PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 144, Number 7, July 2016, Pages 2799–2810
http://dx.doi.org/10.1090/proc/12931
Article electronically published on October 22, 2015

EXPLICIT COMPUTATIONS WITH THE DIVIDED SYMMETRIZATION OPERATOR

TEWODROS AMDEBERHAN

(Communicated by Ken Ono)

Abstract. Given a multi-variable polynomial, there is an associated divided symmetrization (in particular turning it into a symmetric function). Postinkov has found the volume of a permutohedron as a divided symmetrization (DS) of the power of a certain linear form. The main task in this paper is to exhibit and prove closed form DS-formulas for a variety of polynomials and some rational functions. We hope the results to be valuable and available to research practitioners in these areas. In addition, the methods of proof utilized here are simple and amenable to many more analogous computations.

Throughout, let S_n denote the symmetric group of permutations of the n-element set $\{1,2,\dots,n\}$. Given a function $f(\lambda_1,\dots,\lambda_n)$, the *divided symmetrization* (DS) of f is defined by

$$\begin{aligned}\langle f\rangle_n &:= \sum_{\sigma\in S_n}\sigma\cdot\left(\frac{f(\lambda_1,\dots,\lambda_n)}{\prod_{k=1}^{n-1}(\lambda_k-\lambda_{k+1})}\right)\\ &=\sum_{\sigma\in S_n} f(\lambda_{\sigma(1)},\dots,\lambda_{\sigma(n)})\prod_{k=1}^{n-1}\frac{1}{\lambda_{\sigma(k)}-\lambda_{\sigma(k+1)}}.\end{aligned}$$

For starters, a very good reference on the subject would be the monograph by Alain Lascoux [1]. Our motivation comes from a beautiful work [2] of Alexander Postnikov who has found the volume of a permutohedron in terms of divided symmetrization of $(\lambda_1x_1+\cdots+\lambda_{n+1}x_{n+1})^n$; in addition, he offers a combinatorial interpretation of the resulting coefficients. To put in context, we recall:

Proposition 0 (Postnikov). *If f is a polynomial of degree $n-1$ in the variables $\lambda_1,\dots,\lambda_n$, then its divided symmetrization $\langle f\rangle$ is a constant. If $\deg f<n-1$, then $\langle f\rangle=0$.*

Proof. Write $\langle f\rangle = g/\Delta$ where $\Delta=\prod_{i<j}(\lambda_i-\lambda_j)$ is the antisymmetric Vandermonde determinant and g is a some antisymmetric polynomial. Since $\langle f\rangle$ is symmetric, g is divisible by Δ. Because $\deg g=\deg\Delta=\binom{n}{2}$, their quotient must be a constant. A similar argument shows that if $\deg g<n-1$, then $\deg g<\deg\Delta$ and, hence, $g=0$. The proof is complete. $\square$

Received by the editors July 9, 2015 and, in revised form, August 23, 2015.
2010 *Mathematics Subject Classification.* Primary 05E10.

This proposition is applicable to (most) identities discussed in this note. But then, we ask:

Is it possible to obtain the actual values of these constants?

If so, then:

Do these constants have a closed form?

In this paper, we answer both questions for a variety of functions. Many more can be found at `arXiv:1406.0447.` The selection of these results is not meant to be representative, but rather reflects the author's personal taste. As a warm-up, we invite the reader to ponder on proving the following three numerical identities before continuing further on.

Involving products:

$$\sum_{\sigma\in S_{n+1}}\prod_{k=1}^{n}\frac{\sigma(1)+\cdots+\sigma(k)}{\sigma(k)-\sigma(k+1)}=n!,$$

$$\sum_{\sigma\in S_{n+1}}\sigma(1)^m\prod_{k=1}^{n}\frac{\sigma(1)-\sigma(k+1)}{\sigma(k)-\sigma(k+1)}=1^m+2^m+\cdots+(n+1)^m.$$

Involving sums:

$$\sum_{\sigma\in S_{n+1}}\sum_{k=1}^{n}\frac{\sigma(1)+\cdots+\sigma(k)}{\sigma(k)-\sigma(k+1)}=\binom{n+1}{2}n!,$$

which equals the numbers of edges in the Hasse diagram of the weak Bruhat order on S_{n+1}.

From here on, as promised, we state and prove numerous symmetrization formulas.

Lemma 1. *If $\lambda_1,\lambda_2,\ldots$ are variables, then*

$$\sum_{\sigma\in S_{n+1}}\sum_{k=1}^{n}\frac{\lambda_{\sigma(k)}}{\lambda_{\sigma(k)}-\lambda_{\sigma(k+1)}}=\binom{n+1}{2}n!.$$

Proof. Break up the sum according to $\sigma(k)=i,\sigma(k+1)=j$ so that

$$\sum_{\sigma\in S_{n+1}}\sum_{k=1}^{n}\frac{\lambda_{\sigma(k)}}{\lambda_{\sigma(k)}-\lambda_{\sigma(k+1)}}=\sum_{k=1}^{n}\sum_{\substack{i,j=1\\ i\neq j}}^{n+1}\frac{\lambda_i}{\lambda_i-\lambda_j}\sum_{\sigma'\in S_{n-1}}1=(n-1)!\sum_{k=1}^{n}\sum_{\substack{i,j=1\\ i<j}}^{n+1}\frac{\lambda_i-\lambda_j}{\lambda_i-\lambda_j}$$

$$=(n-1)!\cdot\binom{n+1}{2}\sum_{k=1}^{n}1=\binom{n+1}{2}n!.$$

□

Corollary 2. *If $\lambda_1,\lambda_2,\ldots$ are variables, then*

$$\sum_{\sigma\in S_{n+1}}\sum_{k=1}^{n}\frac{\lambda_{\sigma(1)}+\cdots+\lambda_{\sigma(k)}}{\lambda_{\sigma(k)}-\lambda_{\sigma(k+1)}}=\binom{n+1}{2}n!.$$

Proof. If $j\neq k,k+1$, then $\sum_{\sigma\in S_{n+1}}\frac{\lambda_{\sigma(j)}}{\lambda_{\sigma(k)}-\lambda_{\sigma(k+1)}}=0$. Thus by Lemma 1,

$$\sum_{\sigma\in S_{n+1}}\sum_{k=1}^{n}\frac{\lambda_{\sigma(1)}+\cdots+\lambda_{\sigma(k)}}{\lambda_{\sigma(k)}-\lambda_{\sigma(k+1)}}=\sum_{\sigma\in S_{n+1}}\sum_{k=1}^{n}\frac{\lambda_{\sigma(k)}}{\lambda_{\sigma(k)}-\lambda_{\sigma(k+1)}}=\binom{n+1}{2}n!.$$ □

Lemma 3. *If $Y, \lambda_1, \lambda_2, \dots$ are variables, then*

$$\left\langle \prod_{k=1}^{n} (\lambda_k - Y) \right\rangle_{n+1} = 1 \qquad \textit{and} \qquad \left\langle \prod_{k=1}^{n} (Y - \lambda_{k+1}) \right\rangle_{n+1} = 1.$$

Proof. We prove the first identity only; the same argument works for the other. When $n = 1$, the vacuous product confirms the assertion. We induct on n. The base case $n = 1$ is easily verifiable too, since $\frac{\lambda_1 - Y}{\lambda_1 - \lambda_2} + \frac{\lambda_2 - Y}{\lambda_2 - \lambda_1} = 1$. Suppose the statement holds for $n-1$. For the next step n, observe that the polynomial on the left-hand side is of degree at most n, in Y. Let's compute its values at the $n+1$ points $Y \in \{\lambda_1, \dots, \lambda_{n+1}\}$ as follows: for each $1 \leq j \leq n+1$,

$$\begin{aligned}
\sum_{\sigma \in S_{n+1}} \prod_{k=1}^{n} \frac{\lambda_{\sigma(k)} - \lambda_j}{\lambda_{\sigma(k)} - \lambda_{\sigma(k+1)}} &= \sum_{\sigma' \in S_n} \frac{\lambda_{\sigma'(n)} - \lambda_j}{\lambda_{\sigma'(n)} - \lambda_{\sigma(n+1)}} \prod_{k=1}^{n-1} \frac{\lambda_{\sigma'(k)} - \lambda_j}{\lambda_{\sigma'(k)} - \lambda_{\sigma'(k+1)}} \\
&= \sum_{\sigma' \in S_n} \frac{\lambda_{\sigma'(n)} - \lambda_j}{\lambda_{\sigma'(n)} - \lambda_j} \prod_{k=1}^{n-1} \frac{\lambda_{\sigma'(k)} - \lambda_j}{\lambda_{\sigma'(k)} - \lambda_{\sigma'(k+1)}} \\
&= \sum_{\sigma' \in S_n} \prod_{k=1}^{n-1} \frac{\lambda_{\sigma'(k)} - \lambda_j}{\lambda_{\sigma'(k)} - \lambda_{\sigma'(k+1)}} = 1,
\end{aligned}$$

where we used the induction assumption. So, the polynomial is a constant and the proof follows. $\square$

Lemma 4. *As a direct consequence, we obtain the following partial fraction decomposition:*

$$\left\langle \frac{1}{\lambda_j - Y} \right\rangle_{n+1} = (-1)^{n+1-j} \binom{n}{j-1} \prod_{k=1}^{n+1} \frac{1}{\lambda_k - Y}, \qquad j \in \{1, \dots, n+1\}.$$

Proof. We rewrite the assertion into an equivalent formulation and show that

$$\sum_{\sigma \in S_{n+1}} \frac{\lambda_{\sigma(n+1)} - Y}{\lambda_{\sigma(j)} - Y} \prod_{k=1}^{n} \frac{\lambda_{\sigma(k)} - Y}{\lambda_{\sigma(k)} - \lambda_{\sigma(k+1)}} = (-1)^{n+1-j} \binom{n}{j-1}.$$

Denote the sum on the left-hand side by $\Psi_{n,j}(Y)$. The case $j = n+1$ is simply the content of Lemma 3. So, assume $\sigma(n+1) \neq \sigma(j)$ (or $j \leq n$). By dropping off vanishing terms along the way, we compute at $Y = \lambda_\ell$:

$$\begin{aligned}
\Psi_{n,j}(\lambda_\ell) &= \sum_{S_{n+1}} \prod_{k=j}^{n} \frac{\lambda_{\sigma(k+1)} - \lambda_\ell}{\lambda_{\sigma(k)} - \lambda_{\sigma(k+1)}} \prod_{k=1}^{j-1} \frac{\lambda_{\sigma(k)} - \lambda_\ell}{\lambda_{\sigma(k)} - \lambda_{\sigma(k+1)}} \\
&= - \sum_{\substack{\sigma \in S_{n+1} \\ \sigma(j) = \ell}} \prod_{k=j+1}^{n} \frac{\lambda_{\sigma(k+1)} - \lambda_\ell}{\lambda_{\sigma(k)} - \lambda_{\sigma(k+1)}} \prod_{k=1}^{j-2} \frac{\lambda_{\sigma(k)} - \lambda_\ell}{\lambda_{\sigma(k)} - \lambda_{\sigma(k+1)}}.
\end{aligned}$$

Now, form a disjoint partition of the underlying set of permutations as

$$\begin{aligned}
&\{\sigma \in S_{n+1} : \sigma(j) = \ell\} \\
&= \biguplus_{\substack{A \subset \{1, \dots, n+1\} - \{\ell\} \\ (j-1)\text{-subset}}} \{\sigma \in S_{n+1} | \sigma(j) = \ell \text{ and } \{\sigma(1), \dots, \sigma(j-1)\} = A\}.
\end{aligned}$$

With this setup and by Lemma 3, the above summation becomes

$$\begin{aligned}\Psi_{n,j}(\lambda_\ell) &= -\sum_{\substack{A\subset\{1,\ldots,n+1\}-\{\ell\}\\(j-1)\text{-subset}}}\sum_{S_{n-j+1}}\prod_{k=1}^{n-j}\frac{\lambda_{\sigma(k+1)}-\lambda_\ell}{\lambda_{\sigma(k)}-\lambda_{\sigma(k+1)}}\sum_{S_{j-1}}\prod_{k=1}^{j-2}\frac{\lambda_{\sigma(k)}-\lambda_\ell}{\lambda_{\sigma(k)}-\lambda_{\sigma(k+1)}}\\ &= -\sum_{\substack{A\subset\{1,\ldots,n+1\}-\{\ell\}\\(j-1)\text{-subset}}}(-1)^{n-j}\cdot 1 = (-1)^{n-j+1}\binom{n}{j-1}.\end{aligned}$$

The polynomial $\Psi_{n,j}(Y)$, being of degree at most n and attaining the same value at the $n+1$ points $\lambda_1,\ldots,\lambda_{n+1}$, must be constant. The proof follows. □

Lemma 5. *If* $Y,\lambda_1,\lambda_2,\ldots,$ *then*

$$\sum_{j=1}^{n}\prod_{\substack{k=1\\k\neq j}}^{n}\frac{Y-\lambda_j}{\lambda_k-\lambda_j}=1.$$

Proof. This may be derived from the Lagrange interpolation formula. □

Lemma 6. *If* $Y,\lambda_1,\lambda_2,\ldots$ *and* $1\le j\le n+1$, *then*

$$\langle(\lambda_j-Y)^n\rangle_{n+1}=(-1)^{j-1}\binom{n}{j-1}.$$

Proof. Let $j=1$. From Lemma 4 and Lemma 5 (in that order), we obtain

$$\begin{aligned}\sum_{\sigma\in S_{n+1}}\prod_{k=1}^{n}\frac{\lambda_{\sigma(j)}-Y}{\lambda_{\sigma(k)}-\lambda_{\sigma(k+1)}} &= \sum_{\ell=1}^{n+1}\sum_{\sigma'\in S_n}\frac{(\lambda_\ell-Y)^n}{\lambda_\ell-\lambda_{\sigma'(2)}}\prod_{k=2}^{n}\frac{1}{\lambda_{\sigma'(k)}-\lambda_{\sigma'(k+1)}}\\ &= -(-1)^{n-1}\sum_{\ell=1}^{n+1}\prod_{\substack{k=1\\k\neq\ell}}^{n+1}\frac{\lambda_\ell-Y}{\lambda_k-\lambda_\ell}=(-1)^n(-1)^n=1.\end{aligned}$$

For the case $j=n+1$, once more, Lemma 4 and Lemma 5 yield

$$\begin{aligned}\sum_{\sigma\in S_{n+1}}\prod_{k=1}^{n}\frac{\lambda_{\sigma(j)}-Y}{\lambda_{\sigma(k)}-\lambda_{\sigma(k+1)}} &= \sum_{\ell=1}^{n+1}\sum_{\sigma'\in S_n}\frac{(\lambda_\ell-Y)^n}{\lambda_{\sigma'(n)}-\lambda_\ell}\prod_{k=1}^{n-1}\frac{1}{\lambda_{\sigma'(k)}-\lambda_{\sigma'(k+1)}}\\ &= \sum_{\ell=1}^{n+1}\prod_{\substack{k=1\\k\neq\ell}}^{n+1}\frac{\lambda_\ell-Y}{\lambda_k-\lambda_\ell}=(-1)^n.\end{aligned}$$

Suppose that $2 \le j \le n$. By Lemma 4, Lemma 5, and Pascal's recurrence we find that

$$\begin{aligned}
&\sum_{\sigma \in S_{n+1}} \prod_{k=1}^{n} \frac{\lambda_{\sigma(j)} - Y}{\lambda_{\sigma(k)} - \lambda_{\sigma(k+1)}} \\
&\quad = \sum_{\ell=1}^{n+1} \sum_{\sigma' \in S_n} \frac{(\lambda_\ell - Y)^n}{(\lambda_{\sigma'(j-1)} - \lambda_\ell)(\lambda_\ell - \lambda_{\sigma'(j+1)})} \prod_{\substack{k=1\\ k \neq j, j-1}}^{n} \frac{1}{\lambda_{\sigma'(k)} - \lambda_{\sigma'(k+1)}} \\
&\quad = \sum_{\ell=1}^{n+1} \sum_{\sigma' \in S_n} \left[\frac{(\lambda_\ell - Y)^n}{\lambda_{\sigma'(j-1)} - \lambda_\ell}\right] \frac{1}{\lambda_{\sigma'(j-1)} - \lambda_{\sigma'(j+1)}} \prod_{\substack{k=1\\ k \neq j-1, j}}^{n} \frac{1}{\lambda_{\sigma'(k)} - \lambda_{\sigma'(k+1)}} \\
&\qquad + \sum_{\ell=1}^{n+1} \sum_{\sigma' \in S_n} \left[\frac{(\lambda_\ell - Y)^n}{\lambda_\ell - \lambda_{\sigma'(j+1)}}\right] \frac{1}{\lambda_{\sigma'(j-1)} - \lambda_{\sigma'(j+1)}} \prod_{\substack{k=1\\ k \neq j-1, j}}^{n} \frac{1}{\lambda_{\sigma'(k)} - \lambda_{\sigma'(k+1)}} \\
&\quad = \left[(-1)^{n-j+1}\binom{n-1}{j-2} - (-1)^{n-j}\binom{n-1}{j-1}\right] \sum_{\ell=1}^{n+1} \prod_{\substack{k=1\\ k \neq \ell}}^{n+1} \frac{\lambda_\ell - Y}{\lambda_k - \lambda_\ell} \\
&\quad = \left[(-1)^{n-j+1}\binom{n}{j-1}\right](-1)^n = (-1)^{j-1}\binom{n}{j-1}.
\end{aligned}$$

The proof is complete. □

Corollary 7. *If* $\lambda_1, \lambda_2, \ldots$ *are variables and* $e_n(\lambda)$ *is the* n^{th}*-elementary symmetric function, then*

$$\left\langle \prod_{k=1}^{n} \lambda_k^2 \right\rangle_{n+1} = e_n(\lambda_1, \ldots, \lambda_{n+1}).$$

Proof. We prove the equivalent claim $\sum_{\sigma \in S_{n+1}} \frac{1}{\lambda_{\sigma(n+1)}} \prod_{k=1}^{n} \frac{\lambda_{\sigma(k)}}{\lambda_{\sigma(k)} - \lambda_{\sigma(k+1)}} = \sum_{j=1}^{n+1} \frac{1}{\lambda_j}$. To this end,

$$\begin{aligned}
&\sum_{S_{n+1}} \frac{1}{\lambda_{\sigma(n+1)}} \prod_{k=1}^{n} \frac{\lambda_{\sigma(k)}}{\lambda_{\sigma(k)} - \lambda_{\sigma(k+1)}} \\
&\quad = \sum_{S_{n+1}} \frac{\lambda_{\sigma(n)}}{\lambda_{\sigma(n+1)}(\lambda_{\sigma(n)} - \lambda_{\sigma(n+1)})} \prod_{k=1}^{n-1} \frac{\lambda_{\sigma(k)}}{\lambda_{\sigma(k)} - \lambda_{\sigma(k+1)}} \\
&\quad = \sum_{S_{n+1}} \left[\frac{1}{\lambda_{\sigma(n+1)}} + \frac{1}{\lambda_{\sigma(n)} - \lambda_{\sigma(n+1)}}\right] \prod_{k=1}^{n-1} \frac{\lambda_{\sigma(k)}}{\lambda_{\sigma(k)} - \lambda_{\sigma(k+1)}}.
\end{aligned}$$

Next, we separate the last sum into two and apply Lemma 3 to the first summation as follows:

$$\sum_{S_{n+1}} \frac{1}{\lambda_{\sigma(n+1)}} \prod_{k=1}^{n-1} \frac{\lambda_{\sigma(k)}}{\lambda_{\sigma(k)} - \lambda_{\sigma(k+1)}} = \sum_{j=1}^{n+1} \frac{1}{\lambda_j} \sum_{S_n} \prod_{k=1}^{n-1} \frac{\lambda_{\sigma(k)}}{\lambda_{\sigma(k)} - \lambda_{\sigma(k+1)}} = \sum_{j=1}^{n+1} \frac{1}{\lambda_j}.$$

Since $f = \prod_{k=1}^{n-1} \lambda_k$ is of degree $n-1$, Proposition 0 implies the vanishing of the second sum:

$$\sum_{S_{n+1}} \frac{1}{\lambda_{\sigma(n)} - \lambda_{\sigma(n+1)}} \prod_{k=1}^{n-1} \frac{\lambda_{\sigma(k)}}{\lambda_{\sigma(k)} - \lambda_{\sigma(k+1)}} = \sum_{S_{n-1}} \prod_{k=1}^{n-1} \lambda_{\sigma(k)} \prod_{k=1}^{n} \frac{1}{\lambda_{\sigma(k)} - \lambda_{\sigma(k+1)}} = 0.$$

The proof is complete. □

Corollary 8. *If $Y, \lambda_1, \lambda_2, \ldots$ are variables, then*

$$\left\langle \prod_{k=1}^{n} (\lambda_1 + \lambda_2 - \lambda_{k+1} - Y) \right\rangle_{n+1} = 2^{n+1} - n - 2.$$

Proof. Rearrange the left-hand side (LHS, for short), apply Lemma 4, and proceed as

$$\begin{aligned}
\text{LHS} &= \sum_{\substack{i,j=1\\ i\neq j}}^{n+1} \frac{\prod_{k=1}^{n+1} \lambda_i + \lambda_j - \lambda_k - Y}{(\lambda_j - Y)(\lambda_i - \lambda_j)} \sum_{S_{n-1}} \frac{1}{\lambda_j - \lambda_{\sigma(3)}} \prod_{k=3}^{n} \frac{1}{\lambda_{\sigma(k)} - \lambda_{\sigma(k+1)}} \\
&= (-1)^{n+1} \sum_{\substack{i,j=1\\ i\neq j}}^{n+1} \frac{\prod_{k=1}^{n+1} \lambda_i + \lambda_j - \lambda_k - Y}{(\lambda_j - Y)(\lambda_i - \lambda_j)} \prod_{\substack{k=1\\ k\neq i,j}}^{n+1} \frac{1}{\lambda_k - \lambda_j} \\
&= -\sum_{\substack{i,j=1\\ i\neq j}}^{n+1} \prod_{\substack{k=1\\ k\neq i}}^{n+1} (\lambda_k - \lambda_j + Y - \lambda_i) \prod_{\substack{k=1\\ k\neq j}}^{n+1} (\lambda_k - \lambda_j)^{-1} \\
&= -\sum_{i,j=1}^{n+1} \prod_{\substack{k=1\\ k\neq i}}^{n+1} (\lambda_k - \lambda_j + Y - \lambda_i) \prod_{\substack{k=1\\ k\neq j}}^{n+1} (\lambda_k - \lambda_j)^{-1} + \sum_{j=1}^{n+1} \prod_{\substack{k=1\\ k\neq j}}^{n+1} \frac{\lambda_k - 2\lambda_j + Y}{\lambda_k - \lambda_j} \\
&=: \mathrm{I}_n(Y) + \mathrm{II}_{n+1}(Y).
\end{aligned}$$

To determine the second sum, we induct on n. Let's evaluate $\mathrm{II}_{n+1}(\lambda_\ell)$ for each $1 \leq \ell \leq n+1$:

$$\mathrm{II}_{n+1}(\lambda_\ell) = 1 + \sum_{\substack{j=1\\ j\neq \ell}}^{n+1} 2 \prod_{\substack{k=1\\ k\neq j,\ell}}^{n+1} \frac{\lambda_k - 2\lambda_j + \lambda_\ell}{\lambda_k - \lambda_j} = 1 + 2\mathrm{II}_n(\lambda_\ell).$$

After consulting initial conditions, this recurrence implies $\mathrm{II}_{n+1}(\lambda_\ell) = 2^{n+1} - 1$. Since $\deg \mathrm{II}_{n+1} \leq n$, as a polynomial in Y, it must be a constant. Next, we employ Proposition 0 which permits us to work with $\lambda_k = k$ (even ignoring Y) and thus obtain

$$\begin{aligned}
\mathrm{I}_{n+1}(Y) &= \sum_{i,j=1}^{n+1} (-1)^{n+1-j} \binom{i+j-1}{n+1} \binom{n+1}{j} \\
&= \sum_{j=1}^{n+1} \binom{n+1}{j} (-1)^{n+1-j} \sum_{i=1}^{n+1} \binom{i+j-1}{n+1} \\
&= \sum_{j=1}^{n+1} (-1)^{n+1-j} \binom{n+1}{j} \sum_{k=0}^{j-1} \binom{n+1+k}{k} \\
&= \sum_{j=1}^{n+1} (-1)^{n+1-j} \binom{n+1}{j} \binom{n+j+1}{j-1} = n+1.
\end{aligned}$$

The last identity is a consequence of the expansions of $(1-x)^{-Z-3}(1-x)^{Z+1} = (1-x)^{-2}$ which yield $\sum_{j=0}^{n}(-1)^{n-j}\binom{Z+j+2}{j}\binom{Z+1}{n-j} = n+1$, then simply replace $Z=n$ here. In conclusion, we have $-\mathrm{I}_{n+1}(Y) + \mathrm{II}_{n+1}(Y) = -n-1+2^{n+1}-1$ as required. The proof is complete. $\square$

Lemma 9. *Let $Y, Z, \lambda_1, x_1, \lambda_2, x_2, \ldots$ be variables. Then, we have*

$$\left\langle \frac{1}{\lambda_1 - Z} \prod_{k=1}^{n} (\lambda_1 x_1 + \cdots + \lambda_k x_k - Y) \right\rangle_{n+1} = \prod_{k=1}^{n} (Y - (x_1 + \cdots + x_k) Z) \prod_{k=1}^{n+1} (\lambda_k - Z)^{-1}.$$

Proof. We proceed by induction. Assume the statement is valid for n. So, for $n+1$, we find that

$$\begin{aligned}
&\sum_{S_{n+1}} \frac{1}{\lambda_{\sigma(1)} - Z} \prod_{k=1}^{n} \frac{\sum_{i=1}^{k} \lambda_{\sigma(i)} x_i - Y}{\lambda_{\sigma(k)} - \lambda_{\sigma(k+1)}} \\
&= \sum_{j=1}^{n+1} \frac{\lambda_j x_1 - Y}{\lambda_j - Z} \sum_{S_n} \frac{-1}{\lambda_{\sigma(2)} - \lambda_j} \prod_{k=2}^{n} \frac{\sum_{i=2}^{k} \lambda_{\sigma(i)} x_i - (Y - \lambda_j x_1)}{\lambda_{\sigma(k)} - \lambda_{\sigma(k+1)}} \\
&= \sum_{j=1}^{n+1} \frac{Y - \lambda_j x_1}{\lambda_j - Z} \prod_{k=1}^{n-1} (Y - \lambda_j x_1 - \sum_{i=1}^{k} x_{i+1} \lambda_j) \prod_{\substack{k=1 \\ k \neq j}}^{n+1} (\lambda_k - \lambda_j)^{-1} \\
&= \sum_{j=1}^{n+1} \frac{1}{\lambda_j - Z} \prod_{k=1}^{n} (Y - \sum_{i=1}^{k} x_i \lambda_j) \prod_{\substack{k=1 \\ k \neq j}}^{n+1} (\lambda_k - \lambda_j)^{-1} \\
&= \prod_{k=1}^{n} (Y - \sum_{i=1}^{k} x_i Z) \prod_{k=1}^{n+1} (\lambda_k - Z)^{-1}.
\end{aligned}$$

The last equality results from the standard partial fraction decomposition applied to the rational function $\prod_{k=1}^{n}(Y - \sum_{i=1}^{k} x_i Z) \prod_{k=1}^{n+1}(\lambda_k - Z)^{-1}$, in the variable Z. The proof is complete. $\square$

Corollary 10. *Let $Y, \lambda_1, x_, \lambda_2, x_2, \ldots$ be variables. Then, we have*

$$\left\langle \prod_{k=1}^{n} (\lambda_1 x_1 + \cdots + \lambda_k x_k - Y) \right\rangle_{n+1} = \prod_{k=1}^{n} (x_1 + \cdots + x_k).$$

Proof. A reduction of the sum to S_n and a use of Lemma 9 (with $Z \to \lambda_j, Y \to Y - \lambda_j x_1$) yields

$$\begin{aligned}
&\sum_{S_{n+1}} \prod_{k=1}^{n} \frac{\sum_{i=1}^{k} \lambda_{\sigma(i)} x_i - Y}{\lambda_{\sigma(k)} - \lambda_{\sigma(k+1)}} \\
&\quad = \sum_{j=1}^{n+1} (Y - \lambda_j x_1) \sum_{S_n} \frac{1}{\lambda_{\sigma(2)} - \lambda_j} \prod_{k=2}^{n} \frac{\sum_{i=2}^{k} \lambda_{\sigma(i)} x_i - (Y - \lambda_j x_1)}{\lambda_{\sigma(k)} - \lambda_{\sigma(k+1)}} \\
&\quad = \sum_{j=1}^{n+1} (Y - \lambda_j x_1) \prod_{k=1}^{n-1} (Y - \lambda_j x_1 - \sum_{i=1}^{k} x_{i+1} \lambda_j) \prod_{\substack{k=1 \\ k \neq j}}^{n+1} (\lambda_k - \lambda_j)^{-1} \\
&\quad = \sum_{j=1}^{n+1} \prod_{k=1}^{n} (Y - \sum_{i=1}^{k} x_i \lambda_j) \prod_{\substack{k=1 \\ k \neq j}}^{n+1} (\lambda_k - \lambda_j)^{-1} = \prod_{k=1}^{n} (x_1 + \cdots + x_k).
\end{aligned}$$

To justify the last equality, rearrange the partial fraction seen in the proof of Lemma 9 as

$$\sum_{j=1}^{n+1} \frac{\lambda_{n+1} - Z}{\lambda_j - Z} \prod_{k=1}^{n} (Y - \sum_{i=1}^{k} x_i \lambda_j) \prod_{\substack{k=1 \\ k \neq j}}^{n+1} (\lambda_k - \lambda_j)^{-1} = \prod_{k=1}^{n} \frac{Y - \sum_{i=1}^{k} x_i Z}{\lambda_k - Z},$$

and take the limit $Z \to \infty$ on both sides. The proof follows. □

Corollary 11. *We have the identity (including its q-analogue)*

$$\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} (i + a)^n = n!.$$

Proof. Although this result is well known, we present a different approach from the more general relation that we saw in the proof of Corollary 10; that is,

$$\sum_{j=1}^{n+1} \prod_{k=1}^{n} (Y - k\lambda_j) \prod_{\substack{k=1 \\ k \neq j}}^{n+1} (\lambda_k - \lambda_j)^{-1} = n!.$$

Now, replace $Y = 0$ and $\lambda_j = j + a - 1$ to arrive at the desired conclusion. □

Lemma 12. *Let $Y, Z, \lambda_1, \lambda_2, \ldots$ be variables. Then, we have*

$$\left\langle \frac{1}{\lambda_{n+1} - Z} \prod_{k=1}^{n} (Y - \lambda_{k+1} - \cdots - \lambda_{n+1}) \right\rangle_{n+1} = \prod_{k=1}^{n} (Y - kZ) \prod_{k=1}^{n+1} (\lambda_k - Z)^{-1}.$$

Proof. This might be considered a "dual" version with a similar proof to Lemma 9. □

Lemma 13. *Let $Y, \lambda_1, \lambda_2, \ldots$ be variables and $H_n := \sum_{k=1}^{n} \frac{1}{k}$ the harmonic numbers. Then,*

$$\left\langle \prod_{k=1}^{n} \frac{\lambda_1 + \cdots + \lambda_k - Y}{\lambda_k - Y} \right\rangle_{n+1} = \frac{n!(1 - H_n)(\lambda_1 + \cdots + \lambda_{n+1} - Y)}{\prod_{k=1}^{n+1} \lambda_k - Y}.$$

Proof. We convert the left-hand side into an equivalent form and prove that

$$\sum_{S_{n+1}} (\lambda_{\sigma(n+1)} - Y) \prod_{k=1}^{n} \frac{\sum_{i=1}^{k} \lambda_{\sigma(i)} - Y}{\lambda_{\sigma(k)} - \lambda_{\sigma(k+1)}} = n!(1 - H_n) \left(\sum_{k=1}^{n+1} \lambda_k - Y \right).$$

The second half of the last identity follows from Corollary 10, so we only consider the first half with the help of Lemma 12 (use $S_{n+1} \to S_n, Z \to \lambda_j, Y \to \lambda - Y - \lambda_j$) where $|\lambda| := \sum_{i=1}^{n+1} \lambda_i$. Namely,

$$\begin{aligned}
&\sum_{S_{n+1}} \lambda_{\sigma(n+1)} \prod_{k=1}^{n} \frac{\sum_{i=1}^{k} \lambda_{\sigma(i)} - Y}{\lambda_{\sigma(k)} - \lambda_{\sigma(k+1)}} \\
&\quad = \sum_{S_{n+1}} \lambda_{\sigma(n+1)} \prod_{k=1}^{n} \frac{(|\lambda| - Y) - \sum_{i=k+1}^{n+1} \lambda_{\sigma(i)}}{\lambda_{\sigma(k)} - \lambda_{\sigma(k+1)}} \\
&\quad = \sum_{j=1}^{n+1} \lambda_j \sum_{S_n} \frac{(|\lambda| - Y) - \lambda_j}{\lambda_{\sigma(n)} - \lambda_j} \prod_{k=1}^{n-1} \frac{(|\lambda| - Y - \lambda_j) - \sum_{i=k+1}^{n} \lambda_{\sigma(i)}}{\lambda_{\sigma(k)} - \lambda_{\sigma(k+1)}} \\
&\quad = \sum_{j=1}^{n+1} \lambda_j (|\lambda| - Y - \lambda_j) \prod_{k=1}^{n-1} (|\lambda| - Y - (k+1)\lambda_j) \prod_{\substack{k=1 \\ k \neq j}}^{n+1} (\lambda_k - \lambda_j)^{-1} \\
&\quad = \sum_{j=1}^{n+1} \lambda_j \prod_{k=1}^{n} (|\lambda| - Y - k\lambda_j) \prod_{\substack{k=1 \\ k \neq j}}^{n+1} (\lambda_k - \lambda_j)^{-1}.
\end{aligned}$$

Expand the polynomial $P_n(Z) := Z \prod_{k=1}^{n} (|\lambda| - Y - kZ) + n! \prod_{k=1}^{n+1} (\lambda_k - Z)$, of degree n, in the Lagrange interpolating basis:

$$P_n(Z) = \sum_{j=1}^{n+1} \lambda_j \prod_{k=1}^{n} (|\lambda| - Y - k\lambda_j) \prod_{\substack{k=1 \\ k \neq j}}^{n+1} \frac{\lambda_k - Z}{\lambda_k - \lambda_j}.$$

Extracting the coefficients of Z^n from both sides leads to $n!(1 - H_n)|\lambda| + n! H_n Y$. The proof is complete. □

Lemma 14. *Let $Y, Z, \lambda_1, \lambda_2, \ldots$ be variables. Then, we have*

$$\left\langle \frac{1}{(\lambda_1 - Y)(\lambda_{n+1} - Y)} \right\rangle_{n+1} = (Z - Y)^n \prod_{k=1}^{n+1} \frac{1}{(\lambda_k - Y)(\lambda_k - Z)}.$$

Proof. We proceed by induction on n. For $n+1$, the sum over S_{n+1} is established by

$$\begin{aligned}
\mathrm{LHS}_{n+1} &= \sum_{\substack{i,j=1\\ i\neq j}}^{n+1} \frac{-1}{(\lambda_i - Y)(\lambda_j - Z)} \sum_{S_{n-1}} \frac{1}{(\lambda_{\sigma(2)}-\lambda_i)(\lambda_{\sigma(n)}-\lambda_j)} \prod_{k=2}^{n-1} \frac{1}{\lambda_{\sigma(k)}-\lambda_{\sigma(k+1)}} \\
&= \sum_{\substack{i,j=1\\ i\neq j}}^{n+1} \frac{-1}{(\lambda_i - Y)(\lambda_j - Z)} (\lambda_j-\lambda_i)^{n-2} \prod_{\substack{k=1\\ k\neq i,j}}^{n+1} \frac{1}{(\lambda_k-\lambda_i)(\lambda_k-\lambda_j)} \\
&= \sum_{i,j=1}^{n+1} \frac{(\lambda_j-\lambda_i)^n}{(\lambda_i - Y)(\lambda_j - Z)} \prod_{\substack{k=1\\ k\neq i}}^{n+1} \frac{1}{\lambda_k-\lambda_i} \prod_{\substack{k=1\\ k\neq j}}^{n+1} \frac{1}{\lambda_k-\lambda_j} \\
&= (Z-Y)^n \prod_{k=1}^{n+1} \frac{1}{(\lambda_k-Y)(\lambda_k-Z)}.
\end{aligned}$$

The last equality is a consequence of partial fraction decompositions applied to the rational function $(Z-Y)^n \prod_{k=1}^{n+1}(\lambda_k - Y)^{-1}(\lambda_k - Z)^{-1}$, separately, in the variables Y and Z. ∎

Lemma 15. *Denote the Eulerian polynomials by $A_n(t)$. Let $\lambda_1, \lambda_2, \ldots$ be variables. Then, we have*

$$\left\langle \prod_{k=1}^{n} (\lambda_k - t\lambda_{k+1}) \right\rangle_{n+1} = A_{n+1}(t) = \sum_{j=0}^{n} A(n+1, j) t^j.$$

Proof. Rewrite the left-hand side (denoted F_{n+1}) and apply Lemma 3 for a progressive expansion:

$$\begin{aligned}
F_{n+1} &= \sum_{S_{n+1}} \prod_{k=1}^{n} \left(1 + \frac{(1-t)\lambda_{\sigma(k+1)}}{\lambda_{\sigma(k)}-\lambda_{\sigma(k+1)}}\right) \\
&= \sum_{S_{n+1}} \prod_{k=2}^{n} \left(1 + \frac{(1-t)\lambda_{\sigma(k+1)}}{\lambda_{\sigma(k)}-\lambda_{\sigma(k+1)}}\right) \\
&\qquad + \sum_{S_{n+1}} \frac{(1-t)\lambda_{\sigma(2)}}{\lambda_{\sigma(1)}-\lambda_{\sigma(2)}} \prod_{k=2}^{n} \left(1 + \frac{(1-t)\lambda_{\sigma(k+1)}}{\lambda_{\sigma(k)}-\lambda_{\sigma(k+1)}}\right) \\
&= (n+1)F_n + \sum_{S_{n+1}} \frac{(1-t)\lambda_{\sigma(2)}}{\lambda_{\sigma(1)}-\lambda_{\sigma(2)}} \left[1 + \frac{(1-t)\lambda_{\sigma(3)}}{\lambda_{\sigma(2)}-\lambda_{\sigma(3)}}\right] \\
&\qquad \times \prod_{k=3}^{n} \left(1 + \frac{(1-t)\lambda_{\sigma(k+1)}}{\lambda_{\sigma(k)}-\lambda_{\sigma(k+1)}}\right) \\
&= (n+1)F_n + \binom{n+1}{2} \sum_{S_2} \frac{(1-t)\lambda_{\sigma(2)}}{\lambda_{\sigma(1)}-\lambda_{\sigma(2)}} \sum_{S_{n-1}} \prod_{k=3}^{n} \left(1 + \frac{(1-t)\lambda_{\sigma(k+1)}}{\lambda_{\sigma(k)}-\lambda_{\sigma(k+1)}}\right) \\
&\qquad + \sum_{S_{n+1}} \frac{(1-t)^2\lambda_{\sigma(2)}\lambda_{\sigma(3)}}{(\lambda_{\sigma(1)}-\lambda_{\sigma(2)})(\lambda_{\sigma(2)}-\lambda_{\sigma(3)})} \prod_{k=3}^{n} \left(1 + \frac{(1-t)\lambda_{\sigma(k+1)}}{\lambda_{\sigma(k)}-\lambda_{\sigma(k+1)}}\right)
\end{aligned}$$

$$= (n+1)F_n + \binom{n+1}{2}(t-1)F_{n-1}$$
$$+ \sum_{S_{n+1}} \prod_{k=1}^{2} \frac{(1-t)\lambda_{\sigma(k)}}{\lambda_{\sigma(k)} - \lambda_{\sigma(k+1)}} \prod_{k=4}^{n} \left(1 + \frac{(1-t)\lambda_{\sigma(k+1)}}{\lambda_{\sigma(k)} - \lambda_{\sigma(k+1)}}\right)$$
$$+ \sum_{S_{n+1}} \prod_{k=1}^{3} \frac{(1-t)\lambda_{\sigma(k)}}{\lambda_{\sigma(k)} - \lambda_{\sigma(k+1)}} \prod_{k=4}^{n} \left(1 + \frac{(1-t)\lambda_{\sigma(k+1)}}{\lambda_{\sigma(k)} - \lambda_{\sigma(k+1)}}\right)$$
$$= \binom{n+1}{1}F_n + (t-1)\binom{n+1}{2}F_{n-1}$$
$$+ \binom{n+1}{3} \sum_{S_3} \prod_{k=1}^{2} \frac{(1-t)\lambda_{\sigma(k)}}{\lambda_{\sigma(k)} - \lambda_{\sigma(k+1)}} \sum_{S_{n-2}} \prod_{k=4}^{n} \left(1 + \frac{(1-t)\lambda_{\sigma(k+1)}}{\lambda_{\sigma(k)} - \lambda_{\sigma(k+1)}}\right)$$
$$+ \sum_{S_{n+1}} \prod_{k=1}^{3} \frac{(1-t)\lambda_{\sigma(k)}}{\lambda_{\sigma(k)} - \lambda_{\sigma(k+1)}} \prod_{k=4}^{n} \left(1 + \frac{(1-t)\lambda_{\sigma(k+1)}}{\lambda_{\sigma(k)} - \lambda_{\sigma(k+1)}}\right)$$
$$= \binom{n+1}{1}F_n + (t-1)\binom{n+1}{2}F_{n-1} + (t-1)^2\binom{n+1}{3}F_{n-2}$$
$$+ \sum_{S_{n+1}} \left(\prod_{k=1}^{3}\right)\left(\prod_{k=4}^{n}\right).$$

This process terminates with $F_{n+1}(t) = \sum_{j=0}^{n}(t-1)^{n-k}\binom{n+1}{k}F_{n+1-k}(t)$. So, the polynomials $F_n(t)$ fulfill the same recurrence as the Eulerian polynomials $A_n(t)$ and also the same (check!) initial conditions. Hence, $F_n = A_n$ and the proof is complete. ☐

Lemma 16. *Let $Y, \lambda_1, \lambda_2, \ldots$ be variables. Then, we have*

$$\left\langle \prod_{k=1}^{n}(\lambda_1 - t\lambda_{n+1}) \right\rangle_{n+1} = \sum_{k=0}^{n} \binom{n}{k}^2 t^k.$$

Note: Let A_n be the root lattice generated as a monoid by $\{\boldsymbol{e}_i - \boldsymbol{e}_j : 0 \le i, j \le n+1, \text{and } i \ne j\}$. If $P(A_n)$ is the polytope formed by the convex hull of this generating set, then the coefficients of $\sum_k \binom{n}{k}^2 t^k$ are the h-vectors of a unimodular triangulation of $P(A_n)$.

Proof. Single out the values $\sigma(1) = i, \sigma(n+1) = j$ and apply Lemma 14 to obtain

$$\text{LHS} = \sum_{\substack{i,j=1\\ i\ne j}}^{n+1} (\lambda_i - t\lambda_j)^n \sum_{S_{n-1}} \frac{1}{(\lambda_i - \lambda_{\sigma(2)})(\lambda_{\sigma(n)} - \lambda_j)} \prod_{k=2}^{n-1} \frac{1}{\lambda_{\sigma(k)} - \lambda_{\sigma(k+1)}}$$
$$= -\sum_{\substack{i,j=1\\ i\ne j}}^{n+1} (\lambda_i - t\lambda_j)^n (\lambda_j - \lambda_i)^{n-2} \prod_{\substack{k=1\\ k\ne i,j}}^{n+1} \frac{1}{(\lambda_k - \lambda_i)(\lambda_k - \lambda_j)}$$
$$= \sum_{i,j=1}^{n+1} \prod_{\substack{k=1\\ k\ne i}}^{n+1} \frac{\lambda_j - \lambda_i}{\lambda_k - \lambda_i} \prod_{\substack{k=1\\ k\ne j}}^{n+1} \frac{\lambda_i - t\lambda_j}{\lambda_k - \lambda_j}.$$

We now invoke Proposition 0 to utilize invariance and replace $\lambda_k = k$. Some simplification leads to

$$\begin{aligned}\text{LHS} &= \frac{1}{n!^2}\sum_{i,j=0}^{n}(-1)^{i+j}\binom{n}{i}\binom{n}{j}[i+1-t(j+1)]^n(j-i)^n\\ &=\sum_{k=0}^{n}\binom{n}{k}t^k\left[\frac{1}{n!^2}\sum_{i,j=0}^{n}(-1)^{i+j+k}\binom{n}{i}\binom{n}{j}(j+1)^k(i+1)^{n-k}(j-i)^n\right].\end{aligned}$$

Denote the double-sum by $W(n,k)$. We prove $W(n-1,k)+W(n-1,k-1) = W(n,k)$ (the Pascal recurrence):

$$\begin{aligned}&W(n-1,k)+W(n-1,k-1)\\ &\quad=\sum_{i,j\geq 0}(-1)^{i+j+k}\frac{(j+1)^{k-1}(i+1)^{n-1-k}(j-i)^{n-1}}{i!j!(n-i-1)!(n-j-1)!}\,[(j+1)-(i+1)]\\ &\quad=\sum_{i,j\geq 0}(-1)^{i+j+k}\frac{(j+1)^k(i+1)^{n-k}(j-i)^n}{(i+1)!(j+1)!(n-i-1)!(n-j-1)!}\\ &\quad=\sum_{i,j\geq 1}(-1)^{i+j+k}\frac{j^k i^{n-k}(j-i)^n}{i!j!(n-i)!(n-j)!}\\ &\quad=\sum_{i,j\geq 0}(-1)^{i+j+k}\frac{j^k i^{n-k}(j-i)^n}{i!j!(n-i)!(n-j)!}=W(n,k),\end{aligned}$$

where in the last equality once more we took the liberty (due to Proposition 0) of substituting $\lambda_k = k+z$ for a free parameter z. In the present case, $z=-1$ is chosen. Together with the matching (check!) initial conditions, we conclude that $W(n,k)=\binom{n}{k}$ and, hence, the proof is complete. $\square$

Question. Is there a combinatorial or geometrical interpretation (as in [2]) to any of the formulas?

Acknowledgment

The author is grateful to the referee whose suggestions greatly improved the presentation of this paper as well as its contents.

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Department of Mathematics, Tulane University, New Orleans, Louisiana 70118
E-mail address: tamdeber@tulane.edu