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A NOTE ON RIEMANNIAN METRICS ON THE MODULI SPACE OF RIEMANN SURFACES

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Abstract. In this note we show that the moduli space $\mathbb{M}(S_{g,n})$ of surface $S_{g,n}$ of genus g with n punctures, satisfying $3g + n \geq 5$, admits no complete Riemannian metric of nonpositive sectional curvature such that the Teichmüller space $\mathbb{T}(S_{g,n})$ is a mapping class group $\mathrm{Mod}(S_{g,n})$-invariant visibility manifold.

1. Introduction

Let $S_{g,n}$ be a surface of genus g with n punctures, and let $\mathbb{M}(S_{g,n})$ be the moduli space of $S_{g,n}$. There are many canonical metrics on $\mathbb{M}(S_{g,n})$ such as the Teichmüller metric and the Weil-Petersson metric (see [IT92]). Kravetz [Kra59] asserted that the Teichmüller metric has negative curvature. However, a mistake in the proof of Kravetz's theorem was found by Linch [Lin71]. Masur [Mas75] proved that the Teichmüller metric is not nonpositively curved with the exception of a few cases. On the other hand, the Weil-Petersson metric is negatively curved (see [Tro86, Wol86]), but not complete (see [Chu76, Wol75]). The McMullen metric, constructed by McMullen in [McM00], is a complete Kähler-hyperbolic metric in the sense of Gromov. Liu, Sun and Yau in [LSY05] perturbed the Weil-Petersson metric to construct the perturbed Ricci metric, which is complete and has negatively pinched holomorphic sectional curvature. In light of the properties of all these metrics, the following question is phrased in [BF06] (see Question 6.1 in [BF06]).

Question 1.1 (Brock-Farb-McMullen)**.** Does $\mathbb{M}(S_{g,n})$ admit a complete, nonpositively curved Riemannian metric?

Since $\mathbb{M}(S_{g,n})$ is an orbifold, here a Riemannian metric on $\mathbb{M}(S_{g,n})$ means a Riemannian metric on the Teichmüller space $\mathbb{T}(S_{g,n})$, the universal cover of $\mathbb{M}(S_{g,n})$, on which the natural action of the mapping class group $\mathrm{Mod}(S_{g,n})$ is an isometric action.

Recall that Eberlein and O'Neill in [EO73] introduced the so-called visibility manifold M of nonpositive sectional curvature which in some sense means that for any two different points at "infinity" of M, they can be viewed from each other along the space; this phenomenon cannot happen in $\mathbb{R}^n$. Let M be a complete simply connected Riemannian manifold of nonpositive sectional curvature. The visual boundary $M(\infty)$ of M consists of all the equivalent asymptotic geodesic rays (see [BGS85]). We call M a *visibility manifold* if for any $x \neq y \in M(\infty)$ there

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exists a geodesic line $c : (-\infty, +\infty) \to M$ such that $c(-\infty) = x$ and $c(+\infty) = y$. Classical examples for visibility manifolds are complete simply connected Riemannian manifolds with uniformly negative sectional curvatures (see [BGS85, BH99] for details).

If $3g + n \leq 4$ and $\mathbb{M}(S_{g,n})$ has positive dimension, then (g, n) must be one of $\{(1, 0), (1, 1), (0, 4)\}$. For these three cases, it is well known that the Teichmüller metric on $\mathbb{M}(S_{g,n})$ is a complete hyperbolic metric (see [FW10]). In particular, the Teichmüller space $\mathbb{T}(S_{g,n})$, endowed with the Teichmüller metric, is a $\mathrm{Mod}(S_{g,n})$-invariant visibility manifold. Hence, we can always assume that $3g + n \geq 5$ for Question 1.1. Now we are ready to state our result.

Theorem 1.2. *If $3g+n \geq 5$, then $\mathbb{M}(S_{g,n})$ admits no complete Riemannian metric such that $\mathbb{T}(S_{g,n})$ is a $\mathrm{Mod}(S_{g,n})$-invariant visibility manifold.*

We remark that there is no finite volume condition in Theorem 1.2. As introduced above, the universal cover of a complete Riemannian manifold whose sectional curvatures are bounded from above by a negative number (we can always take a rescaling for the metric such that the upper bound for the sectional curvature is given by -1) is a visibility manifold, so the following result follows immediately from Theorem 1.2.

Corollary 1.3. *If $3g + n \geq 5$, $\mathbb{M}(S_{g,n})$ admits no complete Riemannian metric such that the sectional curvature $K(\mathbb{M}(S_{g,n})) \leq -1$.*

Ivanov [Iva88] showed that $\mathbb{M}(S_{g,n})$ $(3g + n \geq 5)$ admits no complete, finite volume Riemannian metric whose sectional curvature is pinched by two negative numbers. McMullen in [McM00] stated (without proof) that $\mathbb{M}(S_{g,n})$ $(3g+n \geq 5)$ admits no complete Riemannian metric whose sectional curvature is pinched by two negative numbers, which was proved by Brock and Farb in [BF06]. Moreover, the authors in [BF06] showed that $\mathbb{M}(S_{g,n})$ $(3g + n \geq 5)$ admits no complete, finite volume Riemannian metric such that the universal covering space is Gromov-hyperbolic. For related topics, one can also see [KN04, LSY04, MP99, MW95, Wu11].

2. Notation and Preliminaries

2.1. **Isometries on manifolds of nonpositive sectional curvatures.** Let M be a complete Riemannian manifold of nonpositive sectional curvature and let γ be an isometry of M. Recall that the *translation length* $|\gamma|$ is defined as

$$|\gamma| := \inf_{p \in M} \mathrm{dist}(p, \gamma \circ p).$$

We call γ *parabolic* if $|\gamma|$ cannot be achieved in M and *hyperbolic* if $|\gamma| > 0$ and $|\gamma|$ is achieved in M. Otherwise, we call γ *elliptic*.

The following lemma will be applied later.

Lemma 2.1. *Let M be a visibility manifold and γ a parabolic isometry of M. Then γ has exactly one fixed point in the visual boundary $M(\infty)$ of M.*

Proof. See Lemma 6.8 in [BGS85]. □

Recall a group G acting on a metric space X is called *proper* if for each compact subset $K \subset X$, the set $K \cap gK$ is nonempty for only finitely many g in G.

Lemma 2.2. *Let M be a visibility manifold and let $G = \mathbb{Z} \oplus \mathbb{Z}$, a free abelian group of rank 2, act properly on M by isometries. Then for any nontrivial $g \in G$, g is parabolic.*

Proof. We argue by contradiction. Assume $g \in G$ is nontrivial and not parabolic. So g is either elliptic or hyperbolic. Since G acts properly on M, the element g cannot be elliptic. So g is hyperbolic. Let $Min(g) := \{p \in M, dist(g \cdot p, p) = |g|\}$. From Theorem 6.8 on page 231 in Chapter II.6 of [BH99] we know that $Min(g)$ is isometric to the product $Y \times \mathbb{R}$ on which g acts trivially on the Y component, where $Y \times \{0\}$ is a closed convex subset in M. Since M is a visibility manifold, Y is bounded, otherwise there exists a flat half-plane $[0, +\infty) \times \mathbb{R}$ in M, which is impossible in a visibility space.

Let $h \in G$ such that the group $\langle g, h \rangle$, generated by g and h, is a free abelian group of rank 2. Since $gh = hg$, $Min(g)$ is h-invariant. Since h is an isometry, it sends geodesic lines to geodesic lines. It is not hard to see that h splits into (h_1, h_2) where h_1 is an isometry on Y and h_2 is an isometry on $\mathbb{R}$. First $Y \times \{0\}$ is a complete Riemannian manifold of nonpositive sectional curvature because $Y \times \{0\}$ is closed convex in M. Since Y is bounded, the classical Cartan Fixed Point Theorem (see Lemma 6.3 in [BGS85]) tells us that there exists $x_1 \in Y$ such that $h_1 \cdot x_1 = x_1$. Hence $x_1 \times \mathbb{R}$ is $\langle g, h \rangle$-invariant. Since G acts properly on M and $\langle g, h \rangle$ is a subgroup of G, $\langle g, h \rangle$ also acts properly on $x_1 \times \mathbb{R}$, which is impossible because the rank of $\langle g, h \rangle$ is 2. □

2.2. **Mapping class groups.** Let $S_{g,n}$ be a Riemann surface of genus g with n punctures, and let $\mathrm{Mod}(S_{g,n})$ be the mapping class group of $S_{g,n}$, i.e., the group of isotopy classes of self-homeomorphisms of $S_{g,n}$ which preserve the orientation and the punctures. The following proposition lists the basic properties of $\mathrm{Mod}(S_{g,n})$, which will be used later.

Proposition 2.3. *If $3g + n \geq 5$ and $\mathrm{Mod}(S_{g,n})$ is the mapping class group of $S_{g,n}$, then*

(1) $\mathrm{Mod}(S_{g,n})$ *acts properly on the Teichmüller space.*

(2) $\mathrm{Mod}(S_{g,n})$ *is finitely generated by Dehn twists along essential non-peripheral simple closed curves.*

(3) *There exists a torsion-free subgroup of finite index in* $\mathrm{Mod}(S_{g,n})$.

Proof. See the details in [FM12]. □

Bestvina, Kapovich and Kleiner in [BKK02] defined the *action dimension* of a group G, denoted by $actdim(G)$, to be the minimum dimension of a contractible manifold on which G acts properly. For example, the action dimension of $\mathbb{Z}$ is 1. More generally, the action dimension of the fundamental group of a closed spherical manifold is the same as the dimension of the manifold. In [BKK02] the authors introduced the so-called *obstructor dimension* of a group G, denoted by $obdim(G)$. This quantity satisfies

Proposition 2.4 (Bestvina-Kapovich-Kleiner)**.** (1) *For any group G,*

$$actdim(G) \geq obdim(G).$$

(2) *The obstructor dimension is a quasi-isometric invariance.*

Proof. See Theorem 1 in [BKK02] for part (1) and remark 11 in [BKK02] for part (2). □

In [Des06] Despotovic proved that the action dimension of the mapping class group satisfies $actdim(\mathrm{Mod}(S_{g,n})) = 6g-6+2n$ if $3g+n \geq 5$ (also see page 6 of [FW10]). In fact, Despotovic proved the obstructor dimension $obdim(\mathrm{Mod}(S_{g,n})) = 6g-6+2n$ and then used part (1) of Proposition 2.4 to conclude $actdim(\mathrm{Mod}(S_{g,n})) = 6g-6+2n$. The following result tells us that the action dimension is preserved by finite index subgroups of the mapping class group. In general, the action dimension is not a quasi-isometric invariance (see [BKK02] for details).

Proposition 2.5 (Despotovic). *If $3g+n \geq 5$, then for any finite index subgroup G of $\mathrm{Mod}(S_{g,n})$, we have $actdim(G) = 6g-6+2n$.*

Proof. Let G be a subgroup of $\mathrm{Mod}(S_{g,n})$ with finite index. So G endowed with a word metric is quasi-isometric to $\mathrm{Mod}(S_{g,n})$. Since $obdim(\mathrm{Mod}(S_{g,n})) = 6g-6+2n$ (see [Des06]), by part (2) of Proposition 2.4 we have $obdim(G) = 6g-6+2n$. Thus from part (1) of Proposition 2.4 we know that $actdim(G) \geq 6g-6+2n$.

On the other hand it is obvious that $actdim(G) \leq 6g-6+2n$ because G acts properly on the Teichmüller space, which is contractible. Hence, $actdim(G) = 6g-6+2n$. ☐

3. Proof of Theorem 1.2

We divide the proof of Theorem 1.2 into several lemmas.

The proof of the following lemma is implicitly included in several places in the literature (see [BF06, KN04, MP99]). For completeness we still give the proof here.

Lemma 3.1. *Let M be a visibility Riemannian manifold. Assume that the mapping class group $\mathrm{Mod}(S_{g,n})$ acts properly on M by isometries. If $3g+n \geq 5$, then there exists a point $x \in M(\infty)$ such that $\gamma \cdot x = x$ for all $\gamma \in \mathrm{Mod}(S_{g,n})$.*

Proof. Let α be an essential non-peripheral simple closed curve on $S_{g,n}$ and let τ_α be the Dehn twist along α (see the definition of Dehn twist in [FM12]). Since $3g+n \geq 5$, there exists an essential non-peripheral simple closed curve β which is disjoint with α. Let τ_β be the Dehn twist along β. Since $\langle \tau_\alpha, \tau_\beta \rangle$ is a free abelian group of rank 2, by Lemma 2.2 the element τ_α is parabolic. Since M is a visibility manifold, by Lemma 2.1 there exists a unique $x \in M(\infty)$ such that $Fix(\tau_\alpha) = \{x\}$.

Claim (1). If $\tau_\alpha \cdot \tau_\beta = \tau_\beta \cdot \tau_\alpha$, then $Fix(\tau_\beta) = Fix(\tau_\alpha) = \{x\}$.

Proof of Claim (1). Since $\tau_\alpha \cdot \tau_\beta = \tau_\beta \cdot \tau_\alpha$ and $\tau_\alpha \cdot x = x$, $\tau_\alpha \cdot (\tau_\beta \cdot x) = \tau_\beta \cdot x$. So $\tau_\beta \cdot x$ is contained in $Fix(\tau_\alpha)$. Thus $\tau_\beta \cdot x = x$. Then the claim $Fix(\tau_\beta) = \{x\}$ follows from the fact that $Fix(\tau_\beta)$ is unique.

Claim (2). For any essential non-peripheral simple closed curve β on $S_{g,n}$, we have $Fix(\tau_\beta) = \{x\}$.

Proof of Claim (2). Let β be a simple closed curve. Since $3g+n \geq 5$, the curve complex is connected (see Theorem 4.3 in [FM12]). In particular, there exists a sequence of essential non-peripheral simple closed curves $\{\alpha_i\}_{i=1}^k$ such that $\alpha_1 = \alpha, \alpha_k = \beta$ and $\alpha_i \cap \alpha_{i+1} = \emptyset$. Hence, by Claim (1),

$$Fix(\tau_\beta) = Fix(\tau_{\alpha_{k-1}}) = Fix(\tau_{\alpha_{k-2}}) = \cdots = Fix(\tau_{\alpha_1}) = \{x\}.$$

Then the conclusion follows from part (2) of Proposition 2.3 and Claim (2). ☐

Lemma 3.2. *Let M be a visibility Riemannian manifold. Assume that the mapping class group* $\mathrm{Mod}(S_{g,n})$ *acts properly on M by isometries. If $3g+n \geq 5$, then any infinite ordered element $\phi \in \mathrm{Mod}(S_{g,n})$ acts as a parabolic isometry.*

Proof. Suppose that there exists an element $\phi \in \mathrm{Mod}(S_{g,n})$ with infinite order which acts on M as a hyperbolic isometry. Then there exists a geodesic line $\gamma : \mathbb{R} \to M$, an axis for ϕ, such that $\phi \cdot \gamma(t) = \gamma(|\phi| + t)$ for all $t \in \mathbb{R}$. Since M is a visibility manifold, it is not hard to see that $Fix(\phi) = \{\gamma(+\infty), \gamma(-\infty)\}$. From Lemma 3.1 we assume that $\gamma(+\infty)$ is fixed by $\mathrm{Mod}(S_{g,n})$. Let $\sigma \in \mathrm{Mod}(S_{g,n})$; since σ fixes $\gamma(+\infty)$ there exists some $C > 0$ such that $dist(\sigma \cdot \gamma(n \cdot |\phi|), \gamma(n \cdot |\phi|)) \leq C$ for all $n > 0$. Hence $dist((\phi^{-n} \cdot \sigma \cdot \phi^n) \cdot \gamma(0), \gamma(0)) \leq C$. Since the action is proper, there exists a subsequence $\{n_i\}$ such that $\phi^{-n_i} \cdot \sigma \cdot \phi^{n_i} \equiv \phi^{-n_1} \cdot \sigma \cdot \phi^{n_1}$, hence $\phi^{n_1 - n_i} \cdot \sigma = \sigma \cdot \phi^{n_1 - n_i}$. Since σ is arbitrary and ϕ has infinite order in $\mathrm{Mod}(S_{g,n})$, we can choose σ to be pseudo-Anosov such that $\{\sigma, \phi\}$ generates a free group of rank 2 (see [Iva92]). Since $\phi^{n_1 - n_i} \cdot \sigma = \sigma \cdot \phi^{n_1 - n_i}$, the group $\langle \sigma, \phi \rangle$ contains a free abelian subgroup of rank 2, which is a contradiction since $\langle \sigma, \phi \rangle$ is a free group. $\square$

Lemma 3.3. *Let M be a visibility Riemannian manifold. Assume that the mapping class group* $\mathrm{Mod}(S_{g,n})$ *acts properly on M by isometries. If $3g+n \geq 5$, then there exists a horosphere H such that every torsion free subgroup of* $\mathrm{Mod}(S_{g,n})$ *acts properly on H.*

Proof. From Lemma 3.1, there exists a point $x \in M(\infty)$ such that $\mathrm{Mod}(S_{g,n})$ fixes x. Let H be a horosphere at x and let G be a torsion free subgroup of $\mathrm{Mod}(S_{g,n})$. By Lemma 3.2 we know that G consists of parabolic isometries except the unit. Hence, by Proposition 8.25 on page 275 in Chapter II.8 of [BH99], H is G-invariant because G fixes x. Let d be the metric of M and let d_H be the induced metric on H. It is obvious that $d_H(p,q) \geq d(p,q)$ for all $p, q \in H$. The conclusion that G acts properly on H follows easily from the facts that G acts properly on M and $d_H(p,q) \geq d(p,q)$. $\square$

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. We argue by contradiction. Assume that $\mathbb{M}(S_{g,n})$ admits a complete Riemannian metric ds^2 such that $\mathbb{T}(S_{g,n})$ is a $\mathrm{Mod}(S_{g,n})$-invariant visibility manifold where the Teichmüller space $\mathbb{T}(S_{g,n})$ is endowed with the pull-back metric ds^2. By part (3) of Proposition 2.3 we may assume that G is a torsion free subgroup of $\mathrm{Mod}(S_{g,n})$ with finite index. By Lemma 3.3, there exists a horosphere H such that G acts properly on H. It is well known that H is homeomorphic to $\mathbb{R}^{6g-7+2n}$, in particular H is a contractible manifold. By the definition of action dimension we know that

$$actdim(G) \leq 6g - 7 + 2n.$$

On the other hand, since $3g+n \geq 5$ and G is a finite index subgroup of $\mathrm{Mod}(S_{g,n})$, by Proposition 2.5, we have

$$actdim(G) = 6g - 6 + 2n,$$

which is a contradiction. $\square$

Remark 3.1. The author is grateful to Benson Farb for pointing out this point; the proof of Theorem 1.2 also yields

Theorem 3.4. *The Teichmüller space* $\mathbb{T}(S_{g,n})$ $(3g+n \geq 5)$ *admits no complete* $\mathrm{Mod}(S_{g,n})$*-invariant nonpositively curved Riemannian metric such that every Dehn twist has only one fixed point in the visual boundary of* $\mathbb{T}(S_{g,n})$*.*

Remark 3.2. Consider the case that $n = 0$ and $g \geq 3$. It is known that $\mathrm{Mod}(S_g)$ is generated by the Dehn twists on certain nonseparating simple closed curves which are pairwise conjugate in $\mathrm{Mod}(S_g)$ (see [FM12]). Then the proof of Theorem 1.2 also yields

Theorem 3.5. *The Teichmüller space* $\mathbb{T}(S_g)$ $(g \geq 3)$ *admits no complete* $\mathrm{Mod}(S_g)$*-invariant nonpositively curved Riemannian metric such that the Dehn twist on some nonseparating simple closed curve has only one fixed point in the visual boundary of* $\mathbb{T}(S_g)$*.*

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