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WHY THE CIGAR CANNOT BE ISOMETRICALLY IMMERSED INTO THE 3-SPACE

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ABSTRACT. In this paper, we study the question if there is an isometric immersion of the cigar soliton into R^3. We show that the answer is negative. This gives a counterexample to the classical Weyl problem on R^2. A similar result in higher dimensions is also true for steady Bryant solitons.

1. INTRODUCTION

The classical Weyl problem asks if there exists a global C^2 isometric embedding $X : (S^2, g) \to (R^3, \sigma)$, where σ is the standard flat metric in R^3, for a two-sphere S^2 with g being a Riemannian metric on S^2 whose Gauss curvature is everywhere positive. This problem is solved affirmatively by L. Nirenberg in [4]. Recall here that such an embedding $X : (S^2, g) \to R^3$ is isometric if in the local coordinates (u^j), $g = g_{ij}du^i du^j$, we have the system $\partial_{u^i} X \cdot \partial_{u^j} X = g_{ij}$. For more recent progress related to the Weyl problem one may see the paper [2]. One may ask a similar question of problem on the plane R^2 with a Riemannian metric g whose Gauss curvature is everywhere positive. We give a counterexample in this short note by considering the cigar soliton in the study of Ricci flow [1]. Our example also shows that a similar Minkowski problem is not true on complete noncompact surfaces in R^3, namely, for a given strictly positive real function f defined on R^2, one cannot find a complete noncompact convex surface $\Sigma \subset R^3$ such that the Gauss curvature of Σ at the point x equals $f(n(x))$, where $n(x)$ denotes the normal to Σ at x.

The problem we consider in this short note is if there is a nontrivial n-dimensional steady Ricci soliton which can be embedded as a hypersurface in R^{n+1}. Recall that steady Ricci solitons are special solutions to Ricci flow introduced by R. Hamilton [3], [1]. In dimension two, the only nontrivial complete Ricci soliton is the cigar. We show that it is impossible to realize it as a surface in R^3. The key step in proving it is to use the deep result of H. Wu [5] about the convex surfaces. A similar result is also true for steady Bryant solitons in higher dimensions. We believe that a similar result is also true for radially symmetric expanding Ricci solitons. We shall use the notation $u = 0(r)$ for large $r > 0$ to denote by $C^{-1}r \leq u \leq Cr$ for some uniform constant $C > 0$ and the uniform constant C may vary from line to line.

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2. The cigar cannot be isometrically immersed into R^3

Recall that the cigar soliton is a two-dimensional Riemannian manifold (R^2, g_Σ) with the Riemannian metric ([1], [3])

$$g_\Sigma = \frac{dx^2+dy^2}{1+x^2+y^2} = \frac{dr^2+r^2d\theta^2}{1+r^2}.$$

It has positive Gauss curvature

$$K = \frac{2}{1+r^2}.$$

If the cigar can be isometrically immersed into R^3 (according to a theorem of Sacksteder-Heijenoort and the main theorem (δ) of H. Wu in [5]), it is the graph of a nonnegative strictly convex function u defined in the plane $\{x_3 = 0\}$.

Recall that the induced metric of the graph of the function $z = u(x_1, x_2)$ is given by

$$g = (\delta_{ij} + u_i u_j)dx^i dx^j = g_{ij}dx^i dx^j$$

with its second fundamental form

$$II = h_{ij}dx^i dx^j,$$

where $(x^i) = (x_i)$, $u_i = \frac{\partial u}{\partial x^i}$, $F(x) = (x, u(x))$, $F_j(x) = e_j + u_j(x)e_{n+1}$,

$$\nu = \frac{(-Du, 1)}{\sqrt{1+|Du|^2}}$$

etc., and

$$h_{ij} = (D_{F_i(x)}\nu, F_j(x)) = \frac{-u_{ij}}{\sqrt{1+|Du|^2}}.$$

Hence,

$$II = h_{ij}dx^i dx^j = \frac{D^2u}{\sqrt{1+|Du|^2}}$$

and for $u = u(r)$,

$$II = \frac{u_{rr}dr^2 + ru_r d\theta^2}{\sqrt{1+u_r^2}}.$$

Then the Gauss curvature of the immersed surface can be computed by

$$K = \det(h_{ij})/\det(g_{ij}) = \det(u_{ij})/(1+|Du|^2)^2.$$

Theorem 1. *The cigar cannot be isometrically immersed into R^3.*

Proof. Assume that we can have such an immersion into R^3.

Since $g = g_\Sigma$ is radially symmetric, we have $z = u(\rho)$ and

$$g = (1+u_\rho^2)d\rho^2 + \rho^2 d\theta^2$$

where $u_\rho = \frac{\partial u}{\partial \rho}$, etc. Hence, we have

$$(1+u_\rho^2)d\rho^2 = \frac{dr^2}{1+r^2} \tag{1}$$

and

$$\rho^2 = \frac{r^2}{1+r^2}. \tag{2}$$

By (2) we have

$$\rho = \frac{r}{\sqrt{1+r^2}}, \quad \frac{d\rho}{dr} = \frac{1}{(\sqrt{1+r^2})^3}.$$

By (1) we have

$$\sqrt{1+u_\rho^2}d\rho = \frac{dr}{\sqrt{1+r^2}},$$

which implies that

$$\sqrt{1+u_\rho^2} \cdot \frac{1}{(\sqrt{1+r^2})^3} = \frac{1}{\sqrt{1+r^2}}.$$

Hence we have

$$u_\rho^2 = r^2$$

and then

$$u_\rho u_{\rho\rho} = r\frac{dr}{d\rho} = r(\sqrt{1+r^2})^3.$$

By direct computation we know that the second fundamental form can be written as

$$II = \frac{1}{\sqrt{1+u_\rho^2}}[u_{\rho\rho}d\rho^2 + \rho u_\rho d\theta^2].$$

This would imply that the Gauss curvature K is

$$K = \frac{u_{\rho\rho}u_\rho\rho}{(1+u_\rho^2)^2\rho^2} = 1,$$

which is absurd. This completes the proof of Theorem 1. □

3. Higher dimensional generalization

It is quite possible to show a higher dimensional analog of the result above. Namely, we may have

Theorem 2. *The n-dimensional radially symmetric Bryant soliton cannot be isometrically immersed into R^{n+1}.*

Recall that the n-dimensional Bryant soliton is (R^n, g), $(n \geq 2)$ with its Riemannian metric ([1])

$$g = dr^2 + w(r)^2 d\theta^2,$$

where $w(r)$ is a smooth function with $w(0) = 0$, $w(r) = 0(r^{1/2})$, and $d\theta^2$ is the metric on S^{n-1}. It is well known that it has its positive sectional curvatures

$$k_1 = -\frac{w''}{w} = 0(r^{-2}), \quad k_2 = \frac{1-(w')^2}{w^2} = 0(r^{-1}), \tag{3}$$

where k_1 is the curvature for the planes tangent to the radial direction $e_1 = \partial_r$ and k_2 is the curvature for the planes tangent to the sphere.

If the n-dimensional Bryant soliton can be imbedded into R^{n+1}, then we can use H. Wu's result [5] as above to have it as the graph of a strictly convex radially symmetric function $u = u(x) = u(\rho), x \in R^3, \rho = |x|$. Then we have

$$g = (1+u_\rho^2)d\rho^2 + \rho^2 d\theta^2$$

and

$$dr = \sqrt{1+u_\rho^2}d\rho, \quad w(r) = \rho = 0(r^{1/2}).$$

Hence, $r = 0(\rho^2)$ and $u_\rho = 0(\rho)$. We then have some uniform constant $C > 0$ such that $u_{\rho\rho} = 0(1)$ for all large $\rho > 0$. Recall that using the components of the second fundamental form (h_{ij}) and $|g| = \rho^{2(n-1)}(1+u_\rho^2)$, the radial Riemannian curvature k_1 with large ρ can also be written as

$$k_1 = \frac{R_{1212}}{|g|} = \frac{u_{\rho\rho}\rho u_\rho}{\rho^{2n-2}(1+u_\rho^2)^2} = 0(\rho^{-2n}) = 0(r^{-n}).$$

We may use this to find a contradiction with (3) as above.

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