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# COMPLETELY BOUNDED $\Lambda_p$ SETS THAT ARE NOT SIDON

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ABSTRACT. In this paper we construct examples of completely bounded $\Lambda_p$ sets, which are not Sidon, on any compact abelian group. As a consequence, we have a new proof of the classical result for the existence of non-Sidon, $\Lambda_p$ sets on any compact abelian group.

## 1. INTRODUCTION

Let $G$ be a compact abelian group and $\Gamma$ its discrete abelian dual group. Lacunary sets on $\Gamma$ play important roles in the study of Fourier analysis on compact abelian groups. J.-P. Kahane [6] first used the term Sidon set and W. Rudin in [9] introduced the concept of $\Lambda_p$ sets. Sidon sets are particular examples of $\Lambda_p$ sets. Among other roles, one of the important features of $\Lambda_p$ sets, for $p > 2$, is that they are the interpolation sets for $L^p(G)$ multipliers.

In the last decade the study of operator spaces and completely bounded multipliers has attracted much attention. G. Pisier assigned a canonical operator space structure on $L^p$ spaces by developing complex interpolation for operator spaces. With this operator space structure on $L^p(G)$ one can define, in the obvious way, the space of all completely bounded multipliers on $L^p(G)$, and then define the notion of completely bounded $\Lambda_p$ sets for $p > 2$ (also known as non-commutative $\Lambda_p$ sets and denoted $\Lambda_p^{cb}$), as the interpolation sets of completely bounded multipliers on $L^p(G)$.

In [5], A. Harcharras extensively studied various properties of $\Lambda_p^{cb}$ sets and established, in particular, the existence of $\Lambda_p$ sets which are not $\Lambda_p^{cb}$. Given that all Sidon sets are $\Lambda_p^{cb}$ for all $p > 2$, it is natural to ask, "Is there a non-Sidon, $\Lambda_p^{cb}$ set"? The analogous question for $\Lambda_p$ sets historically attracted much attention following Rudin's seminal paper, and was finally solved by Edwards, Hewitt and Ross [4] and A. Bonami [3] simultaneously.

Harcharras answered this question for the circle group $\mathbb{T}$ in [1] by showing that for any finite set $Q$ of prime numbers, the set of natural numbers whose prime divisors all lie in $Q$ is $\Lambda_p^{cb}$ for all $p$, but not Sidon. In this paper, we will show that on any compact abelian group there is a set which is not Sidon, but is $\Lambda_p^{cb}$ for all $p$. Our argument is mainly combinatorial and provides a new proof for the classical result of Edwards et al. and Bonami.

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## 2. Preliminaries

2.1. $\Lambda_p$ **sets and** $L^p$ **multipliers.** Throughout this paper we let $G$ be a compact abelian group equipped with normalized Haar measure and $\Gamma$ its dual group. Let $E \subseteq \Gamma$. We call the trigonometric polynomial $f$ an *$E$-polynomial* if the Fourier coefficients of $f$ are supported on $E$.

**Definition 2.1.**

(i) A set $E \subseteq \Gamma$ is called a *Sidon set* if there is a constant $C$ such that $\sum_{\gamma \in E} |\hat{f}(\gamma)| \leq C\|f\|_\infty$ for all $E$-polynomials $f$.

(ii) Let $2 < p < \infty$. A set $E \subseteq \Gamma$ is called a $\Lambda_p$ *set* if there is a constant $C_p$ such that $\|f\|_p \leq C_p\|f\|_2$ for all $E$-polynomials $f$.

The fact that $L^p(G) \subseteq L^q(G)$ if $p \geq q$ ensures that any $\Lambda_p$ set is also a $\Lambda_q$ set for all $q \leq p$. It is known that every Sidon set is a $\Lambda_p$ set for all $p$ [9]; however the converse is not true. Indeed, in every group $\Gamma$ there are examples known of sets that are $\Lambda_p$ for all $p$, but not Sidon [3, 4]. Reference [7] is an excellent survey of basic properties of Sidon and $\Lambda(p)$ sets.

Sidon and $\Lambda_p$ sets play an important role in the study of Fourier multipliers.

**Definition 2.2.** Let $1 \leq p \leq \infty$ The function $\phi \in l^\infty(\Gamma)$ is called a Fourier multiplier for $L^p(G)$, or $L^p$ *multiplier* for short, if the operator $T : L^p(G) \to L^p(G)$ defined by

$$\widehat{Tf}(\gamma) = \phi(\gamma)\hat{f}(\gamma) \ \ \forall f \in L^p \cap L^2(G)$$

is bounded.

We denote the space of $L^p$ multipliers by $M_p(G)$. It is well known that $M_1(G) \simeq M(G)$, the set of finite regular Borel measures on $G$, $M_2(G) \simeq l^\infty(\Gamma)$ and $M_p(G) \simeq M_{p'}(G)$ where $1/p + 1/p' = 1$ (isometrically isomorphic in all cases). A duality argument proves that $E$ is Sidon if and only if whenever $\phi \in l^\infty(E)$, then there exists $\mu \in M(G)$ such that $\hat{\mu}(\gamma) = \phi(\gamma)$ for all $\gamma \in E$. Thus Sidon sets are interpolation sets for $M(G)$. Similarly, it is known that $\Lambda_p$ sets are interpolation sets for $M_p(G)$ when $p > 2$ (see [5] for a proof).

2.2. **Completely bounded multipliers.** We now briefly recall the natural operator space structure on $L^p(X)$ where $X$ is a $\sigma$-finite measure space. For details see [8, Chapter 2]. We consider the canonical $C^*$-algebra, operator space structure on $L^\infty(X)$. The operator space structure on $L^1(X)$ is inherited from the dual of $L^\infty(X)$. With this operator space structure the identification $L^1(X)^* \simeq L^\infty(X)$ is completely isometric [2]. The couple $(L^\infty(X), L^1(X))$ is compatible for operator space interpolation and we consider $L^p(X) = (L^\infty(X), L^1(X))_{1/p}$ as an operator space with this interpolating operator space structure.

If an $L^p$ multiplier $T$ is completely bounded in $L^p$ with this operator space structure, then $T$ is called a *completely bounded*, or *cb-multiplier* on $L^p(G)$. We denote the space of all *cb*-multipliers on $L^p(G)$ by $M_p^{cb}(G)$.

The following result of Pisier [8] provides a characterization of the completely bounded maps on $L^p(G)$. For $1 \leq p < \infty$ let $S_p$ be the space of compact operators on $l_2(\mathbb{Z})$ such that $\|T\|_{S_p} = (tr(T^*T)^{p/2})^{1/p} < \infty$, i.e., $S_p$ is the space of Schattan $p$-class operators. Let us denote by $L^p(G, S_p)$ the space of $S_p$-valued, measurable functions such that

$$\|f\|_{L^p(G,S_p)} = \left(\int_G \|f(x)\|_{S_p}^p dx\right)^{1/p} < \infty.$$

Clearly $L^p(G, S_p)$ is the closure of $L^p(G) \otimes S_p$ in the above-mentioned norm, $\|.\|_{L^p(G,S_p)}$.

**Proposition 2.3** ([8])**.** *Let $1 \le p < \infty$. A linear map $T : L^p(G) \to L^p(G)$ is completely bounded if and only if the mapping $T \otimes I_{S_p}$ is bounded on $L^p(G, S_p)$. Moreover,*

$$\|T\|_{cb} = \|T \otimes I_{S_p}\|_{L^p(G,S_p)\to L^p(G,S_p)}$$

*where $\|\cdot\|_{cb}$ denotes the cb-norm of the operator $T : L^p(G) \to L^p(G)$.*

The notion of completely bounded (or cb) $\Lambda_p$ sets, denoted $\Lambda_p^{cb}$, was introduced in [5] as follows.

**Definition 2.4.** Let $2 < p < \infty$. A subset $E \subseteq \Gamma$ is called a $\Lambda_p^{cb}$ *set* if there exists a constant $C$, depending only on $p$ and $E$, such that

$$\|f\|_{L^p(G,S_p)} \le C \max\left\{ \left\| \sum_{\gamma\in E} \left(\hat{f}(\gamma)^* \hat{f}(\gamma)\right)^{1/2} \right\|_{S_p}, \left\| \sum_{\gamma\in E} \left(\hat{f}(\gamma) \hat{f}(\gamma)^*\right)^{1/2} \right\|_{S_p} \right\}$$

for all $S_p$-valued, $E$-polynomials $f$ defined on $G$. We denote by $\lambda_p^{cb}(E)$ the least constant $C$ for which the above inequality holds.

Of course, any $\Lambda_p^{cb}$ set is $\Lambda_p$. It is a consequence of the non-commutative Khintchine's inequality that the $\Lambda_p^{cb}$ sets are the interpolation sets for $M_p^{cb}$ when $2 < p < \infty$ [5]. The inclusion $M_q^{cb} \subseteq M_p^{cb}$ when $2 < q < p < \infty$ shows that if $E$ is $\Lambda_q^{cb}$, then it is also $\Lambda_p^{cb}$.

A useful combinatorial property for $\Lambda_p^{cb}$ is also given in [5]. Before we state this property let us recall Rudin's [9] combinatorial property that allows one to construct $\Lambda_p$ sets.

*Notation* 2.5. For $p \ge 2$ an integer, let

$$A_p(E) = \sup_{\gamma\in\Gamma} |\{(\gamma_1, \dots, \gamma_p) \in E^p : \ \gamma_1\gamma_2 \dots \gamma_p = \gamma\}| \ \text{ and}$$

$$B_p(E) = \sup_{\gamma\in\Gamma} \left|\{(\gamma_1, \dots, \gamma_p) \in E^p : \ \gamma_1^{-1}\gamma_2 \dots \gamma_p^{(-1)^p} = \gamma\}\right|.$$

Rudin proved that if $A_p(E) < \infty$, then $E$ is a $\Lambda_{2p}$ set. In his proof, if the identity $|f|^{2p} = f^p\overline{f}^p$ is replaced by $|f|^{2p} = g\overline{g}$, where $g = (f\overline{f})^{p/2}$ when $p$ is even or $g = f\overline{f} \cdots f$ ($p$ factors) when $p$ is odd, then one can easily see that $E$ is also $\Lambda_{2p}$ if $B_p(E) < \infty$.

In the above notation, characters $\gamma_j$ may be repeated and this can cause complications with the counting, particularly when the characters have finite order. Motivated by [10], Harcharras in [5] was able to extend this result to $\Lambda_p^{cb}$ sets, under the weaker assumption that the characters $\gamma_j$ were distinct.

**Definition 2.6.** Let $p \ge 2$ be an integer. The set $E$ has the $Z(p)$ property if

$$Z_p(E) = \sup_{\gamma\in\Gamma} \left|\{(\gamma_1, \dots, \gamma_p) \in E^p : \forall i \ne j, \gamma_i \ne \gamma_j, \text{ and } \gamma_1^{-1}\gamma_2 \dots \gamma_p^{(-1)^p} = \gamma\}\right| < \infty.$$

**Theorem 2.7** ([5]). *Let $p \geq 2$ be an integer. Then every subset $E$ of $\Gamma$ with the $Z(p)$ property is a $\Lambda_{2p}^{cb}$ set. Moreover, there exists a constant $C_p$, depending only on $p$, such that $\lambda_{2p}^{cb}(E) \leq C_p Z_p(E)^{1/2p}$ for each $E \subseteq \Gamma$.*

In section 3 we will first prove the existence of a non-Sidon set that is $\Lambda_4^{cb}$ by constructing sets that have property $Z(2)$. The combinatorial arguments are less complicated in this case than for general $2p$, which we take up in section 4. Since $\Lambda_p^{cb}$ sets are always $\Lambda_p$, section 4 provides an alternate proof of the classical result of the existence of non-Sidon sets that are $\Lambda_p$ for all $p < \infty$.

## 3. A non-Sidon, $\Lambda_4^{cb}$ set

Our strategy will be to find size limitations on the arithmetic structures that Sidon sets can contain, and then to construct $\Lambda_p^{cb}$ sets, using Harcharras' sufficient condition, which violate this size limitation. Towards this end we have the following lemma which can also be deduced from the result on 'test families' in [4]. Because of the simplicity of the argument for the special case we need, we have included a proof here.

**Lemma 3.1.** *If $E \subseteq \prod_{j=1}^{\infty} \mathbb{Z}_p^{(j)}$ is a Sidon set, then $\left| E \cap \prod_{j=1}^{N} \mathbb{Z}_p^{(j)} \right| \leq O(N)$, where $\mathbb{Z}_p^{(j)} = \mathbb{Z}_p$ for each $j$.*

*Proof.* Let $\chi_j$ be a generator for $\mathbb{Z}_p^{(j)}$and consider $f_j = 1 + \chi_j + \chi_j^2 + \cdots + \chi_j^{p-1}$. Since $\chi_j(x)$ is a $p$-th root of unity, $f_j(x) = 0$ if $x \neq e$ and thus for any $q' < \infty$ we have $\|f_j\|_{q'}^{q'} = p^{q'-1}$. Hence, if $P_N = \prod_{j=1}^{N} f_j$, then $\|P_N\|_{q'} = p^{\frac{N}{q}}$. Now $\widehat{P_N}(\psi) = 1$ if $\psi \in \prod_{j=1}^{N} \mathbb{Z}_p^{(j)}$ and 0 elsewhere, so $\|\widehat{P_N}|_E\|_2 = |E \cap \prod_{j=1}^{N} \mathbb{Z}_p^{(j)}|^{1/2}$. It is well known (see [7]) that if $E$ is Sidon, then for all $q > 2$ we have $\|\widehat{P_N}|_E\|_2 \leq C\sqrt{q}\|P_N\|_{q'}$. Thus $|E \cap \prod_{j=1}^{N} \mathbb{Z}_p^{(j)}| \leq C^2 q p^{\frac{2N}{q}}$. Take $q = 2N$ to complete the proof. □

It is also well known that Sidon sets can only contain small proportions of any arithmetic progression.

**Lemma 3.2** ([9]). *Let $E \subset \Gamma$ be a Sidon set. If $A$ is any arithmetic progression, then $|E \cap A| \leq O(\log |A|)$.*

The key technical idea is the following elementary combinatorial result that enables us to construct sets with the $Z(2)$ property, which contain more than $O(N)$ elements from predetermined subsets of $2^N$ characters.

**Lemma 3.3.** *Let $E = \{\chi_j\}_{j=1}^{\infty} \subseteq \Gamma$, where $\chi_j^2$ are all distinct and non-trivial. We can choose an infinite subset $E' \subseteq E$ such that $Z_2(E') = 1$ and for sufficiently large $n$,*

$$\left| E' \cap \{\chi_j\}_{2^{n+1}}^{2^{n+2}} \right| \geq n^2.$$

*Proof.* Choose $N$ such that $2(n+1)^9 < 2^n$ for all $n \geq N$. Put $\psi_1 = \chi_{2^N}$. We proceed inductively and assume that we have chosen $\psi_1, \ldots, \psi_{N^3} \in \{\chi_j\}_{j=2^N}^{2^{N+1}}$ and $\psi_{k^3+1}, \ldots, \psi_{(k+1)^3} \in \{\chi_j\}_{j=2^{k+1}+1}^{2^{k+2}}$ for each $N < k < n$ so that $Z_2(\{\psi_j\}_{j=1}^{n^3}) = 1$. We will see how to choose distinct characters $\psi_{n^3+1}, \ldots, \psi_{(n+1)^3} \in \{\chi_j\}_{j=2^{n+1}+1}^{2^{n+2}}$, with $Z_2(\{\psi_j\}_{j=1}^{(n+1)^3}) = 1$.

First, observe that in order to ensure that $Z_2(\{\psi_j\}_{j=1}^{n^3+1}) = 1$, it will be enough to choose $\psi_{n^3+1}$ so that $\psi_{n^3+1}\psi_i^{-1} \neq \psi_j\psi_k^{-1}$ whenever $i,j,k \leq n^3+1$ and $i,j \neq n^3+1$. Thus we will want $\psi_{n^3+1} \neq \psi_j\psi_k^{-1}\psi_i$ for any $i,j,k \leq n^3$ (which also ensures $\psi_{n^3+1} \notin \{\psi_j\}_{j=1}^{n^3}$) and $\psi_{n^3+1}^2 \neq \psi_j\psi_i$ for any $i,j \leq n^3$. (This is sometimes known as the 'greedy' method.) There are $n^9$ and $n^6$ terms, respectively, that $\psi_{n^3+1}$ and $\psi_{n^3+1}^2$ must avoid. As all squares are assumed to be distinct and $2^n > 2n^9$, we can certainly find such a choice in the set $\{\chi_j\}_{j=2^{n+1}+1}^{2^{n+2}}$. Having made such a choice, we now consider the selection of $\psi_{n^3+2}$. Similarly to the arguments for $\psi_{n^3+1}$, the characters $\psi_{n^3+2}$ and $\psi_{n^3+2}^2$ must avoid $(n^3+1)^3$ and $(n^3+1)^2$ terms, respectively, and this is again possible when choosing from the set $\{\chi_j\}_{j=2^{n+1}+1}^{2^{n+2}}$. We repeat this process until $\psi_{(n+1)^3} \in \{\chi_j\}_{j=2^{n+1}+1}^{2^{n+2}}$ has been chosen, which can be done because $2(n+1)^9 < 2^n$.

By construction, the set $E' = \{\psi_j\}$ satisfies $Z_2(E') = 1$ and for all $n \geq N$,

$$\left|E' \cap \{\chi_j\}_{j=2^{n+1}+1}^{2^{n+2}}\right| = \left|\{\psi_j\}_{j=n^3+1}^{(n+1)^3}\right| \geq n^2.$$

□

Observe that if the set $E$ consists only of characters of order 2, then to have $Z_2(E') = 1$ we simply require $\psi_l\psi_i \neq \psi_j\psi_k$ except in the trivial cases $\{i,l\} = \{j,k\}$. Thus similar arguments give the following.

**Lemma 3.4.** *Let $E = \{\chi_j\}_{j=1}^{\infty} \subseteq \Gamma$, where $\chi_j$ are order 2. We can choose an infinite subset $E' \subseteq E$ such that $Z_2(E') = 1$ and for each sufficiently large $n$,*

$$\left|E' \cap \{\chi_j\}_{2^{n+1}}^{2^{n+2}}\right| \geq n^2.$$

Now we state our main result of this section.

**Theorem 3.5.** *Let $G$ be any infinite compact abelian group. Then $\Gamma$ contains a $\Lambda_4^{cb}$ set that is not a Sidon set.*

*Proof.* One of the following three possibilities will occur in $\Gamma$:

(1) $\Gamma$ contains an element $\chi$ of infinite order.
(2) For every integer $N$ there is a character $\chi \in \Gamma$ with order greater than $N$.
(3) There is an integer $N$ such that every element of $\Gamma$ has order less than $N$.

*Case* 1. Let $\chi$ be an element of $\Gamma$ of infinite order and consider $\chi_j = \chi^j$. As $\chi$ has infinite order all $\chi_j^2$ are distinct and non-trivial. Applying Lemma 3.3 we obtain $E' \subseteq \{\chi^j\}$ with $Z_2(E') = 1$ and $\left|E' \cap \{\chi_j\}_{2^{n+1}}^{2^{n+2}}\right| \geq n^2$. By Theorem 2.7, $E'$ is a $\Lambda_4^{cb}$ set, but it is not Sidon by Lemma 3.2.

*Case* 2. For each $j$, choose $\chi_j \in \Gamma$ with order $N_j > \max(2N_{j-1}, 3 \cdot 2^j)$ and consider

$$E = \bigcup_{j=1}^{\infty} E_j \text{ where } E_j = \{\chi_j, \dots, \chi_j^{2^j}\}.$$

If $j > k$ and $\chi_j^2 = \chi_k^2$, then $\chi_j$ would have order dividing $2N_k < N_j$, which is a contradiction. If, instead, $\chi_j^{2l} = \chi_j^{2m}$ where $l, m \le 2^j$, then the order of $\chi_j$ would be at most $2^{j+1}$, again a contradiction. Thus we can apply Lemma 3.3, as in Case 1, to obtain a subset $E'$ which is $\Lambda_4^{cb}$ and with $|E_j \cap E'| \ge j^2$ for large enough $j$. Since the sets $E_j$ are arithmetic progressions of length $2^j$ we again conclude that $E'$ is not Sidon.

*Case* 3. Suppose $\Gamma$ is of bounded order $K$. Being an infinite abelian group, $\Gamma = \bigoplus_{j \in J} \mathbb{Z}_{n_j}$ where $n_j \le K$ and $|J| = \infty$. There must be some integer $n$ with $n = n_j$ for infinitely many indices $j$ and therefore $\Gamma$ contains an infinite subgroup $\bigoplus_{j=1}^{\infty} \mathbb{Z}_p^{(j)}$ for some prime $p$ dividing $n$.

Consider

$$E_n = \prod_{j=n^2+1}^{(n+1)^2} \mathbb{Z}_p^{(j)} \text{ and let } E = \bigcup_{n=1}^{\infty} E_n.$$

The sets $E_n$ are disjoint and have cardinality at least $p^n$. Yet another application of either Lemma 3.3 (if $p \ne 2$) or Lemma 3.4 (if $p = 2$) shows we can choose an infinite subset $E'$ of $E$ which is $\Lambda_4^{cb}$ and such that $|E' \cap E_n| \ge n^2$. By Lemma 3.1, $E'$ is not Sidon. $\square$

## 4. A non-Sidon set that is $\Lambda_p^{cb}$ for all $2 < p < \infty$

In this section, we will construct a non-Sidon set, $E$, with the property that for each integer $s \ge 2$, a cofinite subset of $E$ has the $Z(s)$ property. Consequently, $E$ will be $\Lambda_{2s}^{cb}$ for all integers $s \ge 2$ and since any $\Lambda_{2s}^{cb}$ is $\Lambda_p^{cb}$ for all $p \le 2s$, $E$ will be $\Lambda_p^{cb}$ for all $p < \infty$.

In fact, we will construct sets, $E_s$, with the property that if $\{\chi_j, \beta_j\}_{j=1}^{s} \subseteq E_s$ with $\chi_j \ne \chi_k$ and $\beta_j \ne \beta_k$ for all $j \ne k$, then

$$\chi_1^{-1}\chi_2\chi_3^{-1} \dots \chi_s^{(-1)^s} = \beta_1^{-1}\beta_2\beta_3^{-1} \dots \beta_s^{(-1)^s}$$

if and only if (when $s$ is odd)

$$\{\chi_1, \chi_3, \dots, \chi_s\} = \{\beta_1, \beta_3, \dots, \beta_s\} \text{ and}$$
$$\{\chi_2, \chi_4, \dots, \chi_{s-1}\} = \{\beta_2, \beta_4, \dots, \beta_{s-1}\}$$

(and a similar statement for $s$ even). In this case, we will say that $E_s$ has *alternating s-products distinct up to permutation* and write $Z_s^{\sigma}(E_s) = 1$, for short. Having $Z_s^{\sigma}(E_s) = 1$ is clearly more than enough to ensure $E_s$ has property $Z(s)$. Notice that $Z_2^{\sigma}(E) = 1$ if and only if $Z_2(E) = 1$.

To begin, we will first explain how to modify the construction of Lemma 3.3 to produce a set that, for a given $s$, has alternating $t$-products distinct up to permutation for all integers $t \le s$.

We will assume $\{\chi_j\} \subseteq \Gamma$ is given with all $\chi_j^2 \ne 1$ (the case when all $\chi_j^2 = 1$ is similar and will be briefly discussed later). Put $\psi_1 = \chi_1$ and inductively assume

$Z_t^\sigma(\{\psi_k\}_{k=1}^n) = 1$ for all $t \le s$, where $\{\psi_1, \ldots, \psi_n\}$ is a (given) subset of $\{\chi_j\}$. Choose, if possible, $\psi_{n+1} \in \{\chi_j\}$ such that

$$\psi_{n+1} \neq \prod_{j=1}^{2t-1} \psi_{i_j}^{\epsilon_j} \quad \text{and} \quad \psi_{n+1}^2 \neq \prod_{j=1}^{2t-1} \psi_{i_j}^{\epsilon_j},$$

whenever $\psi_{i_j} \in \{\psi_k\}_{k=1}^n$, $\varepsilon_j = \pm 1$ and $t \le s$. We will verify that $Z_t^\sigma(\{\psi_k\}_{k=1}^{n+1}) = 1$ for all $t \le s$. Towards that end suppose that for some $t \le s$,

$$\psi_{n+1}^{-1}\alpha_2\alpha_3^{-1}\ldots\alpha_t^{(-1)^t} = \beta_1^{-1}\beta_2\beta_3^{-1}\ldots\beta_t^{(-1)^t} \tag{4.1}$$

where $\alpha_j, \beta_j \in \{\psi_k\}_{k=1}^{n+1}$, $\alpha_j \neq \alpha_k$ or $\psi_{n+1}$, and $\beta_j \neq \beta_k$ when $j \neq k$. There are three possibilities to consider: (1) $\psi_{n+1} \neq \beta_j$ for any $j$, (2) $\psi_{n+1} = \beta_j$ for one index $j$ and that index is even, or (3) $\psi_{n+1} = \beta_j$ for one index $j$ and that index is odd.

In case (1), we have $\psi_{n+1} = \prod_{j=1}^{2t-1} \psi_{i_j}^{\epsilon_j}$ for some choice of $\psi_{i_j} \in \{\psi_j\}_{j=1}^n$ and $\varepsilon_j = \pm 1$; but that is not possible by construction. In case (2), we have $\psi_{n+1}^2 = \prod_{j=1}^{2t-2} \psi_{i_j}^{\epsilon_j}$ where $\psi_{i_j} \in \{\psi_j\}_{j=1}^n$, $\varepsilon_j = \pm 1$; again impossible. In case (3), we can assume without loss of generality that $\psi_{n+1} = \beta_1$ and rewrite identity (4.1) as

$$\alpha_2\alpha_3^{-1}\ldots\alpha_t^{(-1)^t} = \beta_2\beta_3^{-1}\ldots\beta_t^{(-1)^t}$$

where $\alpha_j, \beta_k \in \{\psi_k\}_{k=1}^n$. By the inductive assumption this can only occur if the even and odd labelled characters are permutations.

These observations show that this choice for $\psi_{n+1}$ ensures that $\{\psi_k\}_{k=1}^{n+1}$ has alternating $t$-products distinct up to permutation.

Notice that there are at most $(2n)^{2t-1}$ terms for $\psi_{n+1}$ to avoid for each $t \le s$ and $(2n)^{2t-2}$ terms for $\psi_n^2$ to avoid. If all $\chi_j^2$ are distinct, then in any subset of $\{\chi_j\}$ of cardinality at least $\left[(2n)^{4s}\right]^s$ we will be able to find such a choice for $\psi_{n+1}$.

By repeatedly applying this argument it follows that in any subset of $\{\chi_j\}$ of cardinality at least $2^N$, it is possible to choose a subset $E$ of cardinality $N^2$ with $Z_t^\sigma(E) = 1$ for all $t \le s$, provided $2^N > \left[(2N^2)^{4s}\right]^s$.

To summarize:

**Lemma 4.1.** *Let $\{\chi_j\}_{j=1}^\infty \subseteq \Gamma$, where $\chi_j^2 \neq 1$ are all distinct. For any integers $s$, $n$, with $s \ge 2$ and $2^n > \left[(2n^2)^{4s}\right]^s$, we can choose $E_s \subseteq \{\chi_j\}_{j=2^n}^{2^{n+1}}$ with $Z_t^\sigma(E_s) = 1$ for all $t \le s$ and $|E_s| = n^2$.*

If all the characters $\chi_j$ are order two, we need only pick $\psi_{n+1} \neq \prod_{j=1}^{2t-1} \psi_{i_j}$ in the construction above to ensure that $Z_t^\sigma(\{\psi_k\}_{k=1}^{n+1}) = 1$. The details are left to the reader.

We will use this construction to prove the main result of the paper.

**Theorem 4.2.** *Let $G$ be any infinite, compact abelian group. Then $\Gamma$ contains a non-Sidon subset that is $\Lambda_p^{cb}$ for all $2 < p < \infty$.*

*Proof.* To prove this, it will be enough to construct a set that is $\Lambda_{2s}^{cb}$ for all integers $s \ge 2$. As in Theorem 3.5 we consider different cases.

First, suppose $\Gamma$ contains an element $\chi$ of infinite order. The set we construct will be a subset of $\{\chi^j\}_{j=1}^{\infty}$ . Using Lemma 3.3, for suitably large $n_2$ choose $E_2 \subseteq \{\chi^j\}_{2^{n_2}+1}^{2^{n_2+1}}$, with $|E_2| = (n_2)^2$ and all alternating 2-products distinct. We then proceed inductively and assume we have chosen an increasing sequence of integers $n_2 < n_3 < \cdots < n_{J-1}$ and subsets $E_j \subseteq \{\chi_k\}_{k=2^{n_j}+1}^{2^{n_j+1}}$, with $|E_j| = n_j^2$ for each $2 \le j < J$, and having $Z_t^\sigma\left(\bigcup_{j=k}^{J-1} E_j\right) = 1$ for all $t \le k$ and each $2 \le k < J$.

The subset $E_J = \{\psi_1^{(J)}, \dots, \psi_{n_J^2}^{(J)}\}$ will be constructed as follows: If $\psi_1^{(J)}, \dots, \psi_{i-1}^{(J)}$ have already been (appropriately) chosen, then we will choose $\psi_i^{(J)}$ so that for each $2 \le q \le J$ and any $t \le q$,

$$\psi_i^{(J)} \neq \prod_{j=1}^{2t-1} \psi_{i_j}^{\epsilon_j} \quad \text{and} \quad \left(\psi_i^{(J)}\right)^2 \neq \prod_{j=1}^{2t-2} \psi_{i_j}^{\epsilon_j}$$

whenever $\varepsilon_j = \pm 1$ and $\psi_{i_j} \in \{\psi_1^{(J)}, \dots, \psi_{i-1}^{(J)}\} \cup \bigcup_{j=q}^{J-1} E_j$. As explained above, this choice will ensure that

$$Z_t^\sigma\left(\{\psi_1^{(J)}, \dots, \psi_i^{(J)}\} \cup \bigcup_{j=q}^{J-1} E_j\right) = 1 \text{ for all } t \le q.$$

Furthermore, as the number of characters which $\psi_i^{(J)}$ and $\left(\psi_i^{(J)}\right)^2$ must avoid is a polynomial in $n_J$, we can choose $E_J$ from the subset $\{\chi^j\}_{j=2^{n_J}+1}^{2^{n_J+1}}$ provided $n_J$ is sufficiently large. This is the choice we make for $n_J$ and $E_J$.

Now put $E = \bigcup_{j=1}^{\infty} E_j$. For any integer $J \ge 2$, the subset $\bigcup_{j=J}^{\infty} E_j$ has all alternating $J$ -products distinct up to permutation and hence is $\Lambda_{2J}^{cb}$. Being a finite set, $\bigcup_{j=1}^{J-1} E_j$ is also $\Lambda_{2J}^{cb}$, and hence so is the finite union, $E = \bigcup_{j=J}^{\infty} E_j \cup \bigcup_{j=1}^{J-1} E_j$.

However, $E_j$ consists of $n_j^2$ terms chosen from the arithmetic progression $\{\chi^j\}_{j=2^{n_j}+1}^{2^{n_j+1}}$ of length $2^{n_j}$. As $n_j \to \infty$, an application of Lemma 3.2 shows that $E$ is not Sidon, completing the proof of the theorem for this first case.

Otherwise $\Gamma$ does not contain a character of infinite order. As in the proof of Theorem 3.5, we consider separately the case when $\Gamma$ has elements of unbounded order and the case when $\Gamma$ is of bounded order. Both arguments are similar to the first case, using the construction outlined above together with the strategy of cases 2 and 3 in the proof of Theorem 3.5. $\square$

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