-quasiconformal mapping of $\widehat{\mathbb{C}}$ and let μ be its complex dilatation. We remark that $|\mu| \le k$ a.e. in $\mathbb{D}^*$ and $\mu = 0$ in $\mathbb{D}$. By the measurable Riemann mapping theorem, for each $t \in \mathbb{D}$, there exists a unique quasiconformal mapping f_t of $\widehat{\mathbb{C}}$ for which the complex dilatation is $t\mu/k$ and $f_t|_{\mathbb{D}} \in \Sigma^0(p)$. Note here that $f_k = f$. Then f_t has an expansion of the form

$$f_t(z) = \frac{1}{z-p} + \sum_{n=1}^{\infty} a_n(t) z^n, \quad z \in \mathbb{D}.$$

By the holomorphic dependence of the solution to the Beltrami equation, $a_n(t)$ is analytic in $\{t : t \in \mathbb{D}\}$ for every $n \ge 1$. We now consider the sequence $\sigma(t) = \{\sqrt{n} a_n(t)\}_{n=1}^{\infty}$. Theorem B tells us that $\|\sigma(t)\|_{\ell^2} \le 1/(1-p^2)$. Hence, we conclude that $\sigma : \mathbb{D} \to \ell^2$ is a bounded analytic function taking values in the complex Banach space ℓ^2. Since $\sigma(0) = 0$, the (generalized) Schwarz lemma yields the inequality $\|\sigma(t)\|_{\ell^2} \le |t|/(1-p^2)$. In particular, letting $t = k$ gives (1.3).

We must say that this method is conceptually simpler than that of our proof in Section 2. However, this does not provide information about the equality case in an obvious manner.

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