

A NEW CHARACTERIZATION OF GEODESIC SPHERES IN THE HYPERBOLIC SPACE

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ABSTRACT. This paper gives a new characterization of geodesic spheres in the hyperbolic space in terms of a “weighted” higher order mean curvature. Precisely, we show that a compact hypersurface Σ^{n-1} embedded in \mathbb{H}^n with VH_k being constant for some $k = 1, \dots, n-1$ is a centered geodesic sphere. Here H_k is the k -th normalized mean curvature of Σ induced from \mathbb{H}^n and $V = \cosh r$, where r is a hyperbolic distance to a fixed point in \mathbb{H}^n . Moreover, this result can be generalized to a compact hypersurface Σ embedded in \mathbb{H}^n with the ratio $V \left(\frac{H_k}{H_j} \right) \equiv \text{constant}$, $0 \leq j < k \leq n-1$ and H_j not vanishing on Σ .

1. INTRODUCTION

A fundamental question about hypersurfaces in differential geometry is the rigidity of the spheres. Alexandrov [3] studied the rigidity of spheres with constant mean curvature. Precisely, he proved that a compact hypersurface embedded in the Euclidean space with constant mean curvature must be a sphere. This is a well-known Alexandrov theorem. Later, in [28], by exploiting a formula originated by Reilly [27], Ros generalized Alexandrov’s result to the hypersurface embedded in the Euclidean space with constant scalar curvature. Also, Korevaar [22] gave another proof to this result by using the classical reflection method due to Alexandrov [3] and he indicated that Alexandrov’s reflection method works as well for hypersurfaces in the hyperbolic space and the upper hemisphere. Later, Ros [29] extended his result to any constant k -mean curvature and provided another proof in [26] together with Montiel. Explicitly, they proved the following result.

Theorem 1.1 ([26, 29]). *Let Σ^{n-1} be a compact hypersurface embedded in Euclidean space \mathbb{R}^n . If H_k is constant for some $k = 1, \dots, n-1$, then Σ is a sphere.*

There are many works generalizing Theorem 1.1. For instance, In [20, 21], Koh gave a new characterization of spheres in terms of the ratio of two mean curvature which generalized a previous result of Bivens [7]. Aledo-Alías-Romero [2] extended the result to compact space-like hypersurfaces with constant higher order mean curvature in de Sitter space. In [18], He-Li-Ma-Ge studied the compact embedded hypersurfaces with constant higher order anisotropic mean curvatures. Recently, Brendle [8] showed an Alexandrov type result for hypersurfaces with

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constant mean curvature in the warped product spaces including the de-Sitter-Schwarzschild manifold. Thereafter, Brendle-Eichmair [9] extended this result to any star-shaped convex hypersurface with constant higher mean curvature in the de-Sitter-Schwarzschild manifold. For the other generalizing works, see for example [1, 4–6, 19, 24, 25, 30] and the references therein.

In a different direction, there is another kind of mean curvature with a weight V in the hyperbolic space \mathbb{H}^n which has recently attracted much interest. Here $V = \cosh r$, where r is the hyperbolic distance to a fixed point in \mathbb{H}^n . For example, the mean curvature integral $\int_{\Sigma} V H_1 d\mu$ with weight V appears naturally in the definition of the quasi-local mass in \mathbb{H}^n and the Penrose inequality for asymptotically hyperbolic graphs [11]. The higher order mean curvature integrals $\int_{\Sigma} V H_{2k-1} d\mu$ also appear in [16] on the GBC mass for asymptotically hyperbolic graphs. Comparing with the one without weight [14, 15, 23, 31], the corresponding Alexandrov-Fenchel inequalities with weight also hold in the hyperbolic space. Brendle-Hung-Wang [10] and de Lima-Girão [12] established optimal inequalities for the mean curvature integral $\int_{\Sigma} V H_1 d\mu$. More recently, in [16], Ge, Wang and the author established an optimal inequality for the higher order mean curvature integral $\int_{\Sigma} V H_k d\mu$. These inequalities are related to the Penrose inequality for the asymptotically hyperbolic graphs with a horizon type boundary. See [10–12, 16] for instance.

For the hyperbolic space \mathbb{H}^n with the metric $b = dr^2 + \sinh^2 r d\Theta^2$, where $d\Theta^2$ is the standard round metric on \mathbb{S}^{n-1} , the “weight” $V = \cosh r$ appears quite naturally. In fact, it satisfies the following equation:

$$(1.1) \quad \mathbb{N}_b \triangleq \{V \in C^\infty(\mathbb{H}^n) | \text{Hess}^b V = Vb\}.$$

Any element V in \mathbb{N}_b satisfies that the Lorentzian metric $\gamma = -V^2 dt^2 + b$ is a static solution to the vacuum Einstein equation with the negative cosmological constant $\text{Ric}(\gamma) + n\gamma = 0$. In fact, \mathbb{N}_b is an $(n+1)$ -dimensional vector space spanned by an orthonormal basis of the following functions:

$$V_{(0)} = \cosh r, \quad V_{(1)} = x^1 \sinh r, \quad \dots, \quad V_{(n)} = x^n \sinh r,$$

where x^1, x^2, \dots, x^n are the coordinate functions restricted to $\mathbb{S}^{n-1} \subset \mathbb{R}^n$. We equip the vector space \mathbb{N}_b with a Lorentz inner product η with signature $(+, -, \dots, -)$ such that

$$(1.2) \quad \eta(V_{(0)}, V_{(0)}) = 1, \quad \text{and} \quad \eta(V_{(i)}, V_{(i)}) = -1 \quad \text{for} \quad i = 1, \dots, n.$$

Note that the subset \mathbb{N}_b^+ consisting of positive functions is just the interior of the future lightcone. Let \mathbb{N}_b^1 be the subset of \mathbb{N}_b^+ of functions V with $\eta(V, V) = 1$. One can check easily that every function V in \mathbb{N}_b^1 has the following form:

$$V = \cosh \text{dist}_b(x_0, \cdot),$$

for some $x_0 \in \mathbb{H}^n$, where dist_b is the distance function with respect to the metric b . Therefore, in the following we fix $V = V_{(0)} = \cosh r$. Throughout this paper, a centered geodesic sphere means a geodesic sphere with the center x_0 . Comparing with the Euclidean case, $V = 1$ since the corresponding one to (1.1) is 1-dimensional.

It is natural to ask whether the previous rigidity results can be extended to the hypersurface of the hyperbolic space in terms of some “weighted” higher mean curvature. In studying this problem, the existence of function V makes things subtle and it would be difficult to apply the classical Alexandrov reflection method

[3] as in [22] to handle this problem. Hence some new ideas may be needed to attack this problem. Comparing with the proof of Theorem 1.1, the classical Minkowski integral identity (see (2.6) below) is not enough in this case. So one needs to generalize the classical Minkowski integral identity a little bit. Note that in [16], in order to establish the optimal inequality for the “weighted” higher order mean curvature integral, they generalized (2.6) into some inequalities between H_k and H_{k-1} for a convex hypersurface in \mathbb{H}^n . In fact, by the same proof, this condition can be weakened into a k -convex hypersurface. Another ingredient in our proof is a special case of Heintze-Karcher-type inequality due to [8]. Based on these two aspects, we obtain the following results.

Theorem 1.2. *Let Σ^{n-1} be a compact hypersurface embedded in the hyperbolic space \mathbb{H}^n . If VH_k is constant for some $k = 1, \dots, n-1$, then Σ is a centered geodesic sphere.*

The above rigidity result can be generalized to the hypersurface in terms of the ratio of two higher order mean curvatures multiplying by weight.

Theorem 1.3. *Let Σ^{n-1} be a compact hypersurface embedded in the hyperbolic space \mathbb{H}^n . If the ratio $V\left(\frac{H_k}{H_j}\right)$ is constant for some $0 \leq j < k \leq n-1$ and H_j does not vanish on Σ , then Σ is a centered geodesic sphere.*

2. PRELIMINARIES

In this section, first let us recall some basic definitions and properties of higher order mean curvature.

Let σ_k be the k -th elementary symmetry function $\sigma_k : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ defined by

$$\sigma_k(\Lambda) = \sum_{i_1 < \dots < i_k} \lambda_{i_1} \cdots \lambda_{i_k} \quad \text{for } \Lambda = (\lambda_1, \dots, \lambda_{n-1}) \in \mathbb{R}^{n-1}.$$

For a symmetric matrix B , let $\lambda(B) = (\lambda_1(B), \dots, \lambda_n(B))$ be the eigenvalues of B . We set

$$\sigma_k(B) := \sigma_k(\lambda(B)).$$

The Gårding cone Γ_k is defined by

$$(2.1) \quad \Gamma_k = \{\Lambda \in \mathbb{R}^{n-1} \mid \sigma_j(\Lambda) > 0, \quad \forall j \leq k\}.$$

A symmetric matrix B is denoted Γ_k if $\lambda(B) \in \Gamma_k$. Let

$$(2.2) \quad H_k = \frac{\sigma_k}{C_{n-1}^k}$$

be the normalized k -th elementary symmetry function. As a convention, $H_0 = 1$. We have the following Newton-Maclaurin inequalities. For the proof, we refer to a survey of Guan [17].

Lemma 2.1. *For $1 \leq j < k \leq n-1$ and $\Lambda \in \Gamma_k$, we have the following Newton-Maclaurin inequality:*

$$(2.3) \quad H_j \geq (H_k)^{\frac{j}{k}}.$$

Moreover, equality holds in (2.3) at Λ if and only if $\Lambda = c(1, 1, \dots, 1)$.

Next, we recall a Minkowski relation proved in [16]. In order to state this result, let us give some definitions and notation. Let $V = \cosh(r)$ and $p = \langle DV, \xi \rangle > 0$ be the support function. The k -th Newton transformation is defined as follows:

$$(2.4) \quad (T_k)_j^i(B) := \frac{\partial \sigma_{k+1}}{\partial b_j^i}(B),$$

where $B = (b_j^i)$. Recall that we have the following Minkowski identity with weight V (see [4]):

$$(2.5) \quad \nabla_j (T_{k-1}^{ij} \nabla_i V) = -kp\sigma_k + (n-k)V\sigma_{k-1}.$$

Integrating the above equality and recalling (2.2), we get

$$(2.6) \quad \int_{\Sigma} pH_k d\mu = \int_{\Sigma} VH_{k-1} d\mu.$$

This is the classical Minkowski integral identity in \mathbb{H}^n . In [16], (2.6) is generalized to some inequalities between H_k and H_{k-1} , where they are called *Minkowski integral formulae*. These integral inequalities play an important role in this paper. If the principles of a hypersurface belong to the Gårding cone Γ_k , it is called k -convex.

Proposition 2.2 ([16]). *Let Σ^{n-1} be a hypersurface isometric immersed in the hyperbolic space \mathbb{H}^n . We have*

$$(2.7) \quad \int_{\Sigma} pVH_k d\mu = \int_{\Sigma} V^2H_{k-1} d\mu + \frac{1}{kC_{n-1}^k} \int_{\Sigma} (T_{k-1})^{ij} \nabla_i V \nabla_j V d\mu.$$

Moreover, if Σ is k -convex, $k \in \{1, \dots, n-1\}$, then we have

$$(2.8) \quad \int_{\Sigma} pVH_k d\mu \geq \int_{\Sigma} V^2H_{k-1} d\mu.$$

Equality holds if and only if Σ is a centered geodesic sphere in \mathbb{H}^n .

Proof. This has been proved in [16] under the condition that the hypersurface is convex. By the same argument, the result holds for a k -convex hypersurface. For the convenience of the reader, we include the proof. In view of (2.5) and (2.2), we have

$$\frac{1}{kC_{n-1}^k} \nabla_j (T_{k-1}^{ij} \nabla_i V) = -pH_k + H_{k-1}V.$$

Multiplying the above equation by the function V and integrating by parts, one obtains the desired result (2.7). Under the assumption that Σ is k -convex, the $(k-1)$ -th Newton tensor T_{k-1} is positively definite (for the proof see the one of Proposition 3.2 in [6] for instance). That is,

$$(T_{k-1})^{ij} \nabla_i V \nabla_j V \geq 0,$$

thus (2.8) holds. When the equality holds, we have $\nabla V = 0$ which implies that Σ is a geodesic sphere in \mathbb{H}^n . \square

Finally, we need a special case of the Heintze-Karcher-type inequality due to Brendle [8].

Proposition 2.3 (Brendle). *Let Σ be a compact hypersurface embedded in \mathbb{H}^n with positive mean curvature (i.e. $H_1 > 0$), then*

$$\int_{\Sigma} p d\mu \leq \int_{\Sigma} \frac{V}{H_1} d\mu.$$

Moreover, equality holds if and only if Σ is totally umbilical.

To prove this inequality we first choose a suitable geometric flow, then prove the monotonicity, and finally analyze the asymptotical behavior to obtain the desired inequality. This Heintze-Karcher type inequality plays an important role in [12] to establish a weighted Alexandrov-Fenchel equality for mean curvature integral in the hyperbolic space. This inequality is also a main ingredient in this paper. For the proof of Proposition 2.3, we refer the reader to [8].

3. PROOF OF THE MAIN THEOREMS

After all the preparation work, we are ready to prove our main theorems.

Proof of Theorem 1.2. Since the hypersurface Σ is compact, thus at the point where the distance function r of \mathbb{H}^n attains its maximum, all the principal curvatures are positive. This together with the simple fact $V = \cosh r > 0$ yields that VH_k is a positive constant and thus H_k is positive everywhere in Σ . From the result of Gårding [13], we know that the principal curvatures of Σ belong to the Gårding cone Γ_k defined in (2.1).

It follows from (2.8) that

$$(3.1) \quad VH_k \int_{\Sigma} p d\mu = \int_{\Sigma} p VH_k d\mu \geq \int_{\Sigma} V^2 H_{k-1} d\mu.$$

By the Newton-Maclaurin inequality (2.3), we have

$$H_{k-1} \geq H_k^{\frac{k-1}{k}},$$

thus

$$(3.2) \quad \int_{\Sigma} V^2 H_{k-1} d\mu \geq \int_{\Sigma} V^2 H_k^{\frac{k-1}{k}} d\mu = (VH_k)^{\frac{k-1}{k}} \int_{\Sigma} V^{1+\frac{1}{k}} d\mu.$$

Hence (3.1) and (3.2) yield

$$(3.3) \quad \int_{\Sigma} p d\mu \geq (VH_k)^{-\frac{1}{k}} \int_{\Sigma} V^{1+\frac{1}{k}} d\mu,$$

and equality holds if and only if Σ is a geodesic sphere. On the other hand, applying Proposition 2.3 and the Newton-Maclaurin inequality (2.3) we derive that

$$(3.4) \quad \int_{\Sigma} p d\mu \leq \int_{\Sigma} \frac{V}{H_1} d\mu \leq \int_{\Sigma} \frac{V}{H_k^{\frac{1}{k}}} d\mu = (VH_k)^{-\frac{1}{k}} \int_{\Sigma} V^{1+\frac{1}{k}} d\mu.$$

Finally combining (3.3) and (3.4), we complete the proof. \square

Proof of Theorem 1.3. The first step is more or less the same as above. At the point where the distance function r of \mathbb{H}^n attains its maximum, all the principal curvatures are positive. Therefore H_j and H_k are both positive at the point. This together with $V = \cosh r > 0$ yields that $V \left(\frac{H_k}{H_j} \right)$ is a positive constant. Since by the assumption H_j does not vanish on Σ , H_j and H_k are positive everywhere in Σ . From [13], we know that the principal curvatures of Σ belong to Γ_k defined in (2.1).

If $j = 0$, Theorem 1.3 is reduced to the case of Theorem 1.2. In the following, we consider the case $j \geq 1$. Denote the positive constant by α , namely,

$$\alpha := V \left(\frac{H_k}{H_j} \right) > 0.$$

Applying the Newton-Maclaurin inequality (2.3), we note that

$$\frac{H_k}{H_{k-1}} \leq \frac{H_j}{H_{j-1}},$$

which implies

$$(3.5) \quad V \left(\frac{H_{k-1}}{H_{j-1}} \right) \geq \alpha.$$

It follows from (2.8) and (2.6) that

$$(3.6) \quad \int_{\Sigma} V^2 H_{k-1} d\mu \leq \int_{\Sigma} V p H_k d\mu = \alpha \int_{\Sigma} p H_j d\mu = \alpha \int_{\Sigma} V H_{j-1} d\mu.$$

This gives

$$\int_{\Sigma} V (V H_{k-1} - \alpha H_{j-1}) \leq 0.$$

The above together with (3.5) imply

$$V \frac{H_{k-1}}{H_{j-1}} = \alpha,$$

everywhere in Σ . By an iteration argument, one obtains

$$V \frac{H_{k-j}}{H_0} = V H_{k-j} = \alpha,$$

everywhere in Σ . Finally, from Theorem 1.2, we complete the proof. \square

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