

## ON THE EXISTENCE OF BALANCED AND SKT METRICS ON NILMANIFOLDS

ANNA FINO AND LUIGI VEZZONI

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ABSTRACT. On a complex manifold a Hermitian metric which is simultaneously SKT and balanced has to be necessarily Kähler. It has been conjectured that if a compact complex manifold  $(M, J)$  has an SKT metric and a balanced metric both compatible with  $J$ , then  $(M, J)$  is necessarily Kähler. We show that the conjecture is true for nilmanifolds.

### 1. INTRODUCTION

A Riemannian metric  $g$  on a complex manifold  $(M, J)$  is *compatible* with  $J$  (or  *$J$ -Hermitian*) if  $g(J\cdot, J\cdot) = g(\cdot, \cdot)$ . In the present paper we focus on the existence of *special* Hermitian metrics on complex manifolds. More precisely, we study the existence of *strong Kähler with torsion* (shortly SKT) and balanced metrics compatible with the same complex structure. We recall that a  $J$ -Hermitian metric is called *SKT* (or *pluriclosed*) if its fundamental form  $\omega$  satisfies

$$\partial\bar{\partial}\omega = 0,$$

while  $g$  is called *balanced* if  $\omega$  is co-closed, i.e.

$$d^*\omega = 0,$$

where  $d^*$  denotes the formal adjoint operator of  $d$  with respect to the metric  $g$ .

SKT metrics were introduced by Bismut in [6] and further studied in many papers (see e.g. [11, 13, 14, 18–20, 22] and the references therein), while balanced metrics were introduced and first studied by Michelsohn in [17], where their existence is characterized in terms of currents. In a subsequent paper Alessandrini and Bassanelli proved that modifications of compact balanced manifolds are always balanced (see [1, 2]) showing a powerful tool for finding examples of balanced manifolds.

It is well known that if a Hermitian metric  $g$  is simultaneously SKT and balanced, then it is necessarily Kähler (see e.g. [4]). This result has been generalized in [15] showing that a compact SKT conformally balanced manifold has to be Kähler. The next conjecture was stated in [13] and it is about the existence of an SKT metric and a balanced metric both compatible with the same complex structure.

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**Conjecture.** *Every compact complex manifold admitting both an SKT metric and a balanced metric is necessarily Kähler.*

The conjecture has been implicitly already proved in the literature in some special cases. For instance, Verbitsky has shown in [22] that the twistor space of a compact, anti-self-dual Riemannian manifold has no SKT metrics unless it is Kählerian and Chiose has obtained in [7] a similar result for non-Kähler complex manifolds belonging to the Fujiki class. Furthermore, Li, Fu and Yau have proved in [20] that some new examples of SKT manifolds do not admit any balanced metric. A natural source of non-Kähler manifolds admitting balanced metrics and SKT metrics are given by *nilmanifolds*, i.e. by compact manifolds obtained as quotients of a simply connected nilpotent Lie group  $G$  by a cocompact lattice  $\Gamma$ . It is well known that a nilmanifold cannot admit Kähler structures unless it is a torus (see for instance [5, 12]). The aim of the paper is to show that the conjecture is true when  $(M, J)$  is a *complex nilmanifold*. By *complex nilmanifold* we refer to a nilmanifold equipped with an *invariant* complex structure  $J$ , i.e. endowed with a complex structure induced by a left-invariant complex structure on  $G$ .

Our result is the following.

**Theorem 1.1.** *Let  $M = G \backslash \Gamma$  be a nilmanifold equipped with an invariant complex structure  $J$ . Assume that  $(M, J)$  admits a balanced metric  $g$  and an SKT metric  $g'$  both compatible with  $J$ . Then  $(M, J)$  is a complex torus.*

The theorem is trivial in dimension 6 in view of the main result in [10] and it was already proved in [13] when the nilmanifold has dimension 8 by using a classification result proven in [8].

## 2. PROOF OF THEOREM 1.1

In order to prove Theorem 1.1 we need the following lemmas.

**Lemma 2.1.** *Let  $(M = G/\Gamma, J)$  be a complex nilmanifold.*

- *If  $(M, J)$  has a balanced metric, then it also has an invariant balanced metric [9].*
- *If  $(M, J)$  has an SKT metric, then it also has an invariant SKT metric [21].*

**Lemma 2.2** ([8]). *Let  $(M = G/\Gamma, J)$  be a complex nilmanifold of real dimension  $2n$ . If  $(M, J)$  has an SKT metric, then  $G$  is (at most) 2-step nilpotent, and there exists a complex  $(1, 0)$ -coframe  $\{\alpha^1, \dots, \alpha^n\}$  on  $\mathfrak{g}$  satisfying the structure equations*

$$\begin{cases} d\alpha^j = 0, & j = 1, \dots, k, \\ d\alpha^j = \sum_{r,s=1}^k \left( \frac{1}{2} c_{rs}^j \alpha^r \wedge \alpha^s + c_{r\bar{s}}^j \alpha^r \wedge \bar{\alpha}^s \right), & j = k+1, \dots, n, \end{cases}$$

*for some  $k \in \{1, \dots, n-1\}$  and with  $c_{rs}^j, c_{r\bar{s}}^j \in \mathbb{C}$ .*

Now we can prove Theorem 1.1.

*Proof of Theorem 1.1.* Suppose that  $(M = G/\Gamma, J)$  is not a complex torus, i.e. that  $G$  is not abelian, and denote by  $\mathfrak{g}$  the Lie algebra of  $G$ . Assume that  $(M, J)$  admits a balanced metric  $g$  and also an SKT metric  $g'$  both compatible with  $J$ . Then in view of Lemma 2.1, we may assume both  $g$  and  $g'$  invariant and regard them as scalar products on  $\mathfrak{g}$ . This allows us to work at the level of the Lie algebra  $\mathfrak{g}$ .

As a consequence of Lemma 2.2, the existence of the SKT metric implies that the Lie algebra  $\mathfrak{g}$  is 2-step nilpotent and that  $(\mathfrak{g}, J)$  has a  $(1, 0)$ -coframe  $\{\alpha^1, \dots, \alpha^n\}$  satisfying the structure equations

$$(1) \quad \begin{cases} d\alpha^j = 0, & j = 1, \dots, k, \\ d\alpha^j = \sum_{r,s=1}^k \left( \frac{1}{2} c_{rs}^j \alpha^{rs} + c_{r\bar{s}}^j \alpha^{r\bar{s}} \right), \end{cases}$$

for some  $k \in \{1, \dots, n-1\}$  and with  $c_{rs}^j, c_{r\bar{s}}^j \in \mathbb{C}$ . Here we use the notation  $\bar{\alpha}^i = \alpha^{\bar{i}}$  and  $\alpha^{r_1 \dots r_p \bar{s}_1 \dots \bar{s}_q} = \alpha^{s_1} \wedge \dots \wedge \alpha^{s_p} \wedge \alpha^{\bar{s}_1} \wedge \dots \wedge \alpha^{\bar{s}_q}$ . We may assume without restrictions that the coframe  $\{\alpha^i\}$  is unitary with respect to the balanced metric  $g$ . Indeed, since in the Gram-Schmidt process the spaces spanned by the first  $r$  elements of the original basis are preserved, we can modify the coframe  $\{\alpha^r\}$  making it unitary with respect to  $g$  and satisfying the same structure equations as in (1) with different structure constants. In this way  $g$  writes as

$$g = \sum_{r=1}^n \alpha^r \otimes \alpha^{\bar{r}}$$

and the balanced condition can be written in terms of  $c_{r\bar{s}}^l$ 's as

$$(2) \quad \sum_{r=1}^k c_{r\bar{r}}^l = 0,$$

for every  $l > k$  (see also [3, Lemma 2.1]).

Next we focus on the SKT metric  $g'$ . Since  $\mathfrak{g}$  is 2-step nilpotent, we have  $\partial\bar{\partial}\alpha^r = 0$ ,  $r = 1, \dots, n$ . If

$$\omega' = \sum_{i,j=1}^n a_{i\bar{j}} \alpha^i \wedge \alpha^{\bar{j}}$$

is the fundamental form of  $g'$ , then the SKT condition  $\partial\bar{\partial}\omega' = 0$  writes as

$$(3) \quad \sum_{i,j=k+1}^n a_{i\bar{j}} (\bar{\partial}\alpha^i \wedge \partial\alpha^{\bar{j}} - \partial\alpha^i \wedge \bar{\partial}\alpha^{\bar{j}}) = 0.$$

Equation (3) can be written in terms of the structure constants as

$$\sum_{i,j=k+1}^n \sum_{r,s,u,v=1}^k a_{i\bar{j}} \left( c_{r\bar{v}}^i \bar{c}_{u\bar{s}}^j + \frac{1}{4} c_{rs}^i \bar{c}_{uv}^j \right) \alpha^{rs\bar{u}\bar{v}} = 0.$$

By considering the component along  $\alpha^{rs\bar{s}\bar{r}}$  in the above expression we get

$$\begin{aligned} \sum_{i,j=k+1}^n a_{i\bar{j}} \left( c_{r\bar{r}}^i \bar{c}_{s\bar{s}}^j + \frac{1}{4} c_{rs}^i \bar{c}_{sr}^j - c_{s\bar{r}}^i \bar{c}_{s\bar{r}}^j - \frac{1}{4} c_{sr}^i \bar{c}_{sr}^j \right. \\ \left. + c_{s\bar{s}}^i \bar{c}_{r\bar{r}}^j + \frac{1}{4} c_{sr}^i \bar{c}_{rs}^j - c_{r\bar{s}}^i \bar{c}_{r\bar{s}}^j - \frac{1}{4} c_{rs}^i \bar{c}_{rs}^j \right) = 0, \end{aligned}$$

i.e. the condition

$$(4) \quad \sum_{i,j=k+1}^n a_{i\bar{j}} \left( c_{r\bar{r}}^i \bar{c}_{s\bar{s}}^j - c_{s\bar{r}}^i \bar{c}_{s\bar{r}}^j + c_{s\bar{s}}^i \bar{c}_{r\bar{r}}^j - c_{r\bar{s}}^i \bar{c}_{r\bar{s}}^j + c_{rs}^i \bar{c}_{rs}^j \right) = 0.$$

Now by taking the sum of (4) for  $r, s = 1, \dots, k$  and keeping in mind the balanced assumption (2), we get

$$(5) \quad \sum_{i,j=k+1}^n \sum_{r,s=1}^k a_{i\bar{j}} \left( 2c_{s\bar{r}}^i \bar{c}_{s\bar{r}}^j + c_{rs}^i \bar{c}_{rs}^j \right) = 0.$$

Let us consider now the  $(1, 0)$ -vectors  $X_{sr}$  and  $X_{r\bar{s}}$  on  $\mathfrak{g}$  defined by

$$X_{rs} = \sum_{i=k+1}^n c_{rs}^i X_i, \quad X_{r\bar{s}} = \sum_{i=k+1}^n \sqrt{2} c_{r\bar{s}}^i X_i,$$

where  $\{X_1, \dots, X_n\}$  is the dual frame to  $\{\alpha^1, \dots, \alpha^n\}$ . We have

$$\sum_{r,s=1}^k (\omega'(X_{rs}, \bar{X}_{rs}) + \omega'(X_{r\bar{s}}, \bar{X}_{r\bar{s}})) = \sum_{i,j=k+1}^n \sum_{r,s=1}^k a_{i\bar{j}} \left( 2c_{s\bar{r}}^i \bar{c}_{s\bar{r}}^j + c_{rs}^i \bar{c}_{rs}^j \right),$$

and so equation (5) implies  $X_{rs} = X_{r\bar{s}} = 0$  for every  $r, s = 1, \dots, k$ . So every  $\alpha^r$  is closed and  $\mathfrak{g}$  is abelian, contradicting the first assumption in the proof.  $\square$

*Remark 2.3.* Theorem 1.1 is trivial in dimension 6. Indeed, in view of [10] and Lemma 2.1, if a 6-dimensional complex nilmanifold  $(M, J)$  has an SKT metric, then every invariant  $J$ -Hermitian metric on  $(M, J)$  is SKT and so every invariant balanced metric compatible with  $J$  is automatically Kähler. More generally, the same argument works when  $M$  has arbitrary real dimension  $2n$ , but  $k = n - 1$ .

*Remark 2.4.* Another case where Theorem 1.1 is trivial is when  $k = 1$ . In this case the Lie algebra  $\mathfrak{g}$  is necessarily isomorphic to  $\mathfrak{h}_3^{\mathbb{R}} \oplus \mathbb{R}^{2n-3}$ , where  $\mathfrak{h}_3^{\mathbb{R}}$  is the real 3-dimensional Heisenberg algebra, and the result follows.

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DIPARTIMENTO DI MATEMATICA G. PEANO, UNIVERSITÀ DI TORINO, VIA CARLO ALBERTO 10,  
10123 TORINO, ITALY

*E-mail address:* `annamaria.fino@unito.it`

DIPARTIMENTO DI MATEMATICA G. PEANO, UNIVERSITÀ DI TORINO, VIA CARLO ALBERTO 10,  
10123 TORINO, ITALY

*E-mail address:* `luigi.vezzoni@unito.it`