PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 144, Number 6, June 2016, Pages 2565–2573
http://dx.doi.org/10.1090/proc/12958
Article electronically published on March 1, 2016

# SUBGROUP OF INTERVAL EXCHANGES GENERATED BY TORSION ELEMENTS AND ROTATIONS

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(Communicated by Nimish Shah)

ABSTRACT. Denote by $G$ the group of interval exchange transformations (IETs) on the unit interval. Let $G_{per} \subset G$ be the subgroup generated by torsion elements in $G$ (periodic IETs), and let $G_{rot} \subset G$ be the subset of 2-IETs (rotations).

The elements of the subgroup $H = \langle G_{per}, G_{rot} \rangle \subset G$ (generated by the sets $G_{per}$ and $G_{rot}$) are characterized constructively in terms of their Sah-Arnoux-Fathi (SAF) invariant. The characterization implies that a non-rotation type 3-IET lies in $H$ if and only if the lengths of its exchanged intervals are linearly dependent over $\mathbb{Q}$. In particular, $H \subsetneq G$.

The main tools used in the paper are the SAF invariant and a recent result by Y. Vorobets that $G_{per}$ coincides with the commutator subgroup of $G$.

## 1. A GROUP OF IETS

Denote by $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{N}$ the sets of real, rational and natural numbers. By a *standard interval* we mean a finite interval of the form $X = [a, b) \subset \mathbb{R}$ (left closed - right open). We write $|X| = b - a$ for its length.

By an IET (interval exchange transformation) we mean a pair $(X, f)$ where $X = [a, b)$ is a standard interval and $f$ is a right continuous bijection $f \colon X \to X$ with a finite set $D$ of discontinuities and such that the translation function $\gamma(x) = f(x) - x$ is piecewise constant.

The map $f$ itself is referred to as an IET, and then $X = \text{domain}(f)$ and $D = \text{disc}(f)$ denote the domain (also the range) of $f$ and the discontinuity set of $f$, respectively.

Given an IET $f \colon X \to X$, the set $\text{disc}(f)$ partitions $X$ into a finite number of subintervals $X_k$ in such a way that $f$ restricted to each $X_k$ is a translation

$$f|_{X_k} : x \to x + \gamma_k, \tag{1.1}$$

so that the action of $f$ reduces to a rearrangement of the intervals $X_k$. The number $r$ of exchanged intervals $X_k$ can be specified by calling $f$ an $r$-IET.

Denote by $\mathcal{G}$ the set of IETs. Then

$$\mathcal{G} = \bigcup_{r \geq 1} \mathcal{G}_r, \tag{1.2}$$

where $\mathcal{G}_r$ stands for the set of $r$-IETs:

$$\mathcal{G}_r = \{ f \in \mathcal{G} \mid \mathbf{card}(\text{disc}(f)) = r - 1 \}. \tag{1.3}$$

Received by the editors November 14, 2013 and, in revised form, July 26, 2015.
2010 *Mathematics Subject Classification.* Primary 37E05, 37E15, 54H15; Secondary 37A05.
The author was supported in part by research grant: NSF-DMS-1102298.

For a standard interval $X = [a, b)$, the subset

$$G(X) = \{f \in \mathcal{G} \mid \mathrm{domain}(f) = X\} \tag{1.4}$$

forms a group under composition (of bijections of $X$). Its identity is $\mathbf{1}_{G(X)} \in \mathcal{G}_1$, the identity map on $X$. Given two standard intervals $X$ and $Y$, there is a canonical isomorphism

$$\phi_{X,Y} : G(X) \to G(Y) \tag{1.5}$$

defined by the formula

$$\phi_{X,Y}(f) = l \circ f \circ l^{-1} \in G(Y), \quad \text{for } f \in G(X), \tag{1.6}$$

where $l = l_{X,Y}$ stands for the unique affine order preserving bijection $X \to Y$.

By the group of IETs we mean the group $G := G([0, 1))$. (It is isomorphic to $G(X)$, for any standard interval $X$.)

The interval exchange transformations have been a popular subject of study in ergodic theory. (We refer the reader to the book [12] by Marcelo Viana which may serve as a nice introduction and survey reference in the subject.) Most papers on IETs study these as dynamical systems; they concern specific dynamical properties (like minimality, ergodicity, mixing properties, etc.) the IETs may satisfy.

The focus of the present paper is different; we address certain questions on the group-theoretical structure of the group $G$ of IETs. (For recent results on this general area see [13] and [7].) In particular, we discuss possible generator subsets of the group $G$.

It was known for a while that the subgroup $G_{per}$ generated by periodic IETs forms a proper subgroup of $G$; in particular, $G_{per}$ contains no irrational rotations. (The SAF invariant introduced in the next section vanishes on $G_{per}$ but has non-zero value on irrational rotations.) On the other hand, the set $G_{per}$ contains some uniquely ergodic, even pseudo-Anosov (self-similar) IETs (see [2]).

We show that the subgroup $H = \langle G_{per}, G_{rot}\rangle$ generated by $G_{per}$ and the set of rotations $G_{rot} \subset G$ is still a proper subgroup of $G$. On the other hand, $H$ is large enough to contain all rank 2 IETs (see Section 4), in particular the IETs over quadratic number fields.

We present a constructive criterion (in terms of the SAF invariant) for a given IET to lie in $H$. It follows from this criterion that a 3-IET lies in $H$ if and only if the lengths of its exchanged subintervals are linearly independent. Note that 3-IETs generate the whole group of IETs. (More precisely, $\mathcal{G}_3 \cap G$ is a generating set for the group $G$.)

## 2. The SAF invariant and subgroups of $G(X)$

Throughout the paper (whenever vector spaces or linear dependence/independence are discussed) the implied field (if not specified) is always meant to be $\mathbb{Q}$, the field of rationals.

Denote by $\mathbb{T}$ the tensor product of two copies of $\mathbb{R}$ viewed as vector spaces (over $\mathbb{Q}$). Denote by $\mathbb{K}$ the skew symmetric tensor product of two copies of reals:

$$\mathbb{T} := \mathbb{R} \otimes_{\mathbb{Q}} \mathbb{R}; \qquad \mathbb{K} := \mathbb{R} \wedge_{\mathbb{Q}} \mathbb{R} \subset \mathbb{T}.$$

Recall that $\mathbb{K}$ is the vector subspace of $\mathbb{T}$ spanned by the wedge products

$$u \wedge v := u \otimes v - v \otimes u, \qquad u, v \in \mathbb{R}.$$

The *Sah-Arnoux-Fathi (SAF) invariant* (sometimes also called the *scissors congruence invariant*) of $f \in \mathcal{G}_r$ is defined by the formula

$$\mathrm{SAF}(f) := \sum_{k=1}^{r} \lambda_k \otimes \gamma_k \in \mathbb{T}, \tag{2.1}$$

where the vectors $\vec{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_r,)$, $\vec{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_r,) \in \mathbb{R}^r$ encode the lengths $\lambda_k = |X_k|$ of exchanged intervals $X_k$ and the corresponding translation constants $\gamma_k$, respectively (see (1.1)).

The SAF invariant was introduced independently by Sah [9] and Arnoux and Fathi [1]. The following lemma makes this invariant a useful tool in the study of IETs.

**Lemma 1.** *Let $X$ be a standard interval. Then*

(a) $\mathrm{SAF}\colon G(X) \to \mathbb{T}$ *is a group homomorphism.*$\check{2}$
(b) $\mathrm{SAF}\colon G(X) \to \mathbb{K}$ *is a surjective group homomorphism:* $\mathrm{SAF}(G(X)) = \mathbb{K} \subset \mathbb{T}$.$\check{2}$
(c) *If* $f \circ h = h \circ g$ *holds for* $f, g, h \in G(X)$, *then* $\mathrm{SAF}(f) = \mathrm{SAF}(g)$. *(The SAF invariant does not distinguish between conjugate IETs.)*

Clearly, (c) follows from (b) because both $\mathbb{T}$ and $\mathbb{K}$ are Abelian groups.

2.1. **Subgroups of $G(X)$.** Let $X$ be a standard interval. An IET $f \in G(X)$ is called periodic if $f$ is an element of finite order in the group $G(X)$ (defined in (1.4)). Set

$$\mathcal{G}_r(X) := \mathcal{G}_r \cap G(X), \quad r \geq 1, \tag{2.2}$$

(see notation (1.2)). Note that $\mathcal{G}_1(X) = \{\mathbf{1}_{G(X)}\}$ is a singleton.

Consider the following subgroups of $G(X)$:

- $G'(X) = [G(X), G(X)]$ is the commutator subgroup of $G$ (generated by the commutators $f^{-1}g^{-1}fg$, with $f, g \in G(X)$);$\check{3}$
- $G_{per}(X)$ is the subgroup of $G(x)$ generated by periodic IETs $f \in G(X)$;$\check{5}$
- $G_0(X) = \{f \in G(X) \mid \mathrm{SAF}(f) = 0\}$ (see (2.1));$\check{3}$
- $G_{rot}(X) = \mathcal{G}_1(X) \cup \mathcal{G}_2(X) = \mathcal{G}_2(X) \cup \{\mathbf{1}_{G(X)}\}$ is the group of rotations on $X$. (It contains all IETs $f \in G(X)$ with at most one discontinuity.)

Observe that (a) in Lemma 1 implies immediately the inclusions

$$G'(X) \subset G_0(X); \quad G_{per}(X) \subset G_0(X),$$

as well as the fact that the set $G_0(X)$ forms a subgroup of $G(X)$.

An unpublished theorem of Sah [9] (mentioned by Veech in [11]) contains Lemma 1 and the equality $G'(X) = G_0(X)$. This equality has been recently extended by Vorobets [13] to also include $G_{per}(X)$ as an additional set:

$$G'(X) = G_0(X) = G_{per}(X). \tag{2.3}$$

We refer to Vorobets' paper [13] for a nice self-contained introduction to the SAF invariant. In particular, the paper contains the proof of Lemma 1 and of the equality (2.3).

The results of the present paper concern the subgroup

$$H(X) = \langle G_{per}(X), G_{rot}(X) \rangle \subset G(X) \tag{2.4}$$

generated by periodic IETs and the rotations in $G(X)$. The elements of $H(X)$ are classified in terms of their SAF invariant; see Theorem 2. It is shown that "most" 3-IETs do not lie in $H(X)$ (Lemma 5 and Theorem 4). In particular, it follows that $H(X) \neq G(X)$.

## 3. Subgroup $H$ and its SAF invariant characterization

For $u \in \mathbb{R}$, denote

$$\mathbb{K}(u) = \{u \wedge v \mid v \in \mathbb{R}\} \subset \mathbb{K} = \mathbb{R} \wedge_{\mathbb{Q}} \mathbb{R}.$$

$\mathbb{K}(u)$ forms a vector subspace of $\mathbb{K}$. If $u \neq 0$, $\mathbb{K}(u)$ is isomorphic to $\mathbb{R}/\mathbb{Q}$.

In the next lemma we compute the SAF invariant for 2-IETs. This is known and follows immediately from the definition (2.1), but is included for completeness.

In what follows $X$ denotes a standard interval. Recall that $\mathcal{G}_r(X) = \mathcal{G}_r \cap G(X)$ stands for the set of $r$-IETs on $X$.

**Lemma 2.** *Assume that $f \in \mathcal{G}_2(X)$ exchanges two subintervals of lengths $\lambda_1$ and $\lambda_2$ (with $\lambda_1 + \lambda_2 = |X|$). Then*

$$\mathrm{SAF}(f) = |X| \wedge \lambda_1 \in \mathbb{K}(|X|).$$

*Proof.* For such $f$ the traslation constants are $\gamma_1 = -\lambda_2$ and $\gamma_2 = -\lambda_2 + 1 = \lambda_1$ (see (1.1)). It follows that

$$\begin{aligned}\mathrm{SAF}(f) = \lambda_1 \otimes \gamma_1 + \lambda_2 \otimes \gamma_2 &= \lambda_1 \otimes (-\lambda_2) + \lambda_2 \otimes \lambda_1 \\ &= \lambda_2 \wedge \lambda_1 = (\lambda_2 + \lambda_1) \wedge \lambda_1 = |X| \wedge \lambda_1.\end{aligned}$$

□

**Lemma 3.** *Let $\beta \in \mathbb{K}(|X|)$. Then there exists $f \in \mathcal{G}_2(X)$ such that* $\mathrm{SAF}(F) = \beta$.

*Proof.* Let $\beta = |X| \wedge t$. Select a rational $r \in \mathbb{Q}$ such that $0 < r|X| + t < |X|$. Set $\lambda_1 = r|X| + t$ and $\lambda_2 = |X| - \lambda_1$. Take $f \in \mathcal{G}_2(X)$ which exchanges two intervals of lengths $\lambda_1$ and $\lambda_2$. Then, by Lemma 2,

$$\mathrm{SAF}(f) = |X| \wedge \lambda_1 = |X| \wedge (r|X| + t) = |X| \wedge t,$$

completing the proof. □

**Corollary 1.** $\mathrm{SAF}(G_{rot}(X)) = \mathrm{SAF}(\mathcal{G}_2(X)) = \mathbb{K}(|X|)$.

*Proof.* Follows from Lemmas 2 and 3. □

**Theorem 1.** *Let $f \in G(X)$. Assume that* $\mathrm{SAF}(f) \in \mathbb{K}(|X|)$. *Then there exists a rotation $g \in \mathcal{G}_2(X)$ and $h_1, h_2 \in G_{per}(X)$ such that*

$$f = g \circ h_2 = h_1 \circ g.$$

Note that in the above theorem $G_{per}(X)$ can be replaced by $G'(X)$ or $G_0(X)$ (see (2.3)).

*Proof of Theorem* 1. By Lemma 3 there exists $g \in \mathcal{G}_2(X)$ such that $\mathrm{SAF}(g) = \mathrm{SAF}(f) \in \mathbb{K}(|X|)$. Take $h_1 = f \circ g^{-1}$, $h_2 = g^{-1} \circ f$. Then $h_1, h_2 \in G_0(X) = G_{per}(X)$ in view of Lemma 1. □

**Theorem 2.** *Let $f \in G(X)$. Then*

$$f \in H(X) \iff \mathrm{SAF}(f) \in \mathbb{K}(|X|).$$

*Proof.* Direction $\Rightarrow$. In view of (2.4), it is enough to show the following two inclusions:

$$S_1 = \mathrm{SAF}(G_{rot}(X)) \in \mathbb{K}(|X|); \quad S_2 = \mathrm{SAF}(G_0(X)) \in \mathbb{K}(|X|).$$

Both are immediate: $S_1 = \mathbb{K}(|X|)$ by Corollary 1, and $S_2 = 0$.

Direction $\Leftarrow$. Follows from Theorem 1. $\square$

## 4. Rank 2 IETs lie in $H$

By the span of a subset $A \subset \mathbb{R}$ (notation: $\mathbf{span}(A)$) we mean the minimal vector subspace of $\mathbb{R}$ (over $\mathbb{Q}$) containing $A$.

By the rank of an IET $f$ (notation $\mathbf{rank}(f)$) we mean the dimension of the span of the set the lengths of the intervals exchanged by $f$:

$$\mathbf{rank}(f) := \dim_Q(L(f)),$$

where

$$L(f) := \mathbf{span}(\{\lambda_1, \lambda_2, \dots, \lambda_r\}) \quad (\text{if } f \in \mathcal{G}_r). \tag{4.1}$$

The following implication is immediate: $\mathbf{rank}(f) = 1 \implies f \in G_{per} = G_0$.

**Theorem 3.** *For $f \in G(X)$, the following implication holds:*

$$\mathbf{rank}(f) \leq 2 \qquad \Longrightarrow \qquad f \in H(X).$$

*Proof.* We may assume that $\mathbf{rank}(f) = \dim(L(f)) = 2$. Since all $\lambda_k \in L(f)$, it follows that $|X| = \sum_{k=1}^{r} \lambda_k \in L(F)$. Let $B = \{|X|, u\}$ be a basis in $L(F)$. Then

$$\begin{aligned} \mathrm{SAF}(f) \in L(f) \wedge_{\mathbb{Q}} L(f) &= \{q\,|X| \wedge u \mid q \in \mathbb{Q}\} \\ &= \{|X| \wedge qu \mid q \in \mathbb{Q}\} \subset \mathbb{K}(|X|), \end{aligned}$$

whence $f \in H(X)$, in view of Theorem 2. $\square$

## 5. Criterion for 3-IETs to lie in $H$

Let $f \in \mathcal{G}_3(X)$. Then $f$ has 2 discontinuities, and $f$ acts by reversing the order of three subintervals, $X_1$, $X_2$ and $X_3$. Set $\lambda_i = |X_i|$. Then $X = \mathrm{domain}(f) = [a, a + \lambda_1 + \lambda_2 + \lambda_3)$, with some $a \in \mathbb{R}$. Set for convenience $a = 0$ so that $X = [0, \lambda_1 + \lambda_1 + \lambda_3)$.

The corresponding translation constants are easily computed:

$$\begin{aligned} &\gamma_1 = f(0) - 0 = \lambda_2 + \lambda_3 = |X| - \lambda_1; \\ &\gamma_2 = f(\lambda_1) - \lambda_1 = \lambda_3 - \lambda_1; \\ &\gamma_3 = f(\lambda_1 + \lambda_2) - (\lambda_1 + \lambda_2) = 0 - (\lambda_1 + \lambda_2) = \lambda_3 - |X|. \end{aligned}$$

(The above short computation follows general receipt for computing the translation constants $\gamma_k$ in terms of the constants $\lambda_k$ and the permutation specifying the way intervals $X_k$ are rearranged; see e.g. [10, Section 0]. The permutation here must be (321).)

Since $\lambda_2 = |X| - \lambda_1 - \lambda_3$, we can compute $\mathrm{SAF}(X)$ in terms of linear combinations of the wedge products involving only $X$, $\lambda_1$ and $\lambda_3$:

$$\mathrm{SAF}(X) = \sum_{k=1}^{3} \lambda_k \otimes \gamma_k = |X| \wedge (\lambda_1 - \lambda_3) - \lambda_1 \wedge \lambda_3. \tag{5.1}$$

We need the following known basic fact (see e.g. [13, Lemma 3.1]).

**Lemma 4.** *If $v_1, v_2, \ldots, v_n$ are $n \geq 2$ linearly independent real numbers, then the $\frac{n(n-1)}{2}$ wedge products $v_i \wedge v_j$, $1 \leq i < j \leq n$, are linearly independent.*

**Corollary 2.** *Let $v_1, v_2, v_3 \in \mathbb{R}$ be linearly independent. Then $v_1 \wedge v_2 \neq v_3 \wedge u$ for all $u \in \mathbb{R}$.*

*Proof.* Assume to the contrary that

$$v_1 \wedge v_2 = v_3 \wedge u$$

for some $u \in \mathbb{R}$. If $u \notin \mathbf{span}(\{v_1, v_2, v_3\})$, then the set $\{u, v_1, v_2, v_3\}$ is linearly independent, contradicting Lemma 4.

And if $u \in \mathbf{span}(\{v_1, v_2, v_3\})$, then $u = q_1 v_1 + q_2 v_2 + q_3 v_3$ with some $q_i \in \mathbb{Q}$ whence

$$v_1 \wedge v_2 = q_1\, v_3 \wedge v_1 + q_2\, v_3 \wedge v_2,$$

a contradiction with Lemma 4 again. □

**Lemma 5.** *Let $f\colon X \to X$ be a 3-IET and assume that* $\mathbf{rank}(f) = 3$. *(Equivalently, the set $\{\lambda_1, \lambda_1, \lambda_3\}$ is linearly independent.) Then $f \notin H(X)$.*

*Proof.* Assume to the contrary that $f \in H(X)$. Then, by Theorem 2, $\mathrm{SAF}(f) \in \mathbb{K}(|X|)$. It follows from (5.1) that $\lambda_1 \wedge \lambda_3 \in \mathbb{K}(|X|)$, i.e., that

$$\lambda_1 \wedge \lambda_3 = |X| \wedge u, \tag{5.2}$$

for some $u \in \mathbb{R}$. Since the numbers $\lambda_1, \lambda_3$ and $|X| = \lambda_1 + \lambda_2 + \lambda_3$ are linearly independent, (5.2) contradicts Corollary 2. □

**Theorem 4** (Criterion for a 3-IET to lie in $H$)**.** *Let $f\colon X \to X$ be a 3-IET. Then*

$$f \in H(X) \quad \Longleftrightarrow \quad \mathbf{rank}(f) \leq 2.$$

*Proof.* The direction $\Rightarrow$ is a contrapositive restatement of Lemma 5. The direction $\Leftarrow$ is given by Theorem 3. □

## 6. Validation of the membership in classes $G_0(X)$ and $H(X)$

Given $f \in G(X)$, we describe constructive procedures to decide whether $f \in G_{per}(X)$ and whether $f \in H(X)$.

6.1. **Does the inclusion $f \in G_{per}(X)$ hold?** Since $G_{per}(X) = G_0(X)$ (see (2.3)), one only has to test the equality

$$\mathrm{SAF}(f) := \sum_{k=1}^{r} \lambda_k \otimes \gamma_k = 0.$$

We assume that the *linear structure* of $f$ is known. By the linear structure of $f$ we mean:Ǐ

(a) a basis $B = \{v_1, v_2, \ldots, v_n\}$ of the finite dimensional space

$$L(f) := \mathbf{span}(\{\lambda_1, \lambda_2, \ldots, \lambda_r\});$$

(b) the (unique) linear representations $\lambda_k = \sum_{i=1}^{n} q_{k,i} v_i$, $1 \leq k \leq r$, with all $q_{k,i} \in \mathbb{Q}$.

Observe that all translation constants $\gamma_k$ also lie in $L(f)$, and their linear representation in terms of basis $B$ can be computed. This way one can get the representation

$$\mathrm{SAF}(f) = \sum_{1 \leq i < j \leq n} p_{i,j}\ v_i \wedge v_j, \tag{6.1}$$

with known $p_{i,j} \in \mathbb{Q}$. By Lemma 4, $\mathrm{SAF}(f) = 0$ if and only if all the constants $p_{i,j}$ vanish. Then

$$f \in G_{per}(X) = G_0(X) \quad \Longleftrightarrow \quad p_{i,j} = 0, \text{ for all } i, j.$$

6.2. **Does the inclusion $f \in H(X)$ hold?** Again we assume that the linear structure of $f$ is known. To answer the question, we proceed as follows.

First, we modify the basis $B$ of $L(f)$ to make $v_1 = |X|$. Then we proceed just as before and get (6.1) with known $p_{i,j} \in \mathbb{Q}$. By Theorem 2 and Lemma 4,

$$f \in H(X) \quad \Longleftrightarrow \quad \mathrm{SAF}(f) \in \mathbb{K}(|X|) \quad \Longleftrightarrow \quad p_{i,j} = 0, \text{ for } 2 \leq i < j \leq n.$$

## 7. Class $H$ is not preserved under induction

An IET $(X, f)$ is called minimal if its every orbit is dense in $X$. For a sufficient condition for an IET to be minimal see [5].

Let $(X, f)$ be an $r$-IET and assume that $Y \subset X$ is a standard subinterval. It is well known (see e.g. [4] or [5]) that the (first return) map $f_Y : Y \to Y$ induced by $f$ on $Y$ is also an IET (exchanging at most $r+1$ subintervals). It is also known that, under the minimality assumption, the SAF invariant is preserved under induction.

**Proposition 1** ([1, Part II, Proposition 2.13])**.** *Let $(X, f)$ be a minimal IET and assume that $Y \subset X$ is a standard subinterval. Then*

$$\mathrm{SAF}(f) = \mathrm{SAF}(f_Y).$$

**Corollary 3.** *Let $(X, f)$ be a minimal IET and assume that $Y \subset X$ is a standard subinterval. Then*

$$f \in G_{per}(X) \quad \Longleftrightarrow \quad f_Y \in G_{per}(Y).$$

*Proof.* Follows from Proposition 1 and the fact that $G_{per}(X) = G_0(X)$. □

**Corollary 4.** *Let $(X, f)$ be a minimal IET and assume that $Y \subset X$ is a standard subinterval such that $\frac{|Y|}{|X|} \in \mathbb{Q}$. Then*

$$f \in H(X) \quad \Longleftrightarrow \quad f_Y \in H(Y).$$

*Proof.* Follows from Proposition 1 because $\mathbb{K}(|X|) = \mathbb{K}(|Y|)$ assuming that $\frac{|Y|}{|X|} \in \mathbb{Q}$. □

**Theorem 5.** *Let $(X, f)$ be a minimal IET. Then the following two assertions are equivalent:*

*(a) There exists a standard subinterval $Y \subset X$ such that $f_Y \in H(Y)$;*

*(b) $\mathrm{SAF}(X) = u \wedge v$, for some $u, v \in \mathbb{R}$.*

One can show that if (b) of the above theorem holds and $\mathrm{SAF}(X) \neq 0$, then $u, v \in L(f)$.

*Proof.* (a)⇒(b). $\mathrm{SAF}(X) = \mathrm{SAF}(Y) \in \mathbb{K}([Y])$ whence $\mathrm{SAF}(X) = |Y| \wedge v$, for some $v \in \mathbb{R}$.

(b)⇒(a). Let $\mathrm{SAF}(X) = u \wedge v$, for some $u, v \in \mathbb{R}$. Without loss of generality, both $u, v$ can be selected positive (using the identities $u \wedge v = (-v) \wedge u$ and $1 \wedge 1 = 0$). Select $q \in \mathbb{Q}$ so that $0 < qu < |X|$. Select any standard subinterval $Y \subset X$ of length $|Y| = qu$. Then $\mathrm{SAF}(f_Y) = u \wedge v = |Y| \wedge (q^{-1}v) \in \mathrm{SAF}(|Y|)$, and hence $f_Y \in H(Y)$. □

Let $K \subset \mathbb{R}$ be a subfield of reals. An IET $f$ is said to be over $K$ if $L(f) \subset K$ (see (4.1)), i.e., if all (lengths of exchanged intervals) $\lambda_k$ lie in $K$.

We complete the paper by the following result.

**Theorem 6.** *Let $K$ be a real quadratic number field and let $(X, f)$ be a minimal IET over $K$. Let $Y \subset X$ be a standard subinterval. Then*

$$f_Y \in H(Y) \quad \iff \quad |Y| \in K.$$

*Proof.* Since $\mathbf{rank}(f) = \dim(L(f)) \leq \dim(F) = 2$, and $\mathbf{rank}(f) \neq 1$ (because $f$ is not periodic), we conclude that $\mathbf{rank}(f) = 2$ and $L(f) = K$.

*Proof of the ⇐ implication.* Select a basis $B = \{|Y|, u\}$ in $K$. Then

$$\mathrm{SAF}(f_Y) = \mathrm{SAF}(f) \in K \wedge K = \{q\,|Y| \wedge u \mid q \in \mathbb{Q}\} \subset \mathbb{K}(|Y|).$$

By Theorem 2, $f_Y \in H(Y)$.

*Proof of the ⇒ implication.* It has been proved in [3] that minimal rank 2 IETs must be uniquely ergodic. Thus $f$ is uniquely ergodic. By McMullen's result [6, Theorem 2.1], $\mathrm{SAF}(f) \neq 0$. (McMullen uses the "Galois flux" invariant for the IETs over a quadratic number field which, in his setting, is equivalent to the SAF invariant.)

Select a basis $B = \{|X|, u\}$ in $K$. Then

$$\mathrm{SAF}(f_Y) = \mathrm{SAF}(f) \in K \wedge K = \{q\,|X| \wedge u \mid q \in \mathbb{Q}\}.$$

Since $\mathrm{SAF}(f) \neq 0$, $\mathrm{SAF}(f_Y) = q\,|X| \wedge u$, with some $q \in \mathbb{Q}$, $q \neq 0$.

By Theorem 2, $f_Y \in H(Y)$ implies that $q\,|X| \wedge u = |Y| \wedge v$, for some $v \in \mathbb{R}$. This is incompatible with the assumption $Y \notin K$ in view of Corollary 2, completing the proof. □

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