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A REPRESENTATION FOR PSEUDOHOLOMORPHIC SURFACES IN SPHERES

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ABSTRACT. We give a local representation for the pseudoholomorphic surfaces in Euclidean spheres in terms of holomorphic data. Similar to the case of the generalized Weierstrass representation of Hoffman and Osserman for minimal Euclidean surfaces, we assign such a surface in $\mathbb{S}^{2n}$ to a given set of n holomorphic functions defined on a simply-connected domain in $\mathbb{C}$.

A basic tool in the study of minimal surfaces in Euclidean space with codimension higher than one is the generalized Weierstrass representation introduced by Hoffman and Osserman in [9]. Roughly speaking, to a given set of n holomorphic functions defined on a simply-connected domain in $\mathbb{C}$ it assigns a generalized minimal surface in $\mathbb{R}^{n+1}$. Conversely, any simply-connected minimal surface in $\mathbb{R}^{n+1}$ can be obtained in this way.

For minimal surfaces in Euclidean spheres, there is no representation similar to the one in Euclidean space and, maybe, this is not even possible. But inspired by results and methods in [4], [6] and [7], we provide such a representation for a class of minimal surfaces, namely, the generalized pseudoholomorphic surfaces in Euclidean spheres. As above, to a set of n holomorphic functions in a simply-connected domain in $\mathbb{C}$ and some constants of integration we assign a generalized pseudoholomorphic surface in $\mathbb{S}^{2n}$. Here and above, by the term generalized we mean that the metric induced on the surface is singular at most at isolated points.

In a seminal paper due to Calabi [3] (see also Barbosa [1]) it was shown that any two-dimensional sphere minimally immersed in an Euclidean sphere lays in even substantial codimension and has to be pseudoholomorphic. According to Calabi's definition, a minimal surface is pseudoholomorphic means that when parametrized by an isothermal coordinate z the surface $f\colon \mathbb{S}^2 \to \mathbb{S}^{2n} \subset \mathbb{R}^{2n+1}$ satisfies

$$\langle \partial^j f, \partial^k f\rangle = 0, \;\; j+k>0,$$

with respect to the symmetric product where $\partial = \partial/\partial z$ and $\partial^0 f = f$.

There are other definitions for minimal surfaces in spheres to be pseudoholomorphic that are equivalent to the above. For instance, one may require the ellipses of curvature of any order and at any point to be circles; see part (*iii*) in Remarks 5. In turn, this is equivalent to asking the Hopf differentials of all orders to vanish identically; see [10] for details.

We point out that even in the compact case there are plenty of pseudoholomorphic surfaces other than topological spheres. For instance, this is the case of many

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pseudoholomorphic surfaces in the nearly Kaehler sphere $\mathbb{S}^6$. These surfaces were introduced by Bryant [2] and have been intensively studied.

In the sequel, we first describe an algorithm that leads to the representation of the pseudoholomorphic surfaces and after that we state our main result.

Let $\beta_s\colon U \to \mathbb{C}$, $0 \le s \le n-1$, be nonzero holomorphic functions defined on a simply-connected domain $U \subset \mathbb{C}$. Setting $\alpha_0 = \beta_0$, define subsequently isotropic maps with respect to the symmetric product between complex vectors $\alpha_r\colon U \to \mathbb{C}^{2r+1}$, $0 \le r \le n$, by

$$\alpha_{r+1} = \beta_{r+1}\left(1-\phi_r^2, i(1+\phi_r^2), 2\phi_r\right)$$

where $\phi_r = \int^z \alpha_r dz$ and $\beta_n = 1$. Now define $F_s\colon U \to \mathbb{C}^{2n+1}$, $1 \le s \le n+1$, by

$$F_1 = \alpha_n \quad \text{and} \quad F_{r+1} = \partial F_r - \frac{1}{\|F_r\|^2}\langle \partial F_r, \bar{F}_r\rangle F_r,\ 1 \le r \le n. \tag{1}$$

Finally, let $g\colon L^2 \to \mathbb{S}^{2n}$ be the map given by

$$g = \frac{1}{\|\mathrm{Re}(F_{n+1})\|}\mathrm{Re}(F_{n+1})$$

where $L^2 = U$ with the induced metric.

Theorem 1. *The map $g\colon L^2 \to \mathbb{S}^{2n}$ parametrizes a generalized pseudoholomorphic surface. Conversely, any pseudoholomorphic surface in $\mathbb{S}^{2n}$ can be locally obtained in this way.*

To conclude the paper, we show how the above parametrization can be used to parametrize the real Kaehler hypersurfaces in Euclidean space as well as a class of ruled minimal submanifolds with codimension two in spheres.

1. The proof

We first collect some basic facts and definitions about minimal surfaces in space forms and refer to [6] for more details.

Let $f\colon L^2 \to \mathbb{Q}^N$ denote an isometric immersion of a two-dimensional Riemannian manifold into an N-dimensional space form. The *k^{th}-normal space* of f at $x \in L^2$ for $k \ge 1$ is defined as

$$N_k^f(x) = \mathrm{span}\{\alpha_f^{k+1}(X_1,\ldots,X_{k+1}) : X_1,\ldots,X_{k+1} \in T_xL\}$$

where

$$\alpha_f^s\colon TL \times \cdots \times TL \to N_fL,\ \ s \ge 3,$$

denotes the symmetric tensor called the *s^{th}-fundamental form* given inductively by

$$\alpha_f^s(X_1,\ldots,X_s) = \left(\nabla^\perp_{X_s}\ldots,\nabla^\perp_{X_3}\alpha_f(X_2,X_1)\right)^\perp$$

and $\alpha_f\colon TL \times TL \to N_fL$ stands for the standard second fundamental form of f with values in the normal bundle. Here $\nabla^\perp$ denotes the induced connection in the normal bundle N_fL of f and $(\ \ )^\perp$ means taking the projection onto the normal complement of $N_1^f \oplus \ldots \oplus N_{s-2}^f$ in N_fL.

A surface $f\colon L^2 \to \mathbb{Q}^N$ is called *regular* if for each k the subspaces N_k^f have constant dimension and thus form normal subbundles of the normal bundle. Notice that regularity is always verified along connected components of an open dense subset of L^2.

Assume that $f\colon L^2 \to \mathbb{Q}^N$ is a substantial regular minimal surface. Substantial means that the codimension cannot be reduced. Then, the normal bundle of f splits as

$$N_f L = N_1^f \oplus N_2^f \oplus \cdots \oplus N_m^f, \quad m = [(N-1)/2],$$

since all higher normal bundles have rank two except possibly the last one that has rank one if N is odd. The k^{th}-*order curvature ellipse* $\mathcal{E}_k^f(x) \subset N_k^f(x)$ at $x \in L^2$ for each N_k^f, $1 \le k \le m$, is defined by

$$\mathcal{E}_k^f(x) = \{\alpha_f^{k+1}(Z^\varphi, \ldots, Z^\varphi)\colon\ Z^\varphi = \cos\varphi Z + \sin\varphi JZ \text{ and } \varphi \in \mathbb{S}^1\}$$

where $Z \in T_xL$ is any vector of unit length and J is the complex structure in T_xL.

A substantial regular minimal surface $f\colon L^2 \to \mathbb{R}^{2n+1}$ is called *isotropic* if at any point the ellipses of curvature $\mathcal{E}_k^f(x)$, $1 \le k \le n-1$, are circles. From the results in [4] and [7] we have that such a surface can be locally parametrized as $f = \mathrm{Re}(\psi\phi_n)$ where ψ is any nowhere vanishing holomorphic function. The condition that the ellipses of curvature are circles is equivalent to the vector fields $F_1, \partial F_1, \ldots, \partial^{n-1}F_1$ being orthogonal and isotropic. Notice that different functions ψ yield isotropic surfaces with the same Gauss map.

For the proof of Theorem 1 we first give several lemmas.

Lemma 2. *The following facts hold:*

(i) *Outside the zeros $F_1, \ldots, F_n$ span a maximal isotropic subspace of $\mathbb{C}^{2n+1}$.*
(ii) *The vectors $F_1, \ldots, F_{n+1}$ are orthogonal with respect to the Hermitian product.*
(iii) *The vectors $F_{n+1}, \bar{F}_{n+1}$ are collinear.*
(iv) *It holds that*

$$F_{s+1} = \partial F_s - \partial(\log \|F_s\|^2)F_s, \ \ 1 \le s \le n. \tag{2}$$

(v) *It holds that*

$$\partial \bar{F}_s = -\frac{\|F_s\|^2}{\|F_{s-1}\|^2}\bar{F}_{s-1}, \ \ 2 \le s \le n. \tag{3}$$

(vi) *The zeros of $F_1, \ldots, F_{n+1}$ are isolated.*

Proof. (i) This follows from

$$\mathrm{span}_{\mathbb{C}}\{F_1, \ldots, F_n\} = \mathrm{span}_{\mathbb{C}}\{F_1, \partial F_1, \ldots, \partial^{n-1}F_1\},$$

and that n is the dimension of any maximal isotropic subspace in $\mathbb{C}^{2n+1}$.
(ii) We claim that

$$\partial \bar{F}_s \in \mathrm{span}_{\mathbb{C}}\{\bar{F}_1, \ldots, \bar{F}_{s-1}\}, \ \ 2 \le s \le n. \tag{4}$$

In fact, since F_1 is holomorphic we see that

$$\partial \bar{F}_2 = -\partial\left(\langle \bar{\partial}\bar{F}_1, F_1\rangle / \|F_1\|^2\right)\bar{F}_1.$$

Assume that the claim holds for any $2 \le s \le k$. Then, we have

$$\partial \bar{F}_{k+1} = \partial\bar{\partial}\bar{F}_k - \partial\left(\langle \bar{\partial}\bar{F}_k, F_k\rangle / \|F_k\|^2\right)\bar{F}_k - \left(\langle \bar{\partial}\bar{F}_k, F_k\rangle / \|F_k\|^2\right)\partial\bar{F}_k.$$

By assumption, we can write

$$\partial \bar{F}_k = \lambda_1 \bar{F}_1 + \cdots + \lambda_{k-1}\bar{F}_{k-1}.$$

Thus,

$$\partial\bar{\partial}\bar{F}_k = \sum_{j=1}^{k-1} \bar{\partial}(\lambda_j)\bar{F}_j + \sum_{j=1}^{k-1} \lambda_j \bar{\partial}\bar{F}_j$$

and since $\bar{\partial}\bar{F}_j \in \mathrm{span}_{\mathbb{C}}\{\bar{F}_j, \bar{F}_{j+1}\}$, the claim follows.

That $\langle F_2, \bar{F}_1\rangle = 0$ is immediate. Assume that $F_1, \dots, F_s$ are orthogonal with respect to the Hermitian product. For $1 \le j \le s-1$ we obtain using (4) that

$$\begin{aligned}\langle F_{s+1}, \bar{F}_j\rangle &= \langle \partial F_s, \bar{F}_j\rangle - \frac{1}{\|F_s\|^2}\langle \partial F_s, \bar{F}_j\rangle\langle F_s, \bar{F}_j\rangle \\ &= \partial\langle F_s, \bar{F}_j\rangle - \langle F_s, \partial\bar{F}_j\rangle \\ &= 0.\end{aligned}$$

Since also $\langle F_{s+1}, \bar{F}_s\rangle = 0$ is immediate, the result follows.

(iii) It suffices to show that F_{n+1} is orthogonal to $F_j, \bar{F}_j$, $1 \le j \le n$, with respect to the Hermitian inner product. Then the same holds for $\bar{F}_{n+1}$ and the result follows. We have from part *(ii)* that F_{n+1} is orthogonal to F_j, $1 \le j \le n$. The orthogonality with respect to $\bar{F}_j$, $1 \le j \le n$, follows from *(i)*.

(iv) Immediate using (4).

(v) Follows easily using part *(ii)* and (4).

(vi) Since F_1 is holomorphic, we have using (3) that

$$\bar{\partial}\,(F_1 \wedge \cdots \wedge F_{n+1}) = 0,$$

and the result follows. ∎

Lemma 3. *The complexified tangent and normal bundles of* $f = \mathrm{Re}(\phi_n)\colon U \to \mathbb{R}^{2n+1}$ *are given by*

$$\begin{aligned} f_*TU \otimes \mathbb{C} &= \mathrm{span}_{\mathbb{C}}\{F_1, \bar{F}_1\}, \\ N_s^f \otimes \mathbb{C} &= \mathrm{span}_{\mathbb{C}}\{F_{s+1}, \bar{F}_{s+1}\},\ \ 1 \le s \le n-1, \\ N_n^f \otimes \mathbb{C} &= \mathrm{span}_{\mathbb{C}}\{F_{n+1}\}. \end{aligned}$$

Proof. The first equality follows from

$$2\partial f = \partial\phi_n = F_1.$$

We claim that the higher fundamental forms satisfy

$$2\alpha_f^{s+1}(\partial, \dots, \partial) = F_{s+1},\ \ 1 \le s \le n. \tag{5}$$

We proceed by induction. We have that

$$2\alpha_f(\partial,\partial) = 2(\partial\partial f)^{\perp} = (\partial F_1)^{\perp} = \partial F_1 - (\partial F_1)^{f_*TU}.$$

Computing $(\partial F_1)^{f_*TU}$ in the real tangent base $\{F_1 + \bar{F}_1, i(F_1 - \bar{F}_1)\}$ gives

$$(\partial F_1)^{f_*TU} = \frac{1}{\|F_1\|^2}\langle \partial F_1, \bar{F}_1\rangle F_1,$$

and (5) follows for $s = 1$.

Now assume by induction that (5) holds for any $1 \leq k \leq s$. A similar computation as above using parts (i) and (ii) of Lemma 2 gives

$$\begin{aligned} 2\alpha_f^{s+2}(\partial,\ldots,\partial) &= \partial F_{s+1} - (\partial F_{s+1})^{f_*TU} - \sum_{j=1}^{s}(\partial F_{s+1})^{N_j^f} \\ &= \partial F_{s+1} - (\partial F_{s+1})^{N_s^f} \\ &= \partial F_{s+1} - \frac{\langle \partial F_{s+1}, \bar{F}_{s+1}\rangle}{\|F_{s+1}\|^2} F_{s+1} \\ &= F_{s+2}, \end{aligned}$$

and this completes the proof. □

Lemma 4. *The surface $g\colon L^2 \to \mathbb{S}^{2n}$ is minimal and we have:*

$$g_*TL \otimes \mathbb{C} = \operatorname{span}_{\mathbb{C}}\{F_n, \bar{F}_n\}, \tag{6}$$

$$N_s^g \otimes \mathbb{C} = \operatorname{span}_{\mathbb{C}}\{F_{n-s}, \bar{F}_{n-s}\}, \;\; 1 \leq s \leq n-1, \tag{7}$$

and

$$\alpha_g^{s+1}(\partial,\ldots,\partial) = \frac{(-1)^{s+1}}{\|F_{n-s}\|^2}\langle g, F_{n+1}\rangle \bar{F}_{n-s}, \;\; 1 \leq s \leq n-1. \tag{8}$$

Proof. We easily obtain using Lemma 2 that $\partial F_{n+1}, \partial\bar{F}_{n+1}$ are orthogonal to $F_s, \bar{F}_s$, $1 \leq s \leq n-1$. From part (iii) of Lemma 2 we obtain that $\partial F_{n+1}, \partial\bar{F}_{n+1}$ are orthogonal to F_{n+1}, and this gives (6).

Using (6) and Lemma 2 it follows easily that

$$\partial g = -\frac{\langle g, F_{n+1}\rangle}{\|F_n\|^2}\bar{F}_n. \tag{9}$$

Hence,

$$\alpha_g(\partial,\bar{\partial}) = (\partial\bar{\partial}g)^{\perp} = -\frac{\langle g, \bar{F}_{n+1}\rangle}{\|F_n\|^2}(\partial F_n)^{\perp}.$$

However,

$$\begin{aligned} (\partial F_n)^{\perp} &= \partial F_n - \langle \partial F_n, g\rangle g - (\partial F_n)^{g_*TL} \\ &= \partial F_n - \langle F_{n+1}, g\rangle g - \partial(\log\|F_n\|^2)F_n \\ &= F_{n+1} - \langle F_{n+1}, g\rangle g \\ &= 0, \end{aligned}$$

and thus g is minimal.

To prove (7) and (8) we proceed by induction. Using Lemma 2 and (9), we obtain

$$\begin{aligned} \alpha_g(\partial,\partial) &= \partial\partial g - \langle \partial\partial g, g\rangle g - (\partial\partial g)^{g_*TL} \\ &= -\partial\left(\frac{\langle g, F_{n+1}\rangle}{\|F_n\|^2}\right)\bar{F}_n + \frac{\langle g, F_{n+1}\rangle}{\|F_{n-1}\|^2}\bar{F}_{n-1} \\ &\quad +\partial\left(\frac{\langle g, F_{n+1}\rangle}{2\|F_n\|^2}\right)(F_n + \bar{F}_n) - \partial\left(\frac{\langle g, F_{n+1}\rangle}{2\|F_n\|^2}\right)(F_n - \bar{F}_n) \\ &= \frac{1}{\|F_{n-1}\|^2}\langle g, F_{n+1}\rangle \bar{F}_{n-1}. \end{aligned}$$

Now assume that (8) holds for any $1 \leq k \leq s$. Similarly, we have

$$\begin{aligned}
\alpha_g^{s+2}(\partial,\ldots,\partial) &= \partial\alpha_g^{s+1}(\partial,\ldots,\partial) - (\partial\alpha_g^{s+1}(\partial,\ldots,\partial))^{g_*TL} - \sum_{j=1}^{s}(\partial\alpha_g^{s+1}(\partial,\ldots,\partial))^{N_j^g}\\
&= (-1)^{s+1}\partial\left(\frac{\langle g, F_{n+1}\rangle}{\|F_{n-s}\|^2}\right)\bar{F}_{n-s} + \frac{(-1)^s}{\|F_{n-s-1}\|^2}\langle g, F_{n+1}\rangle\bar{F}_{n-s-1}\\
&\quad +(-1)^s\partial\left(\frac{\langle g, F_{n+1}\rangle}{2\|F_{n-s}\|^2}\right)(F_{n-s}+\bar{F}_{n-s})\\
&\quad +(-1)^{s+1}\partial\left(\frac{\langle g, F_{n+1}\rangle}{2\|F_{n-s}\|^2}\right)(F_{n-s}-\bar{F}_{n-s})\\
&= \frac{(-1)^s}{\|F_{n-s-1}\|^2}\langle g, F_{n+1}\rangle\bar{F}_{n-s-1},
\end{aligned}$$

and this completes the proof. $\square$

Proof of Theorem 1. The direct statement follows from the previous lemmas. For the converse, let $g\colon L^2 \to \mathbb{S}^{2n}$ be a simply-connected pseudoholomorphic surface and z a local complex chart. Define $G_0 = g$ and

$$G_{s+1} = \partial G_s - \frac{\langle \partial G_s, \bar{G}_s\rangle}{\|G_s\|^2}G_s, \quad s \geq 0.$$

We have that

$$g_*TL \otimes \mathbb{C} = \mathrm{span}_{\mathbb{C}}\{G_1, \bar{G}_1\}.$$

Clearly, the higher fundamental forms are given by

$$\alpha_g^{s+1}(\partial,\ldots,\partial) = G_{s+1}, \quad 1 \leq s \leq n. \tag{10}$$

Thus $G_1,\ldots,G_n$ span an isotropic subspace of $\mathbb{C}^{2n+1}$. Moreover,

$$N_s^g \otimes \mathbb{C} = \mathrm{span}_{\mathbb{C}}\{G_{s+1}, \bar{G}_{s+1}\}, \quad 1 \leq s \leq n-1. \tag{11}$$

We claim that

$$\partial\bar{G}_s \in \mathrm{span}_{\mathbb{C}}\{\bar{G}_0,\ldots,\bar{G}_{s-1}\}, \quad s \geq 1. \tag{12}$$

Since $\|g\| = 1$ we have $G_1 = \partial g$, and hence $\bar{G}_1 = \bar{\partial} g$. Therefore,

$$\partial\bar{G}_1 = \partial\bar{\partial} g \in \mathrm{span}_{\mathbb{C}}\{g\}.$$

Assume that (12) holds for $1 \leq k \leq s-1$. Then,

$$\partial\bar{G}_{k+1} = \bar{\partial}\partial\bar{G}_k - \partial\left(\frac{\langle \bar{\partial}\bar{G}_k, G_k\rangle}{\|G_k\|^2}\right)\bar{G}_k - \frac{\langle \bar{\partial}\bar{G}_k, G_k\rangle}{\|G_k\|^2}\partial\bar{G}_k,$$

and the claim follows.

Using (11) we see that $G_1,\ldots,G_n$ are orthogonal with respect to the Hermitian product and thus span a maximal isotropic subspace of $\mathbb{C}^{2n+1}$. Hence (12) gives

$$\partial\bar{G}_s = -\frac{\|G_s\|^2}{\|G_{s-1}\|^2}\bar{G}_{s-1}, \quad s \geq 1. \tag{13}$$

We have that

$$\mathbb{C}^{2n+1} = \mathrm{span}_{\mathbb{C}}\{g, G_1,\ldots,G_n,\bar{G}_1,\ldots,\bar{G}_n\} \tag{14}$$

where all vectors are orthogonal with respect to the Hermitian product. It is easy to check that G_{n+1} is orthogonal to all vectors in (14) and thus $G_{n+1} = 0$ or, equivalently, it holds that

$$\partial G_n = \partial \log \|G_n\|^2 G_n,$$

and therefore $\xi = \bar{G}_n/\|G_n\|^2$ is holomorphic.

Consider the map $f\colon L^2 \to \mathbb{R}^{2n+1}$ given by

$$f = \operatorname{Re} \int \xi dz. \tag{15}$$

Then,

$$f_* TL \otimes \mathbb{C} = \operatorname{span}_{\mathbb{C}}\{G_n, \bar{G}_n\}.$$

We have that f is minimal since

$$\alpha_f(\partial, \bar{\partial}) = (\bar{\partial}\partial f)^{\perp} = 0.$$

From (13) we see that

$$\partial^s \bar{G}_n \in \operatorname{span}_{\mathbb{C}}\{\bar{G}_{n-s}, \ldots, \bar{G}_{n-1}\}, \;\; 1 \le s \le n-1.$$

Therefore, the subspace

$$\operatorname{span}_{\mathbb{C}}\{\partial \bar{G}_n, \ldots, \partial^{n-1}\bar{G}_n\}$$

is isotropic, and hence f is an isotropic surface.

We have that

$$F_1 = \partial f = \bar{G}_n/\|G_n\|^2.$$

Using (13) we obtain that F_s, $1 \le s \le n$, defined by (1) satisfies

$$F_s \in \operatorname{span}_{\mathbb{C}}\{\bar{G}_n, \ldots, \bar{G}_{n-s+1}\}.$$

Thus,

$$\operatorname{span}_{\mathbb{C}}\{F_1, \ldots, F_n, \bar{F}_1, \ldots, \bar{F}_n\} = \operatorname{span}_{\mathbb{C}}\{G_1, \ldots, G_n, \bar{G}_1, \ldots, \bar{G}_n\}.$$

Hence,

$$g = \frac{1}{\|\operatorname{Re}(F_{n+1})\|}\operatorname{Re}(F_{n+1}), \tag{16}$$

and this completes the proof. □

Remarks 5. (*i*) Observe that in (15) we can replace ξ by $\psi\xi$ where ψ is any nowhere vanishing holomorphic function. Hence, different holomorphic data may generate the same pseudoholomorphic spherical surface.
(*ii*) That the first two definitions of pseudoholomorphicity given in the introduction are equivalent follows using (10).
(*iii*) The proof of the converse of the theorem can also be obtained from [6] by means of the quite different techniques developed there for the more general context of elliptic surfaces which, in particular, do not yield (16).

2. Applications

In this section, we first provide a local Weierstrass type representation for the real Kaehler Euclidean hypersurfaces free of flat points.

The following result was obtained in [5].

Theorem 6. *Let $g\colon L^2 \to \mathbb{S}^{2n}$, $n \geq 2$, be a pseudoholomorphic surface and $\gamma \in C^\infty(L)$ an arbitrary function. Then the induced metric on the open subset of regular points of the map $\Psi\colon N_gL \to \mathbb{R}^{2n+1}$ given by*

$$\Psi(x,w) = \gamma(x)g(x) + g_*\nabla\gamma(x) + w$$

is Kaehler. Conversely, any real Kaehler hypersurface $f\colon M^{2n} \to \mathbb{R}^{2n+1}, n \geq 2$, free of flat points can be locally parametrized in this way.

We have that

$$\nabla\gamma = \frac{1}{\|\partial\|^2}(\gamma_{\bar z}\partial + \gamma_z\bar\partial),$$

$$g = \frac{1}{\|\mathrm{Re}(F_{n+1})\|}\mathrm{Re}(F_{n+1}) \quad \text{and} \quad \partial g = -\frac{\langle g, F_{n+1}\rangle}{\|F_n\|^2}\bar F_n.$$

Using (7) we see that any $w \in N_gL$ can be written as $w = \sum_{j=1}^{n-1}\mathrm{Re}(w_jF_j)$ where the $w_j = u_j + iv_j$, $1 \leq j \leq n-1$, are complex parameters.

Theorem 7. *Any real Kaehler hypersurface $f\colon M^{2n} \to \mathbb{R}^{2n+1}, n \geq 2$, without flat points can be locally parametrized as*

$$\begin{aligned}\Psi(z,w_j) \;=\; & \frac{\gamma}{\|\mathrm{Re}(F_{n+1})\|}\mathrm{Re}(F_{n+1}) \\ & -\frac{2}{\|\partial\|^2\|F_n\|^2\|\mathrm{Re}(F_{n+1})\|}\mathrm{Re}\left(\gamma_z\langle \mathrm{Re}(F_{n+1}), \bar F_{n+1}\rangle F_n\right) \\ & +\sum_{j=1}^{n-1}(u_j\mathrm{Re}(F_j) - v_j\mathrm{Im}(F_j)).\end{aligned}$$

Proof. A straightforward computation using (9). □

Next, we describe how to parametrize the minimal ruled submanifolds in spheres with codimension two given in [8] but only when associated to a pseudoholomorphic surface.

Let $g\colon L^2 \to \mathbb{S}^{2n}$, $n \geq 3$, be a pseudoholomorphic surface. We constructed in [8] an associated ruled minimal submanifold $F_g\colon M^{2n-2} \to \mathbb{S}^{2n}$ by attaching at each point of g the totally geodesic $(2n-4)$-sphere of $\mathbb{S}^{2n}$ whose tangent space at that point is the fiber of the vector bundle $\Lambda_g = (N_1^g)^\perp$, that is,

$$(p,w) \in \Lambda_g \mapsto F_g(p,w) = \exp_{g(p)} w$$

(outside singular points) where exp is the exponential map of $\mathbb{S}^{2n}$.

As above, any $w \in \Lambda_g$ can be written as $w = \sum_{j=1}^{n-2}\mathrm{Re}(w_jF_j)$ where $w_j = u_j + iv_j$, $1 \leq j \leq n-2$. Then, the submanifold $F_g\colon M^{2n-2} \to \mathbb{S}^{2n}$ in [8] can be parametrized as

$$F_g(z, w_j) = \frac{1}{\|\mathrm{Re}(F_{n+1})\|} \cos \| \sum_{j=1}^{n-2} (u_j \mathrm{Re}(F_j) - v_j \mathrm{Im}(F_j)) \| \mathrm{Re}(F_{n+1})$$
$$+ h(\| \sum_{j=1}^{n-2} (u_j \mathrm{Re}(F_j) - v_j \mathrm{Im}(F_j)) \|) \sum_{j=1}^{n-2} (u_j \mathrm{Re}(F_j) - v_j \mathrm{Im}(F_j))$$

where $h(x) = \frac{1}{x} \sin x$ if $x \neq 0$ and $h(0) = 1$.

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