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A BERNSTEIN RESULT AND COUNTEREXAMPLE FOR ENTIRE SOLUTIONS TO DONALDSON'S EQUATION

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ABSTRACT. We show that convex entire solutions to Donaldson's equation are quadratic, using a result of Weiyong He. We also exhibit entire solutions to the Donaldson equation that are not of the form discussed by He. In the process we discover some nontrivial entire solutions to complex Monge-Ampère equations.

1. Introduction

In this note we show the following.

Theorem 1. *Suppose that u is a convex solution to the Donaldson equation on $\mathbb{R} \times \mathbb{R}^{n-1} = (t, x_2, \dots, x_n)$:*

$$\tilde{\sigma}_2(D^2u) = u_{11}(u_{22} + u_{33} + \cdots + u_{nn}) - u_{12}^2 - \cdots - u_{1n}^2 = 1. \tag{1}$$

Then u is a quadratic function.

Donaldson introduced the operator

$$Q(D^2u) = u_{tt}\Delta u - |\nabla u_t|^2$$

arising in the study of the geometry of the space of volume forms on compact Riemannian manifolds [1]. On Euclidean space, (1) becomes an interesting nonsymmetric fully nonlinear equation. Weiyong He has studied aspects of entire solutions on Euclidean space, and was able to show [2, Theorem 2.1] that if $u_{11} = const$, then the solution can be written in terms of solutions to Laplace equations.

Here we show that any convex solution must also satisfy $u_{11} = const$. It follows quickly that the solution must be quadratic. We also show that, in the absence of the convexity constraint, solutions exist for which $u_{11} = const$ fails.

Theorem 2. *There exist solutions to the Donaldson equation which are not of the form given by He.*

In real dimension 3 we note that solutions of (1) can be extended to solutions of the complex Monge-Ampère equation on $\mathbb{C}^2$

$$\det\left(\partial\bar{\partial}u\right) = 1 \tag{2}$$

and we can conclude the following.

Corollary 3. *There exist a nonflat solution of the complex Monge-Ampère equation (2) on $\mathbb{C}^2$ whose potential depends on only three real variables.*

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2. Proof of Theorem 1

First, we define an operator on symmetric matrices:

Definition 4. Given a matrix M define the quadratic operator

$$\tilde{\sigma}_2(M) = m_{11}(m_{22} + m_{33} + \cdots + m_{nn}) - m_{12}m_{21} - \cdots - m_{1n}m_{n1}.$$

Lemma 5. *Let h be a positive constant and suppose that K_h is the sublevel set $u \leq h$ of a nonnegative solution to*

$$\tilde{\sigma}_2(D^2u) = 1.$$

Let $E \subset K_h$ be an ellipsoid and $A : E \to B_1$ an affine diffeomorphism of the form

$$A(x) = M \cdot x + b, \tag{3}$$

for some matrix M and some vector b. Then

$$\tilde{\sigma}_2(M^2) \geq \frac{1}{4}\frac{1}{h^2}.$$

Proof. Given an ellipsoid $E \subset K_h$, choose an affine diffeomorphism $A : E \to B_1$ of the form (3) and consider the function v on $\mathbb{R}^n$ defined by

$$v(x) = h|A(x)|^2.$$

For x on the boundary of E, since $A(x) \in \partial B_1$ we have

$$v(x) = h \geq u.$$

Differentiating v,

$$\begin{aligned} Dv &= 2hM \cdot (M \cdot x + b), \\ D^2v &= 2hM^2. \end{aligned}$$

Thus

$$\tilde{\sigma}_2(D^2v) = 4h^2\tilde{\sigma}_2(M^2).$$

Now if the matrix M satisfies

$$\tilde{\sigma}_2(M^2) < \frac{1}{4h^2},$$

then

$$\tilde{\sigma}_2(D^2v) < 1 \text{ on } E,$$

so v is a supersolution to the equation and must lie strictly above the solution u. But v must vanish at $A^{-1}(0)$. Because u is nonnegative, this is a contradiction of the strong maximum principle. Thus $\tilde{\sigma}_2(M^2) \geq \frac{1}{4h^2}$. □

Proposition 6. *Suppose that u is an entire convex solution to*

$$\tilde{\sigma}_2(D^2u) = u_{11}(u_{22} + u_{33} + \cdots + u_{nn}) - u_{12}^2 - \cdots - u_{1n}^2 = 1.$$

Then

$$\lim_{t \to \infty} u_1(t, 0, \ldots, 0) = \infty.$$

Proof. Assume not. By convexity the result will follow once we show that

$$\lim_{t\to\infty} \sup u_1(t,0,\ldots,0) = A \tag{4}$$

leads to a contradiction when $A < \infty$. By convexity, $u_{11} \geq 0$, and hence $u_1(t,0,\ldots,0) \leq A$ for all t. We may without loss of generality assume that $u(0) = 0$ and $Du(0) = 0$, adjusting A if necessary. Then

$$u(t,0,\ldots,0) = \int_0^t u_1(s)ds \leq \int_0^t Ads \leq At. \tag{5}$$

Now for $h > 0$ consider the convex sublevel set

$$K_h = \{(t,x) : u(t,x) \leq h\}.$$

By (5)

$$p_1 := (\frac{h}{A}, 0, \ldots, 0) \in K_h.$$

If the level set

$$\partial K_h = \{(t,x) : u(t,x) = h\}$$

intersects the positive x_2 axis at some point p_2, define a_2 by

$$p_2 = (0, a_2, 0, \ldots, 0),$$

which lies on the boundary ∂K_h. If the level set does not intersect the positive x_2 axis, then define

$$\begin{aligned} a_2 &:= 8\sqrt{n-1}A + 1, \\ p_2 &= (0, a_2, 0, \ldots, 0), \end{aligned} \tag{6}$$

which lies in the interior of K_h. Repeat this for each axis, finding $p_3, \ldots, p_n$ such that

$$p_i \in K_h.$$

The sublevel set K_h is convex, so it must contain the simplex given by convex combinations of $\{p_0, p_1, \ldots, p_n\}$ with

$$p_0 = (0, 0, \ldots, 0).$$

In particular, it must contain the rectangle

$$[0, \frac{1}{2}\frac{h}{A}] \times [0, \frac{1}{2}a_2] \times \cdots \times [0, \frac{1}{2}a_n],$$

which in turn must contain the ellipsoid E centered at

$$c = \left(\frac{1}{4}\frac{h}{A}, \frac{1}{4}a_2, \ldots, \frac{1}{4}a_n\right)$$

that is defined by the equation

$$|T(x)|^2 \leq 1$$

where

$$T(x) = M \cdot (x - c),$$

$$M = 4\begin{pmatrix} \frac{A}{h} & & & \\ & \frac{1}{a_2} & & \\ & & \frac{1}{a_3} & \\ & & & \cdots \end{pmatrix}.$$

Now

$$M^2 = 16 \begin{pmatrix} \left(\frac{A}{h}\right)^2 & & & \\ & \left(\frac{1}{a_2}\right)^2 & & \\ & & \left(\frac{1}{a_3^2}\right)^2 & \\ & & & \cdots \end{pmatrix}$$

and

$$\tilde{\sigma}_2(M^2) = 16 \left(\frac{A}{h}\right)^2 \left(\left(\frac{1}{a_2}\right)^2 + \cdots + \left(\frac{1}{a_n}\right)^2\right) \geq \frac{1}{4}\frac{1}{h^2}$$

with the latter inequality following from the previous lemma.

Thus

$$\left(\frac{1}{a_2}\right)^2 + \left(\frac{1}{a_3}\right)^2 + \cdots + \left(\frac{1}{a_n}\right)^2 \geq \frac{1}{64A^2}.$$

It follows that for some i,

$$\frac{1}{a_i^2} \geq \frac{1}{64\,(n-1)\,A^2}.$$

That is,

$$a_i \leq 8\sqrt{n-1}A.$$

Now to finish the argument, let

$$R = 8\sqrt{n-1}A.$$

Now let

$$\bar{U} = \sup_{B_R} u(x).$$

Now by (1) the function u must be strictly convex in some direction, in particular, every sublevel set K_h is nonempty and convex. Choose $h = \bar{U} + 1$. Apply the above argument, and we can find some $p_i \in K_h$ with

$$a_i \leq R = 8\sqrt{n-1}A.$$

By our choice (6) this point must lie on the level set $u = h$. Since $p_i \in B_R$ we have

$$h = u(p_i) \leq \sup_{B_R} u(x) = h - 1,$$

which is clearly a contradiction. Thus (4) cannot hold. □

Note that the equation is invariant under translation in x and also under the symmetry $t \to -t$. Thus we can apply the above proposition for any $x \in \mathbb{R}^{n-1}$ and conclude that

$$\lim_{t\to\infty} u_1(t,x) = \infty,$$
$$\lim_{t\to\infty} u_1(-t,x) = -\infty.$$

In particular, letting $z = u_1(t,x)$, the map

$$\Phi : \mathbb{R}\times\mathbb{R}^{n-1} \to \mathbb{R}\times\mathbb{R}^{n-1}$$
$$\Phi(t,x) = (z,x)$$

is a diffeomorphism. With this in hand, we repeat the argument of He [2, section 3]. For each x fixed, there exists a unique $t = t(z,x)$ such that $z = u_1(t,x)$. Defining

$$\theta(z,x) = t(z,x)$$

the computations in [2, section 3] yield that θ is a harmonic function. It follows that $\frac{\partial\theta}{\partial z} = 1/u_{11}$ is a positive harmonic function, so must be constant. Now we have

$$u(t,x) = at^2 + tb(x) + g(x),$$

which satisfies [2, section 2]

$$\Delta b = 0,$$
$$\Delta g = \frac{1}{2a}\left(1 + |\nabla b|^2\right).$$

Letting $t = 0$ we conclude that g is convex. Letting $t \to \pm\infty$ we conclude that b is convex and concave, so must be linear. It follows that $|\nabla b|^2$ is constant, and

$$\Delta g = \frac{1}{2a}\left(1 + |c|^2\right). \tag{7}$$

Differentiate (7) twice in any direction e, and we see that

$$\Delta g_{ee} = 0.$$

By convexity

$$g_{ee} \geq 0$$

and the classical Liouville theorem tells us the second derivatives of g are constant. Thus g is a quadratic.

3. Counterexamples

We use the method described in [3] and restrict to $n = 3$. Consider

$$u(t,x) = r^2 e^t + h(t)$$

where $r = \left(x_2^2 + x_3^2\right)^{1/2}$. At any point we may rotate $\mathbb{R}^2$ so that $x_2 = r$ and get

$$D^2 u = \begin{pmatrix} r^2 e^t + h''(t) & 2re^t & 0 \\ 2re^t & 2e^t & 0 \\ 0 & 0 & 2e^t \end{pmatrix}.$$

We compute

$$\tilde{\sigma}_2\left(D^2 u\right) = 4e^t\left(r^2 e^t + h''(t)\right) - 4r^2 e^{2t} = 4e^t h''(t).$$

Then

$$u = r^2 e^t + \frac{1}{4}e^{-t}$$

is a solution.

Now defining complex variables

$$z_1 = t + is,$$
$$z_2 = x + iy$$

we can consider the function

$$u(z_1, z_2) = 4\,|z_2|^2\, e^{\mathrm{Re}(z_1)} + e^{-\mathrm{Re}(z_1)}. \tag{8}$$

The complex Hessian becomes

$$\partial_z \partial_{\bar{z}} u = \frac{1}{4}\begin{pmatrix} 4\,|z_2|^2\, e^{\mathrm{Re}(z_1)} + e^{-\mathrm{Re}(z_1)} & 8z_2 e^{\mathrm{Re}(z_1)} \\ 8\bar{z}_2 e^{\mathrm{Re}(z_1)} & 16e^{\mathrm{Re}(z_1)} \end{pmatrix}$$

and it follows that u satisfies the complex Monge-Ampère equation

$$\det\left(\partial_z \partial_{\bar{z}} u\right) = 1.$$

In particular, while the function is not convex, it is plurisubharmonic. The induced Ricci-flat complex metric

$$g_{i\bar{j}} = \partial_{z_i}\partial_{z_{\bar{j}}} u$$

can be neither complete nor flat. Indeed, when $z_2 = 0$, the metric is of the form

$$\frac{1}{4}\begin{pmatrix} e^{-\operatorname{Re}(z_1)} & 0 \\ 0 & 16e^{\operatorname{Re}(z_1)} \end{pmatrix}.$$

Thus we can take a path

$$\begin{aligned} \gamma &: [0,\infty) \to \mathbb{C}^2, \\ t &\mapsto (-t,0,0,0)\,, \end{aligned}$$

which has finite length, but is not contained in any compact set. A standard computation of the sectional curvatures shows that this metric is not flat.

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