PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 144, Number 7, July 2016, Pages 2999–3007
http://dx.doi.org/10.1090/proc/12950
Article electronically published on January 26, 2016

PERIODIC ORBITS OF LAGRANGIAN SYSTEMS WITH PRESCRIBED ACTION OR PERIOD

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(Communicated by Yingfei Yi)

ABSTRACT. We show that for every convex Lagrangian quadratic at infinity there is a real number a_0 such that for every $a > a_0$ the Lagrangian has a periodic orbit with action a. We attain estimates on the period and energy of the periodic orbits obtained. We also show that such a Lagrangian has periodic orbits of every period.

1. INTRODUCTION

Let M be a closed connected smooth Riemannian manifold. Let

$$L : TM \to \mathbb{R}$$

be a smooth convex Lagrangian. This means that L restricted to each T_xM has a positive definite Hessian. We shall also assume that L is quadratic at infinity, that is, there exists $R > 0$ such that for each $x \in M$ and $|v|_x > R$, $L(x, v)$ has the form

$$L(x, v) = \frac{1}{2}|v|_x^2 + \theta_x(v) - V(x),$$

where θ is a smooth 1-form on M and $V : M \to \mathbb{R}$ is a smooth function.

Let Λ be the set of absolutely continuous curves $x : [0, 1] \to M$, $x(0) = x(1)$, such that $\dot{x}$ has finite L^2-norm. It is well known ([Pal63], [Con06]) that Λ has a Hilbert manifold structure compatible with the Riemannian metric on M. Let $\mathbb{R}^+$ stand for the set of positive real numbers. Recall that the free-time action

$$\mathcal{A}_L : \mathbb{R}^+ \times \Lambda \to \mathbb{R}$$

is given by

$$\mathcal{A}_L(b, x) = \int_0^1 b\, L(x(t), \dot{x}(t)/b)\, dt.$$

If the Lagrangian is quadratic at infinity, arguments in Proposition 3.1 in [AS09b] show that $\mathcal{A}_L$ is a $C^{1,1}$ function (i.e. a function with locally Lipschitz derivative).

Periodic orbits with energy larger than Mañé's critical value were obtained in [Con06]. In a previous note ([Pat]) we showed the existence of periodic orbits with large enough abbreviated action. The aim of this note is to complete this picture by extending this kind of result to the action and period of the periodic orbits obtained. We show the following theorems.

Received by the editors April 28, 2015 and, in revised form, August 10, 2015 and August 31, 2015.

2010 *Mathematics Subject Classification.* Primary 37JXX; Secondary 70HXX.

Key words and phrases. Convex Lagrangian, action functional, periodic orbit.

The author was supported by an Anii grant.

Theorem 1. *For every convex Lagrangian quadratic at infinity there are positive numbers a_0 and α such that for every $a > a_0$ the Lagrangian has a periodic orbit with action a so that its period T and energy e satisfy*

$$\frac{1}{\alpha a} \leq T \leq \frac{\alpha}{a} \qquad \textit{and} \qquad \alpha^{-1}a^2 \leq e \leq \alpha a^2.$$

Theorem 2. *For every convex Lagrangian quadratic at infinity there are positive numbers T_0 and σ such that for every $0 < T < T_0$ there is a periodic orbit of period T so that its action a and energy e satisfy*

$$\frac{1}{\sigma T} \leq a \leq \frac{\sigma}{T} \qquad \textit{and} \qquad \frac{1}{\sigma T^2} \leq e \leq \frac{\sigma}{T^2}.$$

Note that the existence of periodic orbits of every period follows from Theorem 2 by taking iterations. In [Ben84] it is shown that a particular class of Lagrangians has an infinite set of periodic orbits satisfying similar bounds.

For the proof of Theorem 1 we shall see that there are real numbers a_0 and $\bar{b}$ such that every $a > a_0$ is a regular value of the restriction of $\mathcal{A}_L$ to $(0, \bar{b}) \times \Lambda$ and therefore the set X_a of those (b, x) such that $\mathcal{A}_L(b, x) = a$ and $0 < b < \bar{b}$ is a $C^{1,1}$ manifold. Consider the function $\mathcal{T}_a : X_a \to \mathbb{R}$ given by

$$\mathcal{T}_a(b, x) = b.$$

In Proposition 5 we show that critical points of $\mathcal{T}_a$ correspond to periodic orbits of the Lagrangian. In the proof of Theorem 1 we show that if a is large enough, then $\mathcal{T}_a$ has a critical point. Theorem 2 is a straightforward adaptation of arguments of [Con06] which we include at the end of this paper for the sake of completeness. (For other results and related approaches see [Abb13, AS09a, AS09b, AS10, BT98, CFP10, CMP04, Con06, GG09, Gin96, GK99, GP12, Maz11, Mer10, Nov82, Taĭ91, Taĭ92] and the references therein.)

2. Periodic orbits with prescribed action

2.1. **Time function.** Recall that the *energy* $E_L : TM \to \mathbb{R}$ is defined by

$$E_L(x, v) = \frac{\partial L}{\partial v}(x, v).v - L(x, v).$$

Since L is autonomous, E_L is a first integral of the Euler-Lagrange flow of L.

Critical points of $\mathcal{A}_{L+k}$ correspond to periodic orbits with energy k:

Proposition 3. *If (b, x) is a critical point of $\mathcal{A}_{L+k}$, then $y : [0, b] \to M$ given by $y(s) = x(s/b)$ is a periodic solution of the Euler-Lagrange equation of L with energy k (see* [AM78, CDI97, CI99, AS09b]*).*

It is useful to define the average energy function

$$e : \mathbb{R}^+ \times \Lambda \to \mathbb{R}$$

by

$$e(b, x) = \int_0^1 E_L(x(t), \dot{x}(t)/b) = \frac{1}{b} \int_0^b E_L(y(s), \dot{y}(s))\, ds.$$

We state the following straightforward remark for future reference.

Remark 4.

$$\frac{\partial \mathcal{A}_L}{\partial b}(b, x) = -e(b, x).$$

Let a_0 and $\bar{b}$ be as in the introduction, i.e. such that every $a > a_0$ is a regular value of the restriction of $\mathcal{A}_L$ to $(0, \bar{b}) \times \Lambda$ (we shall see in Section 4 that such numbers exist).

Define $\mathcal{T}_a : X_a \to \mathbb{R}$ by

$$\mathcal{T}_a(b, x) = b$$

and note that $\mathcal{T}_a$ is the restriction to X_a of the canonical projection $\Pi : \mathbb{R}^+ \times \Lambda \to \mathbb{R}^+$. Then, $\nabla \mathcal{T}_a(b, x)$ is the orthogonal projection of $\nabla \Pi = (1, 0)$ onto $T_{(b,x)} X_a$. We can find the projection by writing

$$\nabla \mathcal{T}_a = (v_b, v_x) \ , \qquad\qquad (1, 0) = \alpha \frac{\nabla \mathcal{A}_L}{||\nabla \mathcal{A}_L||} + (v_b, v_x).$$

It follows that

$$\alpha = \frac{1}{||\nabla \mathcal{A}_L||} \frac{\partial \mathcal{A}_L}{\partial b} \ , \qquad v_b = 1 - \alpha^2 \ , \qquad v_x = -\frac{1}{||\nabla \mathcal{A}_L||^2} \frac{\partial \mathcal{A}_L}{\partial b} \frac{\partial \mathcal{A}_L}{\partial x}. \tag{1}$$

Note that $\nabla \mathcal{T}_a$ is locally Lipschitz.

Proposition 5. *If (b, x) is a critical point of $\mathcal{T}_a$, then it corresponds to a periodic orbit.*

Proof. Let (b, x) be such that $\nabla \mathcal{T}_a(b, x) = 0$. Then $v_b(b, x) = 0$ and $v_x(b, x) = 0$. Therefore $\alpha \neq 0$ and hence

$$\frac{\partial \mathcal{A}_L}{\partial x}(b, x) = 0.$$

□

3. Minimax principle

Definition 6. Let $f : X \to \mathbb{R}$ be a C^1 map where X is an open set of a Hilbert manifold. We say that f satisfies the *Palais-Smale* condition at level c if every sequence $\{x_n\}$ such that $f(x_n) \to c$ and $||d_{x_n} f|| \to 0$ as $n \to \infty$ has a converging subsequence.

The following version of the minimax principle (Proposition 7 below) is a particular case of Proposition 6.3 from [Con06] (which in turn is inspired by that of [HZ94]; see also [Str96]). Let X be an open set in a Hilbert manifold and $f : X \to \mathbb{R}$ be a $C^{1,1}$ map. Observe that if X is not complete or the vector field $Y = -\nabla f$ is not globally Lipschitz, the gradient flow ψ_t of $-f$ is a priori only a local flow. Given $p \in X$, $t > 0$ define

$$\alpha(p) := \sup\{\, a > 0 \,|\, s \mapsto \psi_s(p) \text{ is defined on } s \in [0, a] \,\}.$$

We say that a function $\tau : X \to [0, +\infty)$ is an *admissible time* if τ is differentiable and $0 \leq \tau(x) < \alpha(x)$ for all $x \in X$. Given an admissible time τ and a subset $F \subset X$ define

$$F_\tau := \{\, \psi_{\tau(p)}(p) \,|\, p \in F \,\}.$$

Let $\mathcal{F}$ be a family of subsets $F \subset X$. We say that $\mathcal{F}$ is *forward invariant* if $F_\tau \in \mathcal{F}$ for all $F \in \mathcal{F}$ and any admissible time τ. Define

$$c(f, \mathcal{F}) = \inf_{F \in \mathcal{F}} \sup_{p \in F} f(p).$$

Proposition 7. *Let f be a $C^{1,1}$ function satisfying the Palais-Smale condition at level $c(f,\mathcal{F})$. Assume also that $\mathcal{F}$ is forward invariant under the gradient flow of $-f$. Suppose that $c = c(f,\mathcal{F})$ is finite and that there is ε such that the gradient flow is relatively complete in the set $[c-\varepsilon \le f \le c+\varepsilon]$. Then $c(f,\mathcal{F})$ is a critical value of f.*

4. Proof of Theorem 1

For the sequel of this section, we assume that M is simply connected. At the end we indicate how the nonsimply connected case is treated.

Since M is closed, there is a minimal integer $l > 1$ such that the homotopy group $\pi_l(M)$ is nontrivial. This implies the existence of a nontrivial homotopy class $\mathcal{H}$ of continuous maps $\Gamma : S^{l-1} \to \Lambda$ (see e.g. [Kli78, p. 36]). The proof of Th. 2.1.6, pp. 36-37, in [Kli78] shows that

$$\inf_{\Gamma\in\mathcal{H}} \max_{s} \ \ell(\Gamma(s)) = \rho > 0, \tag{2}$$

where ℓ denotes the length. Since the set of constant paths is a finite dimensional sub-manifold of Λ, there is a map $s \to x_s$ in $\mathcal{H}$ such that $||\dot{x}_s|| > 0$ for every $s \in S^{l-1}$. Let μ_0, μ be positive and such that

$$\mu_0 < ||\dot{x}_s||_{L^2} < \mu \tag{3}$$

for every $s \in S^{l-1}$. The numbers μ_0 and μ shall be fixed for the rest of the paper.

Let $A, B, A_1, B_1, C, D, C_1, D_1$ be positive numbers such that

$$\frac{A}{2}|v|^2 - B \le L(x,v) \le \frac{A_1}{2}|v|^2 + B_1 \quad \text{and} \quad \frac{C}{2}|v|^2 - D \le E_L(x,v) \le \frac{C_1}{2}|v|^2 + D_1$$

for all (x,v) in TM. (These numbers exist because L is quadratic at infinity.) Therefore

$$A\frac{||\dot{x}||_{L^2}^2}{2b} - Bb \le \mathcal{A}_L(b,x) \le A_1\frac{||\dot{x}||_{L^2}^2}{2b} + B_1 b \tag{4}$$

and

$$C\frac{||\dot{x}||_{L^2}^2}{2b^2} - D \le e(b,x) \le C_1\frac{||\dot{x}||_{L^2}^2}{2b^2} + D_1 \tag{5}$$

for every $(b,x) \in \mathbb{R}^+ \times \Lambda$. From (4) and (5) we obtain

$$\frac{C}{A_1}(\frac{\mathcal{A}_L(b,x)}{b} - B_1) - D \le e(b,x) \le \frac{C_1}{A}(\frac{\mathcal{A}_L(b,x)}{b} + B) + D_1 \tag{6}$$

for every $(b,x) \in \mathbb{R}^+ \times \Lambda$. We state the following remark for future reference.

Remark 8. $e(b,x) > 0$ if $b < ||\dot{x}||_{L^2}\sqrt{C/2D}$.

Fix $a_1 > 0$. Let $\bar{b}$ be such that

$$\frac{C}{A_1}(\frac{a_1}{\bar{b}} - B_1) - D > 0 \qquad \text{and} \qquad \bar{b} < \mu_0\sqrt{\frac{C}{2D}}.$$

For every $a > a_1$ define X_a as the set of those (b,x) such that $\mathcal{A}_L(b,x) = a$ and $b < \bar{b}$.

Lemma 9. *X_a does not contain critical points $\mathcal{A}_L$ for every $a > a_1$.*

Proof. By the choice of $\bar{b}$ and (6) we have $e(b,x) > 0$ for every (b,x) in X_a. The lemma is finished by Remark 4. ☐

Let $\mathcal{G}_a$ be the set of continuous maps $\Gamma : S^{l-1} \to \mathbb{R}^+ \times \Lambda$ of the form $\Gamma(s) = (b_s, x_s)$, where $s \to x_s$ belongs to $\mathcal{H}$ and such that $\Gamma(s) \in X_a$ for every $s \in S^{l-1}$. Let $\mathcal{F}_a$ be the family of sets $\Gamma(S^{l-1})$ with $\Gamma \in \mathcal{G}_a$.

Recall that $\mathcal{T}_a : X_a \to \mathbb{R}$ is defined by $\mathcal{T}_a(b, x) = b$.

Lemma 10. *There are positive numbers a_0 and α such that $\mathcal{F}_a$ is nonvoid and*

$$\frac{1}{\alpha a} \leq c(\mathcal{T}_a, \mathcal{F}_a) \leq \frac{\alpha}{a}$$

for every $a > a_0$.

Proof. Let a_0 be such that

$$a_0 > \frac{A_1 \mu^2}{2\bar{b}} + B_1 \bar{b}\,, \qquad a_0 > 2B_1 \bar{b} \qquad \text{and} \qquad 2a > \sqrt{a^2 + 2BA\rho^2}$$

for every $a > a_0$. Let α be such that

$$\alpha > A_1 \mu^2 \qquad \text{and} \qquad \alpha^{-1} < \frac{A\rho^2}{4}.$$

For the sequel of this proof we fix $a > a_0$. We are going to show that for each x such that $\mu_0 < ||\dot{x}||_{L^2} < \mu$ there is $0 < \tau(x) < \bar{b}$ so that $\mathcal{A}_L(\tau(x), x) = a$. Take x such that $\mu_0 < ||\dot{x}||_{L^2} < \mu$ and set

$$g(b) = \mathcal{A}_L(b, x).$$

Then

$$g(\bar{b}) = \mathcal{A}_L(\bar{b}, x) \leq A_1 \frac{||\dot{x}||^2_{L^2}}{2\bar{b}} + B_1 \bar{b} < a_0.$$

By Remarks 4 and 8 we have $g'(b) < 0$ for $b < \bar{b}$. On the other hand, g goes to infinity as b goes to zero by (4) (taking into account that $||\dot{x}||_{L^2} > 0$).

Then there is a unique real number $\tau(x) < \bar{b}$ such that $\mathcal{A}_L(\tau(x), x) = a$. By the implicit function theorem, the map $x \to \tau(x)$ is a real C^1 function defined on the set $\mu_0 < ||\dot{x}||_{L^2} < \mu$ and satisfying $\mathcal{A}_L(\tau(x), x) = a$. Let $s \to x_s$ be a map in $\mathcal{H}$ satisfying (3). Set $b_s = \tau(x_s)$ for every s. Define $\Gamma(s) = (b_s, x_s)$ and note that $\Gamma \in \mathcal{G}_a$, showing that $\mathcal{F}_a$ is nonvoid.

For every $a > a_0$ define $\phi(a)$ as the minimum $b > 0$ satisfying

$$A\frac{\rho^2}{2b} - Bb \leq a.$$

Therefore

$$\phi(a) = \frac{A\rho^2}{\sqrt{a^2 + 2BA\rho^2} + a} > \frac{1}{\alpha a}$$

for every $a > a_0$.

Let $\Gamma \in \mathcal{G}_a$ be such that $\Gamma(s) = (b_s, x_s)$. Let s_0 be such that $\ell(x_{s_0}) \geq \rho$. Then

$$A\frac{\rho^2}{2b_{s_0}} - Bb_{s_0} \leq A\frac{||\dot{x}_{s_0}||^2_{L^2}}{2b_{s_0}} - Bb_{s_0} \leq \mathcal{A}_L(b_{s_0}, x_{s_0}) = a.$$

By the definition of $\phi(a)$ we have

$$\frac{1}{\alpha a} < \phi(a) \leq b_{s_0} = \mathcal{T}_a(\Gamma(s_0)),$$

showing that $1/\alpha a \leq \sup_s \mathcal{T}_a(\Gamma(s))$. Since $\Gamma \in \mathcal{G}_a$ is arbitrary we obtain the leftmost inequality in the statement of the lemma. For the other inequality consider $\Gamma(s) = (b_s, x_s)$, where x_s satisfies $\mu_0 < ||\dot{x}_s||_{L^2} < \mu$. Then

$$a \leq \mathcal{A}_L(b_s, x_s) \leq A_1 \frac{\mu^2}{2b_s} + B_1 \bar{b}$$

for every s, and hence

$$b_s \leq \frac{A_1 \mu^2}{2(a - B_1 \bar{b})} \leq \frac{\alpha}{a}$$

for every s, finishing the lemma. □

Lemma 11. *The flow of $-\nabla \mathcal{T}_a$ is relatively complete in $0 < c_1 \leq \mathcal{T}_a \leq c_2$.*

Proof. By contradiction let $s \to \Gamma(s) = (b(s), x(s))$ be a flow semi-trajectory defined in the maximal interval $[0, \bar{s})$ and contained in $c_1 \leq \mathcal{T}_a \leq c_2$. Let $t_n \in [0, \bar{s})$ be a sequence converging to $\bar{s}$. By the same argument as in Lemma 6.9 of [Con06], the sequence $\Gamma(t_n)$ is a Cauchy sequence, implying that $b(t_n)$ converges to $b_0 \in [0, \infty)$. Since $b(t_n) \geq c_1 > 0$ we know that $b_0 > 0$, and hence the sequence $\Gamma(t_n)$ converges in X_a, which allows the flow semi-trajectory to be extended. □

Lemma 12. *$\mathcal{T}_a$ satisfies the Palais-Smale condition at level $c(\mathcal{T}_a, \mathcal{F}_a)$.*

Proof. Let $\{(b_n, x_n)\}$ be a sequence in X_a such that $\mathcal{T}_a(b_n, x_n) \to c(\mathcal{T}_a, \mathcal{F}_a)$ and such that $||\nabla \mathcal{T}_a(b_n, x_n)|| \to 0$. By Lemma 10, b_n is bounded and bounded away from zero. Then $||\nabla A_L(b_n, x_n)||$ is bounded away from zero since otherwise arguments of Proposition 3.12 in [Con06] (see also [CIPP00] and [Ben86]) show that $\{(b_n, x_n)\}$ has a subsequence converging to a critical point of $\mathcal{A}_L$, which is impossible because a is a regular value of $\mathcal{A}_L$. Since $\mathcal{A}_L(b_n, x_n) = a$ and taking into account that b_n is bounded and bounded away from zero, we conclude, by (4), that $||\dot{x}_n||_{L^2}$ is bounded, which implies, by (5), that $e(b_n, x_n) = -\frac{\partial \mathcal{A}_L}{\partial b}(b_n, x_n)$ is bounded.

Let v_b, v_x, α be as in (1); then $v_b(b_n, x_n)$ converges to 0 and thus $\alpha(b_n, x_n)$ is bounded and bounded away from zero. Then $||\nabla A_L(b_n, x_n)||$ is bounded and $\frac{\partial \mathcal{A}_L}{\partial b}(b_n, x_n)$ is bounded away from zero.

On the other hand, taking into account that $||v_x(b_n, x_n)||$ also converges to 0, we obtain

$$||\frac{\partial \mathcal{A}_L}{\partial x}(b_n, x_n)|| \to 0.$$

Hence, by the argument of Proposition 3.12 of [Con06], the sequence $\{(b_n, x_n)\}$ has a converging subsequence in X_a. In fact Proposition 3.12 assumes that $||d_{(b_n,x_n)} \mathcal{A}_L||$ converges to zero, but it is enough to have $||\frac{\partial \mathcal{A}_L}{\partial x}(b_n, x_n)|| \to 0$ as is shown in Lemma 5.3 of [Abb13]. □

Proof of Theorem 1. Lemmas 10, 11 and 12 allow us to apply Proposition 7, completing the proof in the simply connected case. Note that the orbits obtained are nontrivial: if (b, x) is a critical point such that $b = \mathcal{T}_a(b, x) = c(\mathcal{T}_a, \mathcal{F}_a)$, then x cannot be constant because otherwise (4) would yield $a_0 < a = \mathcal{A}_L(b, x) \leq B_1 c(\mathcal{T}_a, \mathcal{F}_a) \leq B_1 \bar{b}$, contradicting the choice of a_0. The estimates on the period follow from Lemma 10, and the estimates on the energy follow from the estimates on the period and (6) (possibly after taking larger α and a_0).

If M is not simply connected, there exists λ, a connected component of Λ consisting of noncontractible loops. Let $\mathcal{F}$ be the family of sets $F = \{(b, x)\}$ such that

$x \in \lambda$ and $\mathcal{A}_L(b,x) = a$. Since all curves in λ have length bounded away from zero, equation (2) holds and thus all arguments of the simply connected case apply.

5. Periodic orbits with prescribed period

Proof of Theorem 2. Assume that M is simply connected. Let $\mathcal{H}$ be the family of maps of the previous section and let $\mathcal{F}$ be the family of sets $\Gamma(S^{l-1})$ with $\Gamma \in \mathcal{H}$. Take ρ as in (2).

Let T_0 be such that

$$Bb < \frac{A\rho^2}{4b}\,, \qquad B_1 b < \frac{A_1\mu^2}{2b} \qquad \text{and} \qquad B_1 b < \frac{A\rho^2}{2b} - Bb$$

for $0 < b < T_0$. Let σ be such that

$$\sigma > A_1\mu^2 \qquad \text{and} \qquad \sigma^{-1} < \frac{A\rho^2}{4}.$$

Fix $0 < b < T_0$ and set $\mathcal{A}_b(x) = A_L(b,x)$. Let $s \to x_s$ be a map in $\mathcal{H}$ and s_0 be such that $\ell(x_{s_0}) > \rho$. Then

$$\mathcal{A}_b(x_{s_0}) \geq A\frac{\rho^2}{2b} - Bb > A\frac{\rho^2}{4b} > \frac{1}{\sigma b}. \tag{7}$$

Take $s \to x_s$ such that $||\dot{x}_s||_{L^2} < \mu$ for every s. Then

$$\mathcal{A}_b(x_s) \leq A_1\frac{\mu^2}{2b} + B_1 b < A_1\frac{\mu^2}{b} < \frac{\sigma}{b} \tag{8}$$

for every s. By (7) and (8) we conclude that

$$\frac{1}{\sigma b} < c(\mathcal{A}_b, \mathcal{F}) < \frac{\sigma}{b}.$$

The flow of $-\nabla\mathcal{A}_b$ is relatively complete on sets of the form $c_1 \leq \mathcal{A}_b \leq c_2$ because Λ is complete. On the other hand, the Palais-Smale condition for $\mathcal{A}_b$ follows directly by arguments of Proposition 3.12 in [Con06]. Therefore, Proposition 7 completes the theorem in the simply connected case. Observe that the periodic orbits obtained are nontrivial: if x is trivial, then $\mathcal{A}_L(b,x) < B_1 b$, contradicting the inequality $B_1 b < A\rho^2/2b - Bb$ which holds for all $b < T_0$.

The estimates on the energy follow from the estimates on the action and (6) (possibly after taking a larger σ and a smaller T_0).

The nonsimply connected case is treated similarly by taking $\mathcal{F}$, the family of sets $F = \{x\}$ such that x belongs to a connected component of Λ consisting of noncontractible loops.

Acknowledgement

The author is grateful to the anonymous referee for helpful suggestions improving both theorems of this note and for correcting a mistake in Theorem 2.

References

[Abb13] Alberto Abbondandolo, *Lectures on the free period Lagrangian action functional*, J. Fixed Point Theory Appl. **13** (2013), no. 2, 397–430, DOI 10.1007/s11784-013-0128-1. MR3122334

[AM78] Ralph Abraham and Jerrold E. Marsden, *Foundations of mechanics.* Second edition, revised and enlarged. With the assistance of Tudor Raţiu and Richard Cushman, Benjamin/Cummings Publishing Co., Inc., Advanced Book Program, Reading, Mass., 1978. MR515141 (81e:58025)

[AS09a] Alberto Abbondandolo and Matthias Schwarz, *Estimates and computations in Rabinowitz-Floer homology*, J. Topol. Anal. **1** (2009), no. 4, 307–405, DOI 10.1142/S1793525309000205. MR2597650 (2011c:53216)

[AS09b] Alberto Abbondandolo and Matthias Schwarz, *A smooth pseudo-gradient for the Lagrangian action functional*, Adv. Nonlinear Stud. **9** (2009), no. 4, 597–623. MR2560122 (2010m:37099)

[AS10] Alberto Abbondandolo and Matthias Schwarz, *Floer homology of cotangent bundles and the loop product*, Geom. Topol. **14** (2010), no. 3, 1569–1722, DOI 10.2140/gt.2010.14.1569. MR2679580 (2011k:53126)

[Ben84] V. Benci, *Normal modes of a Lagrangian system constrained in a potential well* (English, with French summary), Ann. Inst. H. Poincaré Anal. Non Linéaire **1** (1984), no. 5, 379–400. MR779875 (86h:58048)

[Ben86] Vieri Benci, *Periodic solutions of Lagrangian systems on a compact manifold*, J. Differential Equations **63** (1986), no. 2, 135–161, DOI 10.1016/0022-0396(86)90045-8. MR848265 (88h:58102)

[BT98] Abbas Bahri and Iskander A. Taimanov, *Periodic orbits in magnetic fields and Ricci curvature of Lagrangian systems*, Trans. Amer. Math. Soc. **350** (1998), no. 7, 2697–2717, DOI 10.1090/S0002-9947-98-02108-4. MR1458315 (98k:58051)

[CDI97] Gonzalo Contreras, Jorge Delgado, and Renato Iturriaga, *Lagrangian flows: the dynamics of globally minimizing orbits. II*, Bol. Soc. Brasil. Mat. (N.S.) **28** (1997), no. 2, 155–196, DOI 10.1007/BF01233390. MR1479500 (98i:58093)

[CFP10] Kai Cieliebak, Urs Frauenfelder, and Gabriel P. Paternain, *Symplectic topology of Mañé's critical values*, Geom. Topol. **14** (2010), no. 3, 1765–1870, DOI 10.2140/gt.2010.14.1765. MR2679582 (2011k:53127)

[CI99] Gonzalo Contreras and Renato Iturriaga, *Global minimizers of autonomous Lagrangians*, 22° Colóquio Brasileiro de Matemática. [22nd Brazilian Mathematics Colloquium], Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 1999. MR1720372 (2001j:37113)

[CIPP00] G. Contreras, R. Iturriaga, G. P. Paternain, and M. Paternain, *The Palais-Smale condition and Mañé's critical values*, Ann. Henri Poincaré **1** (2000), no. 4, 655–684, DOI 10.1007/PL00001011. MR1785184 (2001k:37101)

[CMP04] Gonzalo Contreras, Leonardo Macarini, and Gabriel P. Paternain, *Periodic orbits for exact magnetic flows on surfaces*, Int. Math. Res. Not. **8** (2004), 361–387, DOI 10.1155/S1073792804205050. MR2036336 (2005a:37103)

[Con06] Gonzalo Contreras, *The Palais-Smale condition on contact type energy levels for convex Lagrangian systems*, Calc. Var. Partial Differential Equations **27** (2006), no. 3, 321–395, DOI 10.1007/s00526-005-0368-z. MR2260806 (2007i:37116)

[GG09] Viktor L. Ginzburg and Başak Z. Gürel, *Periodic orbits of twisted geodesic flows and the Weinstein-Moser theorem*, Comment. Math. Helv. **84** (2009), no. 4, 865–907, DOI 10.4171/CMH/184. MR2534483 (2010j:53185)

[Gin96] Viktor L. Ginzburg, *On closed trajectories of a charge in a magnetic field. An application of symplectic geometry*, Contact and symplectic geometry (Cambridge, 1994), Publ. Newton Inst., vol. 8, Cambridge Univ. Press, Cambridge, 1996, pp. 131–148. MR1432462 (97j:58128)

[GK99] Viktor L. Ginzburg and Ely Kerman, *Periodic orbits in magnetic fields in dimensions greater than two*, Geometry and topology in dynamics (Winston-Salem, NC, 1998/San Antonio, TX, 1999), Contemp. Math., vol. 246, Amer. Math. Soc., Providence, RI, 1999, pp. 113–121, DOI 10.1090/conm/246/03778. MR1732375 (2000k:37087)

[GP12] Nancy Guelman and Miguel Paternain, *Periodic orbits of convex Lagrangians and average energy*, Bull. Braz. Math. Soc. (N.S.) **43** (2012), no. 2, 201–219, DOI 10.1007/s00574-012-0010-6. MR2928169

[HZ94] Helmut Hofer and Eduard Zehnder, *Symplectic invariants and Hamiltonian dynamics*, Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks], Birkhäuser Verlag, Basel, 1994. MR1306732 (96g:58001)

[Kli78] Wilhelm Klingenberg, *Lectures on closed geodesics*, Springer-Verlag, Berlin-New York, 1978. Grundlehren der Mathematischen Wissenschaften, Vol. 230. MR0478069 (57 #17563)

[Maz11] Marco Mazzucchelli, *The Lagrangian Conley conjecture*, Comment. Math. Helv. **86** (2011), no. 1, 189–246, DOI 10.4171/CMH/222. MR2745280 (2012f:37131)

[Mer10] Will J. Merry, *Closed orbits of a charge in a weakly exact magnetic field*, Pacific J. Math. **247** (2010), no. 1, 189–212, DOI 10.2140/pjm.2010.247.189. MR2718211 (2012b:37168)

[Nov82] S. P. Novikov, *The Hamiltonian formalism and a multivalued analogue of Morse theory* (Russian), Uspekhi Mat. Nauk **37** (1982), no. 5(227), 3–49, 248. MR676612 (84h:58032)

[Pal63] Richard S. Palais, *Morse theory on Hilbert manifolds*, Topology **2** (1963), 299–340. MR0158410 (28 #1633)

[Pat] Miguel Paternain, *Periodic orbits with prescribed abbreviated action*, Proc. Amer. Math. Soc. **143** (2015), no. 9, 4001–4008, DOI 10.1090/S0002-9939-2015-12597-X. MR3359588

[Str96] Michael Struwe, *Variational methods*, 2nd ed., Applications to nonlinear partial differential equations and Hamiltonian systems. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 34, Springer-Verlag, Berlin, 1996. MR1411681 (98f:49002)

[Taĭ91] I. A. Taĭmanov, *Non-self-intersecting closed extremals of multivalued or not-everywhere-positive functionals* (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **55** (1991), no. 2, 367–383; English transl., Math. USSR-Izv. **38** (1992), no. 2, 359–374. MR1133303 (92i:58036)

[Taĭ92] I. A. Taĭmanov, *Closed extremals on two-dimensional manifolds* (Russian, with Russian summary), Uspekhi Mat. Nauk **47** (1992), no. 2(284), 143–185, 223, DOI 10.1070/RM1992v047n02ABEH000880; English transl., Russian Math. Surveys **47** (1992), no. 2, 163–211. MR1185286 (93k:58050)

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