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THE CHOWLA-SELBERG FORMULA FOR QUARTIC ABELIAN CM FIELDS

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Abstract. We provide explicit evaluations of the Chowla-Selberg formula for quartic abelian CM fields due to Barquero-Sanchez and Masri. These identities relate values of a Hilbert modular function at CM points to values of Euler's gamma function Γ and an analogous function Γ_2 at rational numbers. Our work consists of two main parts. First, we implement an algorithm in SageMath to compute the CM points. Second, we exhibit families of quartic abelian CM fields for which the part of the formula involving values of Γ and Γ_2 takes a particularly simple form.

1. Introduction and statement of results

The Chowla-Selberg formula [4, 5] relates values of the Dedekind eta function η at CM points associated to imaginary quadratic fields to values of Euler's gamma function Γ at rational numbers. In [1], Barquero-Sanchez and Masri generalized the Chowla-Selberg formula to abelian CM fields. Their formula relates values of a Hilbert modular function at CM points associated to an abelian CM field to values of both Γ and an analogous function Γ_2 at rational numbers. In this paper we will provide explicit evaluations of the Chowla-Selberg formula for quartic abelian CM fields. This consists of two parts: (1) calculating the CM points, and (2) identifying families of field extensions for which the side of the formula involving Γ and Γ_2 takes a particularly simple form. We provide several examples of our formulas for specific quartic fields.

We briefly recall the classical Chowla-Selberg formula. Let $K = \mathbb{Q}(\sqrt{d})$ be a quadratic field of discriminant Δ, $\mathcal{O}_K$ be the ring of integers, $\mathrm{CL}(K)$ be the ideal class group, h_d be the class number, $w_d = \#\mathcal{O}_K^\times$ be the number of units (for $d < 0$), ϵ_d be the fundamental unit (for $d > 0$), and $\chi_d(k) = \left(\frac{\Delta}{k}\right)$ be the Kronecker symbol. Let

$$\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad q = e^{2\pi i z}, \quad z = x + iy \in \mathbb{H},$$

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be the Dedekind eta function, where $\mathbb{H}$ is the complex upper half-plane. Define the $\mathrm{SL}_2(\mathbb{Z})$-invariant function

$$G(z) = \sqrt{y}|\eta(z)|^2.$$

The Chowla-Selberg formula is the identity

$$\prod_{[\mathfrak{a}]\in \mathrm{CL}(K)} G(z_{\mathfrak{a}}) = \left(\frac{1}{4\pi\sqrt{|\Delta|}}\right)^{\frac{h_d}{2}} \prod_{k=1}^{|\Delta|} \Gamma\left(\frac{k}{|\Delta|}\right)^{\frac{w_d \chi_d(k)}{4}} \tag{1.1}$$

where $d < 0$ and the product is taken over a complete set of CM points $z_{\mathfrak{a}}$ of discriminant Δ.

We now state the analog of (1.1) for abelian CM fields proved in [1]. We refer the reader to the introduction of that paper for an overview of the proof. Let $F/\mathbb{Q}$ be a totally real field of degree n with embeddings $\tau_1, \ldots, \tau_n$. Let $\mathcal{O}_F$ be the ring of integers, $\mathcal{O}_F^{\times}$ be the group of units, d_F be the absolute value of the discriminant, ∂_F be the different ideal, $\zeta_F(s)$ be the Dedekind zeta function, and

$$\zeta_F^*(s) = d_F^{s/2}\pi^{-ns/2}\Gamma(s/2)^n\zeta_F(s)$$

be the completed Dedekind zeta function of F.

Let

$$z = (z_1, \ldots, z_n) \in \mathbb{H}^n.$$

The Hilbert modular group $\mathrm{SL}_2(\mathcal{O}_F)$ acts componentwise on $\mathbb{H}^n$ by linear fractional transformations. There exists an $\mathrm{SL}_2(\mathcal{O}_F)$-invariant function $H\colon \mathbb{H}^n \to \mathbb{R}^+$ analogous to $G(z)$ which arises from a renormalized Kronecker limit formula for the non-holomorphic Hilbert modular Eisenstein series. More precisely, define the non-holomorphic Hilbert modular Eisenstein series

$$E(z,s) = \sum_{M\in\Gamma_\infty\backslash \mathrm{SL}_2(\mathcal{O}_F)} N(y(Mz))^s, \quad z \in \mathbb{H}^n, \quad \mathrm{Re}(s) > 1,$$

where

$$\Gamma_\infty = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}_F) \right\}$$

and

$$N(y(z)) = \prod_{j=1}^{n} y_j$$

is the product of the imaginary parts of the components of $z \in \mathbb{H}^n$. Barquero-Sanchez and Masri [1, Proposition 3.1] derived the renormalized Kronecker limit formula

$$E\left(z, \frac{s+1}{2}\right) = 1 + \log(H(z))(s+1) + O((s+1)^2).$$

Explicitly,

$$H(z) = \sqrt{N(y(z))}\phi(z)$$

where $\phi\colon \mathbb{H}^n \to \mathbb{R}^+$ is

$$\phi(z) = \exp\left(\frac{\zeta_F^*(-1)N(y(z))}{r_F} + \sum_{\substack{\mu \in \partial_F^{-1}/\mathcal{O}_F^\times \\ \mu \neq 0}} \frac{\sigma_1(\mu\partial_F)e^{2\pi i T(\mu,z)}}{r_F \left|N_{F/\mathbb{Q}}(\mu\partial_F)N_{F/\mathbb{Q}}(\mu)\right|^{1/2}} \right).$$

Here r_F is the residue of $\zeta_F^*(s)$ at $s = 0$,

$$\sigma_1(\mu\partial_F) = \sum_{\mathfrak{b} \mid \mu\partial_F} N_{F/\mathbb{Q}}(\mathfrak{b}),$$

and

$$T(\mu, z) = \sum_{j=1}^n \tau_j(\mu)x_j + i\sum_{j=1}^n |\tau_j(\mu)|y_j.$$

Let E be a CM extension of F with class number h_E and let $\Phi = (\sigma_1, \ldots, \sigma_n)$ be a CM type for E. Assume that F has narrow class number 1. For each fractional ideal class $[\mathfrak{a}] \in \mathrm{CL}(E)$, there exists a CM point $z_{\mathfrak{a}} \in E^\times$ of type (E, Φ) which is unique up to $\mathrm{SL}_2(\mathcal{O}_F)$-equivalence. Let

$$\mathcal{CM}(E, \Phi, \mathcal{O}_F) = \{z_{\mathfrak{a}} : [\mathfrak{a}] \in \mathrm{CL}(E)\}$$

be a complete set of CM points of type (E, Φ) on the Hilbert modular variety $\mathrm{SL}_2(\mathcal{O}_F)\backslash\mathbb{H}^n$ (see Section 5 for more details).

Let $L \subset \mathbb{Q}(\zeta_m)$ be an abelian field, where $\zeta_m = e^{2\pi i/m}$ is a primitive mth root of unity. Let H_L be the subgroup of $G = \mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$ which fixes L. Recall that $G \cong (\mathbb{Z}/m\mathbb{Z})^\times$, where we identify the map $\sigma_t \in G$ determined by $\sigma_t(\zeta_m) = \zeta_m^t$ with the class $[t] \in (\mathbb{Z}/m\mathbb{Z})^\times$. We can then define the group of Dirichlet characters associated to L by

$$X_L = \{\chi \in \widehat{(\mathbb{Z}/m\mathbb{Z})^\times} : \chi|_{H_L} \equiv 1\}.$$

Since the CM field E is abelian over $\mathbb{Q}$, by the Kronecker-Weber theorem E embeds into some cyclotomic field $\mathbb{Q}(\zeta_m)$. As $H_E \leq H_F$, it follows that $X_F \leq X_E$. For our purposes, the choice of cyclotomic field containing E does not matter. Let $L(\chi, s)$ be the L-function of the primitive Dirichlet character of conductor c_χ which induces $\chi \in X_E$ and define the Gauss sum

$$\tau(\chi) = \sum_{k=1}^{c_\chi} \chi(k)\zeta_{c_\chi}^k, \quad \zeta_{c_\chi} = e^{2\pi i/c_\chi}.$$

Let

$$\zeta(s, x) = \sum_{n=0}^\infty \frac{1}{(n + x)^s}, \quad x > 0, \quad \mathrm{Re}(s) > 1,$$

be the Hurwitz zeta function, and consider the Taylor expansion

$$\zeta(s, x) = \frac{1}{2} - x + \partial_s\zeta(0, x)s + \partial_s^2\zeta(0, x)s^2 + O(s^3).$$

By the Bohr-Mollerup theorem, the function $\log(\Gamma(x)/\sqrt{2\pi})$ is the unique function $f\colon \mathbb{R}^+ \to \mathbb{R}$ such that

$$f(x+1) - f(x) = \log(x),$$

$f(1) = \zeta'(0) = -\log\sqrt{2\pi}$, and $f(x)$ is convex on $\mathbb{R}^+$. One can show that $\partial_s \zeta(0,x)$ satisfies these three properties, which yields Lerch's identity [10]

$$\partial_s \zeta(0,x) = \log(\Gamma(x)/\sqrt{2\pi}).$$

Deninger [6] showed that $\partial_s^2 \zeta(0,x)$ is the unique function $g\colon \mathbb{R}^+ \to \mathbb{R}$ such that

$$g(x+1) - g(x) = \log^2(x),$$

$g(1) = -\zeta''(0)$, and $g(x)$ is convex on (e,∞). Define

$$\Gamma_2(x) = \exp(\partial_s^2 \zeta(0,x)), \quad x > 0,$$

which by the preceding discussion can be viewed as analogous to $\Gamma(x)/\sqrt{2\pi}$.

We can now state the Chowla-Selberg formula for abelian CM fields.

Theorem 1.1 ([1, Theorem 1.1])**.** *Let $F/\mathbb{Q}$ be a totally real field of degree n with narrow class number 1. Let E/F be a CM extension with $E/\mathbb{Q}$ abelian. Let Φ be a CM type for E and*

$$\mathcal{CM}(E,\Phi,\mathcal{O}_F) = \{z_{\mathfrak{a}} : [\mathfrak{a}] \in \mathrm{CL}(E)\}$$

be a set of CM points of type (E,Φ). Then

$$\prod_{[\mathfrak{a}]\in \mathrm{CL}(E)} H(\Phi(z_{\mathfrak{a}})) = c_1(E,F,n) \prod_{\chi \in X_E \setminus X_F} \prod_{k=1}^{c_\chi} \Gamma\left(\frac{k}{c_\chi}\right)^{\frac{h_E \chi(k)}{2L(\chi,0)}} \times \prod_{\substack{\chi\in X_F \\ \chi \neq 1}} \prod_{k=1}^{c_\chi} \Gamma_2\left(\frac{k}{c_\chi}\right)^{\frac{h_E \tau(\chi)\overline{\chi}(k)}{2c_\chi L(\chi,1)}} \tag{1.2}$$

where

$$c_1(E,F,n) := \left(\frac{d_F}{2^{n+1}\pi\sqrt{d_E}}\right)^{\frac{h_E}{2}}.$$

Our first result is the following explicit version of (1.2) for a family of biquadratic CM fields.

Theorem 1.2. *Let $p \equiv 1 \pmod 4$ and $q \equiv 3 \pmod 4$ be primes. Let $F = \mathbb{Q}(\sqrt{p}), E = \mathbb{Q}(\sqrt{p},\sqrt{-q})$, and Φ be a CM type for E. If F has narrow class number 1, then*

$$\prod_{[\mathfrak{a}]\in \mathrm{CL}(E)} H(\Phi(z_{\mathfrak{a}})) = \left(\frac{1}{8\pi q}\right)^{\frac{h_E}{2}} \prod_{k=1}^{q} \Gamma\left(\frac{k}{q}\right)^{\frac{h_E \chi_{-q}(k) w_{-q}}{4h_{-q}}} \times \prod_{k=1}^{pq} \Gamma\left(\frac{k}{pq}\right)^{\frac{h_E \chi_{-pq}(k) w_{-pq}}{4h_{-pq}}} \prod_{k=1}^{p} \Gamma_2\left(\frac{k}{p}\right)^{\frac{h_E \chi_p(k)}{4\log(\epsilon_p)}}.$$

Let $B_1(\chi)$ be the first generalized Bernoulli number attached to a Dirichlet character χ. Given a prime $p \equiv 1 \pmod 4$, the pth cyclotomic field $\mathbb{Q}(\zeta_p)$ contains a unique cyclic quartic subfield E. We will show that if $p \equiv 5 \pmod 8$, then E is a CM field with totally real quadratic subfield $F = \mathbb{Q}(\sqrt{p})$ (see Lemma 3.2).

Our analogous result for this family of cyclic quartic CM fields is as follows.

Theorem 1.3. *Let $p \equiv 5 \pmod 8$ be a prime. Let $F = \mathbb{Q}(\sqrt{p})$, E be the unique cyclic quartic subfield of $\mathbb{Q}(\zeta_p)$, and Φ be a CM type for E. If F has narrow class number 1, then*

$$\prod_{[\mathfrak{a}]\in \mathrm{CL}(E)} H(\Phi(z_{\mathfrak{a}})) = c_1(E,F,2)\prod_{k=1}^{p}\Gamma\left(\frac{k}{p}\right)^{-h_E \operatorname{Re}\frac{\chi(k)}{B_1(\chi)}}\prod_{k=1}^{p}\Gamma_2\left(\frac{k}{p}\right)^{\frac{h_E\chi_p(k)}{4\log(\epsilon_p)}}.$$

Here χ is any choice of character of $(\mathbb{Z}/p\mathbb{Z})^\times$ that sends a primitive root modulo p to a primitive fourth root of unity.

Remark 1.4. For the fields in Theorem 1.3 we have $E = \mathbb{Q}\left(\sqrt{-(p+B\sqrt{p})}\right)$ where $p = B^2 + C^2$ with $B > 0$, $C > 0$ integers and $B \equiv 2 \pmod 4$. This can be deduced from the main theorem in [11] which gives the conductor of a cyclic quartic field in terms of certain integers that define a primitive element, and the fact that $\mathbb{Q}(\zeta_p)$ has at most one subfield of a given degree. We will need this fact to compute a complete set of CM points $\mathcal{CM}(E,\Phi,\mathcal{O}_F)$ using Algorithm 6.1.

Remark 1.5. Because the Dirichlet character χ in Theorem 1.3 takes values in $\{0,\pm1,\pm i\}$, the exponents on the right side of Theorem 1.3 are rational except for the regulator $\log(\epsilon_p)$. In particular,

$$B_1(\chi) = \frac{1}{p}\sum_{k=1}^{p}\chi(k)k.$$

Organization. This paper is organized as follows. In Section 2, we give several examples of our main theorems. In Section 3, we calculate the groups of characters appearing in the right side of Theorem 1.1 for the families of fields in Theorems 1.2 and 1.3, and in Section 4 we prove these two theorems using Theorem 1.1. In Section 5, we review some relevant facts about CM points, and in Section 6 we give a modified version of an algorithm of Streng [13] to compute the CM points for a quartic abelian CM field. In Section 7, we give some tables of CM points for fields of small class number and for some of the larger examples in Section 2. Finally, in the appendix we explain how to use an implementation of this algorithm in SageMath [12].

2. Examples

We now give several examples of Theorems 1.2 and 1.3, including explicit CM points obtained by Algorithm 6.1. We have also included a cyclic quartic field not covered by either of these theorems.

Example 2.1 (Theorem 1.2, $p = 13$, $q = 3$)**.** Let $E = \mathbb{Q}(\sqrt{13},\sqrt{-3})$, which has totally real subfield $F = \mathbb{Q}(\sqrt{13})$. Note that F has narrow class number 1. Then $h_E = 2, h_{-3} = 1, h_{-39} = 4, w_{-3} = 6$, and $w_{-39} = 2$. A fundamental unit is $\epsilon_{13} = (3+\sqrt{13})/2$. Let $\Phi = \{\mathrm{id},\sigma\}$ be the CM type where id is the identity map and σ is the automorphism such that $\sigma(\sqrt{13}) = -\sqrt{13}$ and $\sigma(\sqrt{-3}) = \sqrt{-3}$. A complete set of CM points of type (E, Φ) is given in Table 1 in Section 7. By Theorem 1.2, we compute

$$H\left(\frac{\sqrt{-3}-\sqrt{13}}{2}, \frac{\sqrt{-3}+\sqrt{13}}{2}\right) H\left(\frac{\sqrt{-3}-3\sqrt{13}}{\sqrt{13}+5}, \frac{\sqrt{-3}+3\sqrt{13}}{-\sqrt{13}+5}\right) = \frac{\Gamma\left(\frac{1}{3}\right)^3}{24\pi\Gamma\left(\frac{2}{3}\right)^3}$$
$$\times \left(\frac{\Gamma\left(\frac{1}{39}\right)\Gamma\left(\frac{2}{39}\right)\Gamma\left(\frac{4}{39}\right)\Gamma\left(\frac{5}{39}\right)\Gamma\left(\frac{8}{39}\right)\Gamma\left(\frac{10}{39}\right)\Gamma\left(\frac{11}{39}\right)\Gamma\left(\frac{16}{39}\right)\Gamma\left(\frac{20}{39}\right)\Gamma\left(\frac{22}{39}\right)\Gamma\left(\frac{25}{39}\right)}{\Gamma\left(\frac{7}{39}\right)\Gamma\left(\frac{14}{39}\right)\Gamma\left(\frac{17}{39}\right)\Gamma\left(\frac{19}{39}\right)\Gamma\left(\frac{23}{39}\right)\Gamma\left(\frac{28}{39}\right)\Gamma\left(\frac{29}{39}\right)\Gamma\left(\frac{31}{39}\right)\Gamma\left(\frac{34}{39}\right)\Gamma\left(\frac{35}{39}\right)\Gamma\left(\frac{37}{39}\right)}\right)^{\frac{1}{4}}$$
$$\times \frac{\Gamma\left(\frac{32}{39}\right)^{\frac{1}{4}}}{\Gamma\left(\frac{38}{39}\right)^{\frac{1}{4}}} \left(\frac{\Gamma_2\left(\frac{1}{13}\right)\Gamma_2\left(\frac{3}{13}\right)\Gamma_2\left(\frac{4}{13}\right)\Gamma_2\left(\frac{9}{13}\right)\Gamma_2\left(\frac{10}{13}\right)\Gamma_2\left(\frac{12}{13}\right)}{\Gamma_2\left(\frac{2}{13}\right)\Gamma_2\left(\frac{5}{13}\right)\Gamma_2\left(\frac{6}{13}\right)\Gamma_2\left(\frac{7}{13}\right)\Gamma_2\left(\frac{8}{13}\right)\Gamma_2\left(\frac{11}{13}\right)}\right)^{\frac{1}{2\log\left(\frac{3+\sqrt{13}}{2}\right)}}.$$

Example 2.2 (Theorem 1.2, $p = 29$, $q = 31$)**.** For an example involving a larger class number, consider the biquadratic CM field $E = \mathbb{Q}(\sqrt{29}, \sqrt{-31})$. The real quadratic subfield $F = \mathbb{Q}(\sqrt{29})$ has narrow class number 1. We have $h_E = 21, h_{-31} = 3, h_{-29\cdot 31} = 14, w_{-31} = 2, w_{-29\cdot 31} = 2$ and $\epsilon_{29} = (5+\sqrt{29})/2$. Similarly to Example 2.1, we will use the CM type $\Phi = \{\mathrm{id}, \sigma\}$ where $\sigma(\sqrt{29}) = -\sqrt{29}$ and $\sigma(\sqrt{-31}) = \sqrt{-31}$. Table 3 gives a complete set of CM points $\mathcal{CM}(E, \Phi, \mathcal{O}_F)$. Then Theorem 1.2 gives

$$\prod_{[\mathfrak{a}]\in \mathrm{CL}(E)} H(\Phi(z_{\mathfrak{a}})) = \left(\frac{1}{248\pi}\right)^{\frac{21}{2}} \prod_{k=1}^{31} \Gamma\left(\frac{k}{31}\right)^{\frac{7}{2}\chi_{-31}(k)}$$
$$\times \prod_{k=1}^{899} \Gamma\left(\frac{k}{899}\right)^{\frac{3}{4}\chi_{-899}(k)} \prod_{k=1}^{29} \Gamma\left(\frac{k}{29}\right)^{\frac{21\chi_{29}(k)}{4\log\left(\frac{5+\sqrt{29}}{2}\right)}}.$$

Example 2.3 (Theorem 1.3, $p = 13$)**.** Consider the real quadratic field $F = \mathbb{Q}(\sqrt{13})$, which has narrow class number 1. The unique cyclic quartic subfield of $\mathbb{Q}(\zeta_{13})$ is $E = \mathbb{Q}(\sqrt{\Delta})$, where $\Delta = -(13 + 2\sqrt{13})$ (the fact that Δ is a generator follows by Remark 1.4). Let $\Phi = \{\mathrm{id}, \tau\}$ be the CM type consisting of the identity map and the map τ such that $\tau(\sqrt{\Delta}) = \sqrt{\Delta'}$ where $\Delta' = -(13 - 2\sqrt{13})$. The number 2 is a primitive root modulo 13. Let χ be the character of $(\mathbb{Z}/13\mathbb{Z})^\times$ such that $\chi(2) = i$. A calculation shows $B_1(\chi) = -1 - i$. We have $h_E = 1$, and a fundamental unit is $\epsilon_{13} = (3+\sqrt{13})/2$. From Table 2, $(\sqrt{\Delta} - 3\sqrt{13})/(\sqrt{13}+5)$ is a CM point corresponding to the only fractional ideal class in E. We have $d_F = 13$ and $d_E = 2197$. Now by Theorem 1.3,

$$H\left(\frac{\sqrt{-(13+2\sqrt{13})}-3\sqrt{13}}{\sqrt{13}+5}, \frac{\sqrt{-(13-2\sqrt{13})}+3\sqrt{13}}{-\sqrt{13}+5}\right)$$
$$= \frac{1}{2\cdot 13^{\frac{1}{4}}\sqrt{2\pi}} \left(\frac{\Gamma\left(\frac{1}{13}\right)\Gamma\left(\frac{2}{13}\right)\Gamma\left(\frac{3}{13}\right)\Gamma\left(\frac{5}{13}\right)\Gamma\left(\frac{6}{13}\right)\Gamma\left(\frac{9}{13}\right)}{\Gamma\left(\frac{4}{13}\right)\Gamma\left(\frac{7}{13}\right)\Gamma\left(\frac{8}{13}\right)\Gamma\left(\frac{10}{13}\right)\Gamma\left(\frac{11}{13}\right)\Gamma\left(\frac{12}{13}\right)}\right)^{\frac{1}{2}}$$
$$\times \left(\frac{\Gamma_2\left(\frac{1}{13}\right)\Gamma_2\left(\frac{3}{13}\right)\Gamma_2\left(\frac{4}{13}\right)\Gamma_2\left(\frac{9}{13}\right)\Gamma_2\left(\frac{10}{13}\right)\Gamma_2\left(\frac{12}{13}\right)}{\Gamma_2\left(\frac{2}{13}\right)\Gamma_2\left(\frac{5}{13}\right)\Gamma_2\left(\frac{6}{13}\right)\Gamma_2\left(\frac{7}{13}\right)\Gamma_2\left(\frac{8}{13}\right)\Gamma_2\left(\frac{11}{13}\right)}\right)^{\frac{1}{4\log\left(\frac{3+\sqrt{13}}{2}\right)}}.$$

Example 2.4 (Theorem 1.3, $p = 109$)**.** Let $F = \mathbb{Q}(\sqrt{109})$. The unique cyclic quartic subfield of $\mathbb{Q}(\zeta_{109})$ is $E = \mathbb{Q}(\sqrt{\Delta})$ where $\Delta = -(109 + 10\sqrt{109})$. The class number of E is $h_E = 17$, and F has narrow class number 1. Let $\Phi = \{\mathrm{id}, \tau\}$ be the CM type consisting of the identity map and the map τ such that $\tau(\sqrt{\Delta}) = \sqrt{\Delta'}$ where $\Delta' = -(109 - 10\sqrt{109})$. The number 6 is a primitive root modulo 109, so let χ be the character of $(\mathbb{Z}/109\mathbb{Z})^\times$ such that $\chi(6) = i$. Then $B_1(\chi) = -5 - 3i$. A fundamental unit is $\epsilon_{109} = (261 + 25\sqrt{109})/2$, and we have $d_F = 109$, $d_E = 109^3$. Substituting this data into Theorem 1.3, we get

$$\prod_{[\mathfrak{a}]\in \mathrm{CL}(E)} H(\Phi(z_\mathfrak{a})) = \left(\frac{1}{8\pi\sqrt{109}}\right)^{\frac{17}{2}} \prod_{k=1}^{108} \Gamma\left(\frac{k}{109}\right)^{\frac{\alpha(k)}{2}} \prod_{k=1}^{108} \Gamma_2\left(\frac{k}{109}\right)^{\frac{17\chi_{109}(k)}{4\log\left(\frac{261+25\sqrt{109}}{2}\right)}}$$

where the 17 CM points are given in Table 4 and

$$\alpha(k) = \begin{cases} 5, & \text{if } \chi(k) = 1, \\ -5, & \text{if } \chi(k) = -1, \\ 3, & \text{if } \chi(k) = i, \\ -3, & \text{if } \chi(k) = -i. \end{cases}$$

Example 2.5. For an example of a cyclic quartic CM field not covered by Theorem 1.3, let $E = \mathbb{Q}(\sqrt{\Delta})$ where $\Delta = -3(5 + \sqrt{5})$, and $F = \mathbb{Q}(\sqrt{5})$, which has narrow class number 1. The field E is cyclic of degree 4 by the introduction to [11], and hence E is a CM field because it has at least one complex embedding. We have $d_F = 5$, $d_E = 2^6 \cdot 3^2 \cdot 5^3$, $h_E = 4$, and $\epsilon_5 = (1 + \sqrt{5})/2$. Let $\Phi = \{\mathrm{id}, \tau\}$ be the CM type such that $\tau(\sqrt{\Delta}) = \sqrt{\Delta'}$ where $\Delta' = -3(5 - \sqrt{5})$. We can use Magma [2] to compute the primitive Dirichlet characters inducing the Dirichlet L-functions in the factorization of the Dedekind zeta function of E, which are the characters in X_E. The non-trivial character in X_F is the Kronecker symbol of $\mathbb{Q}(\sqrt{5})$. It is also possible to determine the characters by using the explicit description of a generator of E found in [11, Lemma 2] in terms of biquadratic Gauss sums. Using either method, we find that $X_E \setminus X_F = \{\chi, \overline{\chi}\}$ where χ is the character of conductor 120 such that

$$\chi(31) = -1, \quad \chi(61) = -1, \quad \chi(41) = -1, \quad \chi(97) = i.$$

Proceeding in the same way as in the proof of Theorem 1.3, we find

$$\prod_{[\mathfrak{a}]\in \mathrm{CL}(E)} H(\Phi(z_\mathfrak{a})) = \frac{1}{2^{12} \cdot 3^2 \cdot 5 \cdot \pi^2} \prod_{k=1}^{120} \Gamma\left(\frac{k}{120}\right)^{\alpha(k)} \left(\frac{\Gamma_2\left(\frac{1}{5}\right)\Gamma_2\left(\frac{4}{5}\right)}{\Gamma_2\left(\frac{2}{4}\right)\Gamma_2\left(\frac{3}{4}\right)}\right)^{\frac{1}{\log\left(\frac{1+\sqrt{5}}{2}\right)}}$$

where

$$\alpha(k) = \begin{cases} 1, & \text{if } \chi(k) = -1 \text{ or } \chi(k) = -i, \\ -1, & \text{if } \chi(k) = 1 \text{ or } \chi(k) = i, \\ 0, & \text{if } \gcd(k, 120) > 1, \end{cases}$$

and the four CM points computed using Algorithm 6.1 are

$$z_1 = \sqrt{\Delta}, \qquad z_2 = \frac{\sqrt{\Delta}}{2}, \qquad z_3 = \frac{2\sqrt{\Delta}}{\sqrt{5}+5}, \qquad z_4 = \frac{\sqrt{\Delta}}{3}.$$

3. Groups of characters for quartic abelian CM fields

In this section we compute the groups of characters necessary to obtain Theorems 1.2 and 1.3. First, we recall a couple of classical results about quadratic Gauss sums. For an odd prime p and integer k not divisible by p, let

$$g(k,p) = \sum_{x=1}^{p-1} \left(\frac{x}{p}\right) e^{2\pi i k x/p}$$

denote the quadratic Gauss sum. Then we have the following:

$$g(k,p) = \left(\frac{k}{p}\right) g(1,p), \qquad [7, \text{Prop. } 6.3.1] \tag{3.1}$$

$$g(1,p) = \begin{cases} \sqrt{p}, & \text{if } p \equiv 1 \pmod 4, \\ i\sqrt{p}, & \text{if } p \equiv 3 \pmod 4, \end{cases} \qquad [7, \text{Ch. } 6, \text{Theorem } 1] \tag{3.2}$$

where the square roots have positive real part. In [1, Theorem 1.4], the authors give a more general version of Theorem 1.2 for multiquadratic extensions. Their computation of the group of characters for a multiquadratic extension relies on a certain Galois-theoretic property of the Kronecker symbol, which is more or less equivalent to the fact that an analogue of (3.2) holds when p is replaced by a squarefree integer (see [8, §5.2(d)-5.2(e)]). We will give an alternative computation of the group of characters for the family of biquadratic extensions in Theorem 1.2 using only (3.1) and (3.2).

Starting with the biquadratic case, let $p \equiv 1 \pmod 4$ and $q \equiv 3 \pmod 4$ be primes. Let $F = \mathbb{Q}(\sqrt{p})$, $E = \mathbb{Q}(\sqrt{p}, \sqrt{-q})$, and $m = pq$. By (3.2) it is clear that $E \subset \mathbb{Q}(\zeta_m)$ where $\zeta_m = e^{2\pi i/m}$. For a finite abelian group G and a subgroup H, there is a natural isomorphism $H^\perp \cong \widehat{G/H}$ where $H^\perp = \{\chi \in \widehat{G} : \chi|_H \equiv 1\}$. Take $G = \mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$, and let H_F and H_E denote the subgroups that fix F and E respectively. Then we have $X_F = H_F^\perp$ and $X_E = H_E^\perp$. Recall that we identify the automorphism $\sigma_t \in G$ determined by $\sigma_t(\zeta_m) = \zeta_m^t$ with the element $[t] \in (\mathbb{Z}/m\mathbb{Z})^\times$.

Lemma 3.1. *Let $p \equiv 1 \pmod 4$ and $q \equiv 3 \pmod 4$ be primes. Let $F = \mathbb{Q}(\sqrt{p})$, $E = \mathbb{Q}(\sqrt{p}, \sqrt{-q})$. Then X_F and X_E may be taken to be subsets of $\widehat{(\mathbb{Z}/pq\mathbb{Z})^\times}$, and*

$$X_F = \{\chi_1, \chi_2\}, \quad X_E = \{\chi_1, \chi_2, \chi_3, \chi_4\},$$

where

$$\chi_1 \equiv 1, \qquad \chi_2(k) = \left(\frac{k}{p}\right), \qquad \chi_3(k) = \left(\frac{k}{q}\right), \qquad \chi_4(k) = \left(\frac{k}{p}\right)\left(\frac{k}{q}\right).$$

Proof. We first determine H_F and H_E by using quadratic Gauss sums. Since $\zeta_p = \zeta_m^q$, then $\sigma_t(\zeta_p) = \zeta_p^t$. Hence by (3.1) and (3.2),

$$\sigma_t(\sqrt{p}) = \sigma_t(g(1,p)) = g(t,p) = \left(\frac{t}{p}\right)\sqrt{p}.$$

Thus σ_t fixes F if and only if t is a quadratic residue modulo p. The same argument shows

$$\sigma_t(\sqrt{-q}) = \left(\frac{t}{q}\right)\sqrt{-q}.$$

It follows that

$$H_F = \left\{ [t] \in (\mathbb{Z}/m\mathbb{Z})^\times : \left(\frac{t}{p}\right) = 1 \right\}, \; H_E = \left\{ [t] \in (\mathbb{Z}/m\mathbb{Z})^\times : \left(\frac{t}{p}\right) = \left(\frac{t}{q}\right) = 1 \right\}.$$

Now to determine X_E and X_F, by Galois theory $G/H_F \cong \mathrm{Gal}(F/\mathbb{Q})$ and $G/H_E \cong \mathrm{Gal}(E/\mathbb{Q})$. A finite abelian group is non-canonically isomorphic to its character group, so by the remarks preceding this lemma we have $X_F \cong \mathrm{Gal}(F/\mathbb{Q})$ and $X_E \cong \mathrm{Gal}(E/\mathbb{Q})$. It is immediate from the descriptions of H_F and H_E that the characters χ_i for $1 \leq i \leq 4$ lie in the groups X_F and X_E as claimed above, so by size considerations we are done. □

Moving on to the family of fields in Theorem 1.3, we first verify that these fields are in fact cyclic quartic CM fields.

Lemma 3.2. *Let $p \equiv 5 \pmod 8$ be a prime and let $F = \mathbb{Q}(\sqrt{p})$. Then $\mathbb{Q}(\zeta_p)$ has a unique degree 4 subfield E, and furthermore E is a cyclic quartic CM field with totally real quadratic subfield F.*

Proof. The Galois group G of $\mathbb{Q}(\zeta_p)$ is cyclic of order $p-1$. Hence by Galois theory the subfield E fixed by the unique subgroup H of G of index 4 is a cyclic quartic subfield. By (3.2) it is clear that $F \subset \mathbb{Q}(\zeta_p)$, and by a similar uniqueness argument we conclude $F \subset E$. Because $|G| \equiv 4 \pmod 8$, H does not contain the complex conjugation automorphism, which has order 2. Thus E is not fixed by complex conjugation and so E has a complex embedding. Since E is Galois over $\mathbb{Q}$, then every embedding of E is complex, so E is totally imaginary. □

We can explicitly determine X_F and X_E for the family of cyclic quartic CM fields in Theorem 1.3 without doing any computations in F or E.

Lemma 3.3. *Let $p \equiv 5 \pmod 8$ be a prime and let a be a primitive root modulo p. Let $F = \mathbb{Q}(\sqrt{p})$ and let E be the unique cyclic quartic subfield of $\mathbb{Q}(\zeta_p)$. Then X_F and X_E may be taken to be subsets of $\widehat{(\mathbb{Z}/p\mathbb{Z})^\times}$, and*

$$X_F = \{\chi_1, \chi_2\}, \quad X_E = \{\chi_1, \chi_2, \chi_3, \overline{\chi}_3\},$$

where

$$\chi_1 \equiv 1, \qquad \chi_2(k) = \left(\frac{k}{p}\right), \qquad \chi_3 \textit{ is such that } \chi_3(a) = i.$$

Proof. If we make the standard identification $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}) \cong (\mathbb{Z}/p\mathbb{Z})^\times$, then by Galois theory we have $H_E \cong \langle a^4 \rangle$ and $H_F \cong \langle a^2 \rangle$. Since the above characters χ_i for $1 \leq i \leq 4$ all have order dividing 4 each of them is in X_E. Similarly χ_1, $\chi_2 \in X_F$. As in the proof of Lemma 3.1 we are done because $X_F \cong \mathrm{Gal}(F/\mathbb{Q})$ and $X_E \cong \mathrm{Gal}(E/\mathbb{Q})$. □

4. Proof of main results

We are now prepared to evaluate the right side of (1.2) for the families of quartic CM fields under consideration.

Proof of Theorem 1.2. We have

$$X_F = \{\chi_1, \chi_2\}, \qquad X_E = \{\chi_1, \chi_2, \chi_3, \chi_4\},$$

where χ_1, χ_2, χ_3, χ_4 are as in Lemma 3.1. Recall that for the purposes of evaluating the right side of (1.2), we use the primitive characters which induce the characters in X_E and X_F. Note that χ_2 has conductor p, χ_3 has conductor q, and χ_4 has conductor pq. It is well known that χ_2 is the Kronecker symbol associated to $F = \mathbb{Q}(\sqrt{p})$, χ_3 is the Kronecker symbol associated to $\mathbb{Q}(\sqrt{-q})$, and χ_4 is the Kronecker symbol associated to $\mathbb{Q}(\sqrt{-pq})$. The residue of the Dedekind zeta function of F at $s = 1$ is equal to $L(\chi_2, 1)$, so by the class number formula

$$L(\chi_2, 1) = \frac{2h_p \log \epsilon_p}{\sqrt{p}}. \tag{4.1}$$

Recall that $h_p = 1$ by the narrow class number 1 assumption, and $\tau(\chi_2) = \sqrt{p}$ by (3.2). To evaluate $L(\chi_3, 0)$, let $\zeta_K^*(s)$ be the completed Dedekind zeta function of the field $K = \mathbb{Q}(\sqrt{-q})$. Combining the class number formula with the functional equation for $\zeta_K^*(s)$ gives

$$L(\chi_3, 0) = \frac{2h_{-q}}{w_{-q}}.$$

Similarly

$$L(\chi_4, 0) = \frac{2h_{-pq}}{w_{-pq}}.$$

Since $p \equiv 1 \pmod 4$ and $-q \equiv 1 \pmod 4$, then $d_F = p$ and $d_K = q$. As E is the compositum of the linearly disjoint fields F and K which have relatively prime discriminants, then we have $d_E = (d_F d_K)^2 = p^2 q^2$ [9, Ch. 3, Prop. 17]. It is now straightforward to compute the right side of (1.2) in terms of the data given. □

Proof of Theorem 1.3. In the notation of Lemma 3.3,

$$X_E = \{\chi_1, \chi_2, \chi_3, \overline{\chi}_3\}, \qquad X_F = \{\chi_1, \chi_2\}.$$

Hence in this case all characters appearing on the right side of (1.2) have conductor p. As in the proof of Theorem 1.2, the character χ_2 is the Kronecker symbol associated to the field $\mathbb{Q}(\sqrt{p})$, so $L(\chi_2, 1)$ is given by (4.1). We cannot use the class number formula for quadratic fields to evaluate the L-functions $L(\chi_3, s)$ and $L(\overline{\chi}_3, s)$ at $s = 0$, but we can write these values in terms of generalized Bernoulli numbers [7, Prop. 16.6.2]. Specifically,

$$L(\chi_3, 0) = -B_1(\chi_3)$$

where $B_1(\chi_3)$ is the first generalized Bernoulli number attached to χ_3. Similarly we have that $L(\overline{\chi}_3, 0) = -B_1(\overline{\chi}_3) = -\overline{B_1(\chi_3)}$. We can now evaluate the right side of (1.2) and arrive at Theorem 1.3. □

5. CM points

Before presenting an algorithm to compute the CM points appearing in (1.2), we give more detailed information on CM 0-cycles on a Hilbert modular variety, essentially following the exposition of Bruinier and Yang [3, Section 3] and specializing to the case F has narrow class number 1. Let F be a totally real number field of degree n with embeddings $\tau_1, \ldots, \tau_n$ and assume that F has narrow class number 1. Then $\mathrm{SL}_2(\mathcal{O}_F)$ acts on $\mathbb{H}^n$ via

$$Mz = (\tau_1(M)z_1, \ldots, \tau_n(M)z_n).$$

The quotient space $X(\mathcal{O}_F) = \mathrm{SL}_2(\mathcal{O}_F)\backslash\mathbb{H}^n$ is the open Hilbert modular variety associated to $\mathcal{O}_F$. The variety $X(\mathcal{O}_F)$ parametrizes isomorphism classes of principally polarized abelian varieties (A, ι) with real multiplication $\iota : \mathcal{O}_F \hookrightarrow \mathrm{End}(A)$.

Let E be a CM extension of F and $\Phi = (\sigma_1, \dots, \sigma_n)$ be a CM type for E. A point $z = (A, \iota)$ in $X(\mathcal{O}_F)$ is a CM point of type (E, Φ) if one of the following equivalent conditions holds:

(1) As a point $z \in \mathbb{H}^n$, there is a point $\tau \in E$ such that

$$\Phi(\tau) = (\sigma_1(\tau), \dots, \sigma_n(\tau)) = z$$

and

$$\Lambda_\tau = \mathcal{O}_F + \mathcal{O}_F\tau$$

is a fractional ideal of E.

(2) There exists a pair (A, ι') that is a CM abelian variety of type (E, Φ) with complex multiplication $\iota' : \mathcal{O}_E \hookrightarrow \mathrm{End}(A)$ such that $\iota = \iota'|_{\mathcal{O}_F}$.

By [3, Lemma 3.2] and the narrow class number 1 assumption, there is a bijection between $\mathrm{CL}(E)$ and the CM points of type (E, Φ) defined as follows: given an ideal class $C \in \mathrm{CL}(E)$, there exists a fractional ideal $\mathfrak{a} \in C^{-1}$ and $\alpha, \beta \in E^\times$ such that

$$\mathfrak{a} = \mathcal{O}_F\alpha + \mathcal{O}_F\beta \tag{5.1}$$

and

$$z = \frac{\beta}{\alpha} \in E^\times \cap \mathbb{H}^n = \{z \in E^\times : \Phi(z) \in \mathbb{H}^n\}.$$

Then z represents a CM point in $X(\mathcal{O}_F)$ in the sense that $\mathbb{C}^n/\Lambda_z$ is a principally polarized abelian variety of type (E, Φ) with complex multiplication by $\mathcal{O}_E$. Conversely, every principally polarized abelian variety of type (E, Φ) with complex multiplication by $\mathcal{O}_E$ arises from a decomposition as in (5.1) for some $\mathfrak{a}$ in a unique fractional ideal class in $\mathrm{CL}(E)$. We denote the CM 0-cycle consisting of the set of CM points of type (E, Φ) by $\mathcal{CM}(E, \Phi, \mathcal{O}_F)$ and identify it with the set

$$\{z_\mathfrak{a} \in E^\times \cap \mathbb{H}^n : [\mathfrak{a}] \in \mathrm{CL}(E)\}$$

under the bijection just described.

6. Algorithm to compute CM points

In his thesis, Streng [13] gives a couple of algorithms for enumerating principally polarized abelian varieties admitting complex multiplication by a CM field. In particular, [13, Algorithm 2.5, Appendix 2] produces a list of elements $z_i \in E^\times$ such that for every fractional ideal class $C \in \mathrm{CL}(E)$ and any choice of CM type Φ, there is some $\mathfrak{a} \in C^{-1}$ and z_i such that $\mathfrak{a} = \mathcal{O}_F + \mathcal{O}_F z_i$ and the element $\epsilon = (z_i - \overline{z}_i)^{-1}\delta^{-1}$ is such that $\Phi(\epsilon) \in E^\times \cap \mathbb{H}^n$, where δ is a generator of the different ideal of F. The points z_i produced by this algorithm give us the decomposition (5.1), so we simply need to modify the algorithm to select z_i such that $\Phi(z_i) \in E^\times \cap \mathbb{H}^n$ rather than $\Phi(\epsilon) \in E^\times \cap \mathbb{H}^n$.

Below we give a modified version of Streng's algorithm, where the only differences are that we have specialized to the case of a quadratic totally real subfield with narrow class number 1, and we are imposing the CM type on a different point as just explained. See the appendix for access to an implementation of Algorithm 6.1 in SageMath.

Algorithm 6.1

INPUT: A quartic abelian CM field $E = F(\sqrt{\Delta})$ and a CM type Φ where $\Delta \in F$, F is totally real of degree 2, and F has narrow class number 1.
OUTPUT: A complete set of representatives $\mathcal{CM}(E, \Phi, \mathcal{O}_F)$.

(1) Compute a set of representatives of the ideal class group of E.
(2) Compute an integral basis of $\mathcal{O}_E$.
(3) Write each element in the integral basis of $\mathcal{O}_E$ in the form $x_i + y_i\sqrt{\Delta}$ where $x_i, y_i \in F$.
(4) Compute all elements $a \in \mathcal{O}_F$ up to multiplication by $(\mathcal{O}_F^\times)^2$ such that

$$|N_{F/\mathbb{Q}}(a)| \leq \sqrt{|N_{F/\mathbb{Q}}(\Delta)|\, d_F \frac{6}{\pi^2}}.$$

(5) For each a, compute a complete set of representatives T_a for $\mathcal{O}_F/(a)$.
(6) For each a, compute all $b \in T_a$ for which a divides $b^2 - \Delta$.
(7) For each pair (a, b) and each basis element $x_i + y_i\sqrt{\Delta}$ of $\mathcal{O}_E$, check if $y_i a \in \mathcal{O}_F$, $x_i \pm y_i b \in \mathcal{O}_F$, and $a^{-1}y_i(\Delta - b^2) \in \mathcal{O}_F$ hold. Remove the pair (a, b) from the list if one of these conditions is not satisfied.
(8) For each pair (a, b), let

$$z = \frac{\sqrt{\Delta} - b}{a}.$$

Check if $\Phi(z) \in E^\times \cap \mathbb{H}^2$ and if

$$\mathfrak{a} = \mathcal{O}_F + \mathcal{O}_F z$$

is a fractional ideal of E. If so, then z is a CM point corresponding to the fractional ideal class of $\mathfrak{a}$. Search through the pairs (a, b) until a CM point has been found for each fractional ideal class in E.

7. Tables of CM points

In this section we give several tables of CM points computed using the implementation of Algorithm 6.1 referenced in the appendix.

Table 1 shows a complete set of representatives of the CM points for the biquadratic CM fields $E = \mathbb{Q}(\sqrt{p}, \sqrt{-q})$ and all primes p, $q < 50$ such that $p \equiv 1 \pmod 4$, $q \equiv 3 \pmod 4$, the class number of E is less than 4, and F has narrow class number 1. Here the CM type is given by $\Phi = \{\mathrm{id}, \sigma\}$ where σ is the automorphism of E such that $\sigma(\sqrt{p}) = -\sqrt{p}$ and $\sigma(\sqrt{-q}) = \sqrt{-q}$. Table 2 shows a complete set of representatives of the CM points for the cyclic quartic CM fields $E = \mathbb{Q}(\sqrt{\Delta})$ where $\Delta = -\left(p + B\sqrt{p}\right)$, the numbers p, B are as in Remark 1.4, and $p \leq 101$. The CM type is $\Phi = \{\mathrm{id}, \tau\}$ such that $\tau(\sqrt{\Delta}) = \sqrt{\Delta'}$ where $\Delta' = -(p - B\sqrt{p})$. In both cases, the CM points z are such that each $\mathcal{O}_F$-module $\mathcal{O}_F + \mathcal{O}_F z$ belongs to a distinct fractional ideal class in E. For each field E, the first CM point listed corresponds to the fractional ideal class of $\mathcal{O}_E$.

TABLE 1. CM points for biquadratic fields $E = \mathbb{Q}(\sqrt{p}, \sqrt{-q})$ with small class number.

Primes	h_E	CM points		
$p = 5, \quad q = 3$	1	$\dfrac{\sqrt{-3}-\sqrt{5}}{2}$		
$p = 5, \quad q = 7$	1	$\dfrac{\sqrt{-7}-\sqrt{5}}{2}$		
$p = 5, \quad q = 11$	2	$\dfrac{\sqrt{-11}-\sqrt{5}}{2}$	$\dfrac{\sqrt{-11}+\sqrt{5}}{4}$	
$p = 5, \quad q = 23$	3	$\dfrac{\sqrt{-23}-\sqrt{5}}{2}$	$\dfrac{\sqrt{-23}-1}{4}$	$\dfrac{\sqrt{-23}-2\sqrt{5}-1}{4}$
$p = 13, \quad q = 3$	2	$\dfrac{\sqrt{-3}-\sqrt{13}}{2}$	$\dfrac{\sqrt{-3}-3\sqrt{13}}{\sqrt{13}+5}$	
$p = 13, \quad q = 7$	1	$\dfrac{\sqrt{-7}-\sqrt{13}}{2}$		
$p = 13, \quad q = 19$	3	$\dfrac{\sqrt{-19}-\sqrt{13}}{2}$	$\dfrac{\sqrt{-19}+\sqrt{13}}{4}$	$\dfrac{\sqrt{-19}-\sqrt{13}}{4}$
$p = 13, \quad q = 31$	3	$\dfrac{\sqrt{-31}-\sqrt{13}}{2}$	$\dfrac{\sqrt{-31}-1}{4}$	$\dfrac{\sqrt{-31}-2\sqrt{13}-1}{4}$
$p = 17, \quad q = 3$	1	$\dfrac{\sqrt{-3}-\sqrt{17}}{2}$		
$p = 17, \quad q = 11$	1	$\dfrac{\sqrt{-11}-\sqrt{17}}{2}$		
$p = 17, \quad q = 19$	2	$\dfrac{\sqrt{-19}-\sqrt{17}}{2}$	$\dfrac{\sqrt{-19}+\sqrt{17}}{6}$	
$p = 29, \quad q = 3$	3	$\dfrac{\sqrt{-3}-\sqrt{29}}{2}$	$\dfrac{\sqrt{-3}+\sqrt{29}}{4}$	$\dfrac{\sqrt{-3}-\sqrt{29}}{4}$
$p = 29, \quad q = 7$	2	$\dfrac{\sqrt{-7}-\sqrt{29}}{2}$	$\dfrac{\sqrt{-7}-7\sqrt{29}}{2\sqrt{29}+12}$	
$p = 37, \quad q = 7$	2	$\dfrac{\sqrt{-7}-\sqrt{37}}{2}$	$\dfrac{\sqrt{-7}-7\sqrt{37}}{3\sqrt{37}+19}$	
$p = 41, \quad q = 3$	1	$\dfrac{\sqrt{-3}-\sqrt{41}}{2}$		
$p = 41, \quad q = 11$	3	$\dfrac{\sqrt{-11}-\sqrt{41}}{2}$	$\dfrac{\sqrt{-11}+3\sqrt{41}}{4\sqrt{41}+26}$	$\dfrac{\sqrt{-11}-3\sqrt{41}}{4\sqrt{41}+26}$

TABLE 2. CM points for cyclic quartic fields $E = \mathbb{Q}(\sqrt{\Delta})$ as in Theorem 1.3 where $\Delta = -(p + B\sqrt{p})$ and $p \leq 101$.

Parameters	h_E	CM points		
$p = 5, \quad B = 2$	1	$\dfrac{\sqrt{\Delta}-\sqrt{5}}{2}$		
$p = 13, \quad B = 2$	1	$\dfrac{\sqrt{\Delta}-3\sqrt{13}}{\sqrt{13}+5}$		
$p = 29, \quad B = 2$	1	$\dfrac{\sqrt{\Delta}-5\sqrt{29}}{\sqrt{29}+7}$		
$p = 37, \quad B = 6$	1	$\dfrac{\sqrt{\Delta}-\sqrt{37}}{2}$		
$p = 53, \quad B = 2$	1	$\dfrac{\sqrt{\Delta}-7\sqrt{53}}{\sqrt{53}+9}$		
$p = 61, \quad B = 6$	1	$\dfrac{\sqrt{\Delta}-5\sqrt{61}}{-\sqrt{61}+9}$		
$p = 101, \quad B = 10$	5	$\dfrac{\sqrt{\Delta}-\sqrt{101}}{2}$	$\dfrac{\sqrt{\Delta}+3\sqrt{101}}{19\sqrt{101}+191}$	$\dfrac{\sqrt{\Delta}-3\sqrt{101}}{19\sqrt{101}+191}$
		$\dfrac{\sqrt{\Delta}+3\sqrt{101}}{\sqrt{101}+11}$	$\dfrac{\sqrt{\Delta}-3\sqrt{101}}{\sqrt{101}+11}$	

The biquadratic field $\mathbb{Q}(\sqrt{29}, \sqrt{-31})$ has class number 21 and the cyclic quartic field $\mathbb{Q}(\sqrt{\Delta})$ where $\Delta = -(109+10\sqrt{109})$ has class number 17. Tables 3 and 4 show CM points for these fields, where the CM points and CM types are as in Tables 1 and 2.

TABLE 3. CM points for $\mathbb{Q}(\sqrt{29}, \sqrt{-31})$ (see Example 2.2).

$\dfrac{\sqrt{-31}-\sqrt{29}}{2}$	$\dfrac{\sqrt{-31}-1}{4}$	$\dfrac{\sqrt{-31}-2\sqrt{29}-1}{4}$
$\dfrac{\sqrt{-31}+\sqrt{29}}{4\sqrt{29}+22}$	$\dfrac{\sqrt{-31}-\sqrt{29}}{4\sqrt{29}+22}$	$\dfrac{\sqrt{-31}+\sqrt{29}}{\sqrt{29}+7}$
$\dfrac{\sqrt{-31}-\sqrt{29}}{\sqrt{29}+7}$	$\dfrac{\sqrt{-31}+5\sqrt{29}}{2\sqrt{29}+12}$	$\dfrac{\sqrt{-31}-5\sqrt{29}}{2\sqrt{29}+12}$
$\dfrac{\sqrt{-31}+5\sqrt{29}}{3\sqrt{29}+17}$	$\dfrac{\sqrt{-31}-5\sqrt{29}}{3\sqrt{29}+17}$	$\dfrac{\sqrt{-31}+\sqrt{29}}{6}$
$\dfrac{\sqrt{-31}-\sqrt{29}}{6}$	$\dfrac{\sqrt{-31}+4\sqrt{29}-1}{8\sqrt{29}+44}$	$\dfrac{\sqrt{-31}+2\sqrt{29}-1}{8\sqrt{29}+44}$
$\dfrac{\sqrt{-31}+19\sqrt{29}}{3\sqrt{29}+19}$	$\dfrac{\sqrt{-31}-19\sqrt{29}}{3\sqrt{29}+19}$	$\dfrac{\sqrt{-31}+19\sqrt{29}}{7\sqrt{29}+39}$
$\dfrac{\sqrt{-31}-19\sqrt{29}}{7\sqrt{29}+39}$	$\dfrac{\sqrt{-31}+10\sqrt{29}-1}{4\sqrt{29}+24}$	$\dfrac{\sqrt{-31}-8\sqrt{29}-1}{4\sqrt{29}+24}$

TABLE 4. CM points for $\mathbb{Q}(\sqrt{\Delta})$ where $\Delta = -(109+10\sqrt{109})$ (see Example 2.4).

$\dfrac{\sqrt{\Delta}-3\sqrt{109}}{-\sqrt{109}+11}$	$\dfrac{\sqrt{\Delta}-2\sqrt{109}+1}{6}$	$\dfrac{\sqrt{\Delta}-4\sqrt{109}-1}{6}$	$\dfrac{\sqrt{\Delta}+3\sqrt{109}}{511\sqrt{109}+5335}$
$\dfrac{\sqrt{\Delta}-3\sqrt{109}}{511\sqrt{109}+5335}$	$\dfrac{\sqrt{\Delta}+3\sqrt{109}}{\sqrt{109}+13}$	$\dfrac{\sqrt{\Delta}-3\sqrt{109}}{\sqrt{109}+13}$	$\dfrac{\sqrt{\Delta}+3\sqrt{109}}{218\sqrt{109}+2276}$
$\dfrac{\sqrt{\Delta}-3\sqrt{109}}{218\sqrt{109}+2276}$	$\dfrac{\sqrt{\Delta}+4\sqrt{109}+1}{3\sqrt{109}+33}$	$\dfrac{\sqrt{\Delta}-4\sqrt{109}-1}{3\sqrt{109}+33}$	$\dfrac{\sqrt{\Delta}+4\sqrt{109}+1}{-3\sqrt{109}+33}$
$\dfrac{\sqrt{\Delta}-2\sqrt{109}+1}{-3\sqrt{109}+33}$	$\dfrac{\sqrt{\Delta}+2\sqrt{109}-1}{-3\sqrt{109}+33}$	$\dfrac{\sqrt{\Delta}-4\sqrt{109}-1}{-3\sqrt{109}+33}$	$\dfrac{\sqrt{\Delta}+21\sqrt{109}}{143\sqrt{109}+1493}$
$\dfrac{\sqrt{\Delta}-21\sqrt{109}}{143\sqrt{109}+1493}$			

8. APPENDIX

A text file containing the code for an implementation of Algorithm 6.1 in SageMath can be found at `http://www.ms.uky.edu/~raca225/code.txt`. The code may be run in a SageMath worksheet. To explain how to use the code, we must first describe the way in which we parametrize cyclic quartic fields. Following the introduction in [11], let E be a cyclic quartic extension of $\mathbb{Q}$. Then there exist unique integers A, B, C, d_1 such that

$$E = \mathbb{Q}\left(\sqrt{A(d_1+B\sqrt{d_1})}\right)$$

where

$$A \text{ is squarefree and odd,} \tag{8.1}$$

$$d_1 = B^2+C^2 \text{ is squarefree and } B, C > 0, \tag{8.2}$$

$$\gcd(A, d_1) = 1. \tag{8.3}$$

Moreover, each choice of integers A, B, C, d_1 satisfying these conditions defines a cyclic quartic field. The unique quadratic subfield is always $\mathbb{Q}(\sqrt{d_1})$. The cyclic CM fields of degree 4 are precisely those fields with $A < 0$.

In the first lines of the code, the user can edit the existing values for the relatively prime squarefree integers `d1` $= d_1 > 1$ and `d2` $= d_2 \geq 1$ and compute the CM points for the biquadratic extension $\mathbb{Q}(\sqrt{d_1}, \sqrt{-d_2})$. Alternatively, the user can define `d1` $= d_1$, `A` $= A < 0$, and `B` $= B$ as in (8.1)-(8.3) and compute the CM points for the cyclic quartic extension $\mathbb{Q}(\sqrt{\Delta})$ where $\Delta = A(d_1 + B\sqrt{d_1})$. In both cases the CM type is the same as in Tables 1 and 2. Line 8 should read `biquadratic = true` when considering a biquadratic extension, and the line should read `biquadratic = false` when considering a cyclic extension. The output is a list of ordered pairs $[a, b]$ such that $a, b \in E$ and

$$z = \frac{\sqrt{\Delta} - b}{a}$$

gives the decomposition

$$\mathfrak{a} = \mathcal{O}_F + \mathcal{O}_F z,$$

where $\mathfrak{a} \subset E$ is the fractional ideal generated by 1 and z. One such z is given for each ideal class in E. Here $\Delta = -d_2$ in the biquadratic case, and $\Delta = A(d_1 + B\sqrt{d_1})$ in the cyclic case. The variable `f` appearing in the output is $\sqrt{d_1}$. Different versions of SageMath may produce different representatives of the CM points. The CM points in this paper were computed using SageMath Version 6.7.

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