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A BOUND FOR THE ORDER OF THE FUNDAMENTAL GROUP OF A COMPLETE NONCOMPACT RICCI SHRINKER

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Abstract. In the case of a complete noncompact shrinking gradient Ricci soliton, building upon the works of Derdzinski, Fernández-López and García-Rio, Lott, Naber, and Wylie, we obtain a bound for the order of its fundamental group π_1 in terms of the dimension n and the logarithmic Sobolev constant. Under the additional assumption of being strongly κ-noncollapsed, a bound for the order of π_1 is κ^{-1} times a function of n.

Let $(\mathcal{M}^n, g)$ be a complete Riemannian manifold and let X be a vector field on $\mathcal{M}$ satisfying the inequality $\mathrm{Rc} + \mathcal{L}_X g \geq \frac{1}{2} g$. Here, Rc is the Ricci tensor of g and $\mathcal{L}$ is the Lie derivative. Regarding the fundamental group of $\mathcal{M}$, the strongest result is by Wylie [11], who proved that $\pi_1(\mathcal{M})$ is finite. Naber [9] proved this assuming that $X = \frac{1}{2}\nabla f$ for some function f (i.e., a Bakry–Emery manifold) and $|\mathrm{Rc}|$ is bounded. Fernández-López and García-Rio [4] proved that if X is bounded, then $\mathcal{M}$ is compact. By passing to the universal cover, this implies that $\pi_1(\mathcal{M})$ is finite; for an alternative proof, see Zhang [13]. Earlier, Lott [6] obtained the same conclusion assuming that $\mathcal{M}$ is compact and $X = \frac{1}{2}\nabla f$ for some f. We also mention two other works. Derdzinski [3] proved that if $\mathrm{Rc} + \mathcal{L}_X g > 0$, then $\pi_1(\mathcal{M})$ has only finitely many conjugacy classes. Yokota [12] has some previous work on $\pi_1(\mathcal{M})$ of ancient solutions of Ricci flow using the reduced volume which appears related to ours.

In this paper, for the special case of shrinking gradient Ricci solitons (GRS), i.e., the equality case $\mathrm{Rc} + \mathcal{L}_X g = \frac{1}{2} g$ with $X = \frac{1}{2}\nabla f$, we discuss a quantification of the above results regarding the fundamental group. Throughout the rest of this paper $(\mathcal{M}^n, g, f)$ shall denote a complete noncompact shrinking GRS, where $\mathrm{Rc} + \nabla^2 f = \frac{1}{2} g$ and f is normalized by $R + |\nabla f|^2 = f$. Note that $\int_{\mathcal{M}} e^{-f} d\mu$ is finite and that the logarithmic Sobolev constant, that is, the minimum of Perelman's entropy at scale 1, is $\mu_0(g, f) \doteqdot -\ln\left(\int_{\mathcal{M}} (4\pi)^{-n/2} e^{-f} d\mu\right)$; see Carrillo and Ni [2].

Our main result is the following.

Theorem 1. *If $(\mathcal{M}^n, g, f)$ is a complete noncompact shrinking gradient Ricci soliton, then $|\pi_1(\mathcal{M})| \leq C(n, \mu_0(g, f))$.*

Proof. Let $(\tilde{\mathcal{M}}^n, \tilde{g}, \tilde{f})$ be the universal covering shrinking GRS, that is, $\pi : \tilde{\mathcal{M}} \to \mathcal{M}$ is the universal covering map, $\tilde{g} = \pi^* g$, and $\tilde{f} = f \circ \pi$. Then we have that $\mathrm{Rc}_{\tilde{g}} + \nabla^2_{\tilde{g}} \tilde{f} = \frac{1}{2}\tilde{g}$ and $R_{\tilde{g}} + |\nabla \tilde{f}|^2_{\tilde{g}} = \tilde{f}$, so that the potential function $\tilde{f}$ is normalized.

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Let $\gamma \in \pi_1(\mathcal{M}) - \{e\}$. Then γ corresponds to a deck transformation $\gamma : \tilde{\mathcal{M}} \to \tilde{\mathcal{M}}$ that is a fixed-point-free isometry of the metric $\tilde{g}$. Since $\pi \circ \gamma = \pi$, we have $\tilde{f} \circ \gamma = \tilde{f}$. In particular, γ is a fixed-point-free isometry of any level or sublevel set of $\tilde{f}$. Hence $\mathrm{Vol}_{\tilde{g}}(\{\tilde{f} \leq s\}) = |\pi_1(\mathcal{M})| \, \mathrm{Vol}(\{f \leq s\})$ for any $s > 0$. Let O be a minimum point of f. By Cao and Zhou [1] and its improvement of constants by Haslhofer and Müller [5], we have

$$\frac{1}{4}\left((d(x,O) - 5n)_+\right)^2 \leq f(x) \leq \frac{1}{4}(d(x,O) + \sqrt{2n})^2 \tag{1}$$

for g. They also proved that $\mathrm{Vol}_{\tilde{g}}(\{\tilde{f} \leq s\}) \leq C(n)s^{n/2}$. We remark that by using the Riccati equation and a clever integration by parts, Munteanu and Wang [8] proved the slightly improved inequality that $\mathrm{Vol}\, B_r(O) \leq \omega_n e^{n/2} r^n$ for $r > 0$, where $\omega_n = \mathrm{Vol}_{\mathbb{R}^n} B_1$. On the other hand, by Munteanu and Wang [7], there exists a constant $c(n, \int_{\mathcal{M}} e^{-f} d\mu) > 0$ such that $\mathrm{Vol}\, B_r(O) \geq cr$ for $r \geq 1$.

By the above, we have that

$$C(n)(2n)^{\frac{n}{2}} \geq \mathrm{Vol}_{\tilde{g}}(\{\tilde{f} \leq 2n\}) = |\pi_1(\mathcal{M})| \, \mathrm{Vol}(\{f \leq 2n\}).$$

We also have the inequality

$$\mathrm{Vol}(\{f \leq 2n\}) \geq \mathrm{Vol}(B_{\sqrt{2n}}(O)) \geq c(n, \int_{\mathcal{M}} e^{-f} d\mu). \tag{2}$$

The theorem follows. ☐

Remark 2. For shrinking GRS $(\mathcal{M}_i, g_i, f_i)$, $i = 1, 2$, we have the product rule

$$\mu_0(g_1 \times g_2, f_1 \circ p_1 + f_2 \circ p_2) = \mu_0(g_1, f_1) + \mu_0(g_2, f_2),$$

where $p_i : \mathcal{M}_1 \times \mathcal{M}_2 \to \mathcal{M}_i$ are the projection maps. For a Gaussian shrinking GRS we have $\mu_0 = 0$. Thus, if $\mathcal{M}$ is compact, then we may apply the theorem to $\mathcal{M} \times \mathbb{R}$.

Assume that, whenever $x \in \mathcal{M}$ and $r \in (0, 1)$ are such that the scalar curvature satisfies $R \leq r^{-2}$ in $B_r(x)$, then we have $\mathrm{Vol}\, B_s(x) \geq \kappa s^n$ for $0 < s \leq r$. In this case we say that $(\mathcal{M}, g)$ is strongly κ-noncollapsed below the scale 1. For a shrinking GRS that is also the singularity model of a finite time singular solution to the Ricci flow on a closed manifold, there is such a $\kappa > 0$ by the improved κ-noncollapsing result of Perelman [10]. We have the following.

Proposition 3. If $(\mathcal{M}^n, g, f)$ is a complete noncompact shrinking gradient Ricci soliton which is strongly κ-noncollapsed below the scale 1, then $|\pi_1(\mathcal{M})| \leq C(n)\kappa^{-1}$.

Proof. Let $x \in B_{\frac{1}{\sqrt{2n}}}(O)$. From (1), we have $R(x) \leq f(x) \leq 2n$. By our hypothesis, this implies $\mathrm{Vol}\, B_{\frac{1}{\sqrt{2n}}}(O) \geq (2n)^{-\frac{n}{2}}\kappa$. Using this to replace the last inequality in (2), the proposition then follows. ☐

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