PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 144, Number 7, July 2016, Pages 2985–2998
http://dx.doi.org/10.1090/proc/13054
Article electronically published on March 1, 2016

ESTIMATES OF KOLMOGOROV, GELFAND AND LINEAR n-WIDTHS ON COMPACT RIEMANNIAN MANIFOLDS

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(Communicated by Alexander Iosevich)

Abstract. We determine lower and exact estimates of Kolmogorov, Gelfand and linear n-widths of unit balls in Sobolev norms in L_p-spaces on compact Riemannian manifolds. As it was shown by us previously these lower estimates are exact asymptotically in the case of compact homogeneous manifolds. The proofs rely on two-sides estimates for the near-diagonal localization of kernels of functions of elliptic operators.

1. Introduction and the main results

The goal of this paper is to determine lower and exact estimates of Kolmogorov, Gelfand and linear n-widths of unit balls in Sobolev norms in $L_p(\mathbf{M})$-spaces on a compact connected Riemannian manifold $\mathbf{M}$.

Let us recall [15], [17] that for a given subset H of a normed linear space Y, the Kolmogorov n-width $d_n(H,Y)$ is defined as

$$d_n(H,Y)=\inf_{Z_n}\sup_{x\in H}\inf_{z\in Z_n}\|x-z\|_Y$$

where Z_n runs over all n-dimensional subspaces of Y. The linear n-width $\delta_n(H,Y)$ is defined as

$$\delta_n(H,Y)=\inf_{A_n}\sup_{x\in H}\|x-A_nx\|_Y$$

where A_n runs over all bounded operators $A_n : Y\to Y$ whose range has dimension n. The Gelfand n-width of a subset H in a linear space Y is defined by

$$d^n(H,Y)=\inf_{Z^n}\sup\{\|x\|:\ x\in H\cap Z^n\},$$

where the infimum is taken over all subspaces $Z^n\subset Y$ of codimension $\le n$. The width d_n characterizes the best approximative possibilities by approximations by n-dimensional subspaces, and the width δ_n characterizes the best approximative possibilities of any n-dimensional linear method. The width d^n plays a key role in questions about interpolation and reconstruction of functions.

In our paper the notation S_n will stay for either Kolmogorov n-width d_n or linear n-width δ_n; the notation s_n will be used for either d_n or Gelfand n-width d^n; S^n will be used for either d_n, d^n, or δ_n.

Received by the editors August 30, 2015.
2010 *Mathematics Subject Classification.* Primary 43A85, 42C40, 41A17; Secondary 41A10.
Key words and phrases. Compact manifold, Laplace-Beltrami operator, Sobolev space, eigenfunctions, kernels, n-widths.

If $\gamma \in \mathbf{R}$, we write $S^n(H,Y) \ll n^{\gamma}$, $n \in \mathbb{N}$, to mean that one has the upper estimate $S^n(H,Y) \geq Cn^{\gamma}$ where C is independent of n. We say that one has the lower estimate if $S^n(H,Y) \gg cn^{\gamma}$ for $n>0$ where $c>0$ is independent of n. In the case where we have both estimates we write $S^n(H,Y) \asymp cn^{\gamma}$ and call it the exact estimate. More generally, for two functions $f(t), g(t)$ notation $f(t) \asymp g(t)$ means the existence of two unessential positive constants c, C for which $cg(t) \leq f(t) \leq Cg(t)$ for all admissible t.

Let $(\mathbf{M}, g)$ be a smooth, connected, compact Riemannian manifold without boundary with Riemannian measure dx. Let $L_q(\mathbf{M}) = L_q(\mathbf{M})$, $1 \leq q \leq \infty$, be the regular Lebesgue space constructed with the Riemannian density. Let L be an elliptic smooth second-order differential operator L which is self-adjoint and positive definite in $L_2(\mathbf{M})$. For such an operator all the powers L^r, $r>0$, are well defined on $C^{\infty}(\mathbf{M}) \subset L_2(\mathbf{M})$ and continuously map $C^{\infty}(\mathbf{M})$ into itself. Using duality every operator L^r, $r>0$, can be extended to distributions on $\mathbf{M}$. The Sobolev space $W_p^r = W_p^r(\mathbf{M})$, $1 \leq p \leq \infty$, $r>0$, is defined as the space of all $f \in L_p(\mathbf{M}), 1 \leq p \leq \infty$, for which the following graph norm is finite:

$$\|f\|_{W_p^r(\mathbf{M})} = \|f\|_p + \|L^{r/2} f\|_p. \tag{1.1}$$

Our objective is to obtain asymptotic estimates of $S_n\left(H, L_q(\mathbf{M})\right)$, where H is the unit ball $B_p^r(\mathbf{M})$ in the Sobolev space $W_p^r = W_p^r(\mathbf{M}), 1 \leq p \leq \infty$, $r>0$, Thus,

$$B_p^r = B_p^r(\mathbf{M}) = \left\{ f \in W_p^r(\mathbf{M}) \ : \ \|f\|_{W_p^r(\mathbf{M})} \leq 1 \right\}.$$

It is important to remember that in all our considerations the inequality

$$\frac{r}{s} > \left(\frac{1}{p} - \frac{1}{q}\right)_+ \tag{1.2}$$

with $s = dim\, \mathbf{M}$ will be assumed. Thus, by the Sobolev embedding theorem the set $B_p^r(\mathbf{M})$ is a subset of $L_q(\mathbf{M})$. Moreover, since $\mathbf{M}$ is compact, the Rellich-Kondrashov theorem implies that the embedding of $B_p^r(\mathbf{M})$ into $L_q(\mathbf{M})$ will be compact.

Our main result is the following theorem which is proved in section 4.

Theorem 1.1. *For any compact Riemannian manifold $\mathbf{M}$ of dimension s, any elliptic second-order smooth operator L, any $1 \leq p,\ q \leq \infty$, if S_n is either of d_n or δ_n and S^n is either of d_n, δ_n or d^n, then the following holds for any r that satisfies* (1.2)*:*

(1) *if $1 \leq q \leq p \leq \infty$, then*

$$S^n\left(B_p^r(\mathbf{M}), L_q(\mathbf{M})\right) \gg n^{-\frac{r}{s}}, \tag{1.3}$$

(2) *if $1 \leq p \leq q \leq 2$, then*

$$S_n(B_p^r(\mathbf{M}), L_q(\mathbf{M})) \gg n^{-\frac{r}{s}+\frac{1}{p}-\frac{1}{q}}, \tag{1.4}$$

and

$$d^n\left(B_p^r(\mathbf{M}), L_q(\mathbf{M})\right) \gg n^{-\frac{r}{s}}, \tag{1.5}$$

(3) *if $2 \leq p \leq q \leq \infty$, then*

$$d_n\left(B_p^r(\mathbf{M}), L_q(\mathbf{M})\right) \gg n^{-\frac{r}{s}}, \tag{1.6}$$

$$d^n(B_p^r(\mathbf{M}), L_q(\mathbf{M})) \gg n^{-\frac{r}{s}+\frac{1}{p}-\frac{1}{q}}, \tag{1.7}$$

$$\delta_n(B_p^r(\mathbf{M}), L_q(\mathbf{M})) \gg n^{-\frac{r}{s}+\frac{1}{p}-\frac{1}{q}}, \tag{1.8}$$

(4) *if* $1 \le p \le 2 \le q \le \infty$, *then*

$$d_n(B_p^r(\mathbf{M}), L_q(\mathbf{M})) \gg n^{-\frac{r}{s}+\frac{1}{p}-\frac{1}{2}}, \tag{1.9}$$

$$d^n(B_p^r(\mathbf{M}), L_q(\mathbf{M})) \gg n^{-\frac{r}{s}+\frac{1}{2}-\frac{1}{q}}, \tag{1.10}$$

$$\delta_n(B_p^r(\mathbf{M}), L_q(\mathbf{M})) \gg max\left(n^{-\frac{r}{s}+\frac{1}{2}-\frac{1}{q}}, n^{-\frac{r}{s}+\frac{1}{q}-\frac{1}{2}}\right). \tag{1.11}$$

As it was shown in [8] all the estimates of this theorem are exact if $\mathbf{M}$ is a compact homogeneous manifold. Let's recall that every compact homogeneous manifold is of the form G/H where G is a compact Lie group and H is its closed subgroup. For compact homogeneous manifolds we obtained in [8] exact asymptotic estimates for d_n and δ_n for all $1 \le p, q \le \infty$, and some restrictions on r.

To compare our lower estimates with known upper estimates, let us recall an inequality which was proved in our previous papers [7], [8].

Theorem 1.2. *For any compact Riemannian manifold, any L, any $1 \le p, q \le \infty$, and any r which satisfies* (1.2) *if S_n is either of d_n or δ_n, then the following holds:*

$$S_n(B_p^r(\mathbf{M}), L_q(\mathbf{M})) \ll n^{-\frac{r}{s}+\left(\frac{1}{p}-\frac{1}{q}\right)_+}. \tag{1.12}$$

By comparing these two theorems we obtain the following exact estimates.

Theorem 1.3. *For any compact Riemannian manifold $\mathbf{M}$ of dimension s, any elliptic second-order smooth operator L, any $1 \le p,\ q \le \infty$, if S_n is either of d_n or δ_n, then the following holds for any r which satisfies* (1.2)*:*

(1) *if* $1 \le q \le p \le \infty$, *then*

$$S_n\left(B_p^r(\mathbf{M}), L_q(\mathbf{M})\right) \asymp n^{-\frac{r}{s}},$$

(2) *if* $1 \le p \le q \le 2$, *then*

$$S_n\left(B_p^r(\mathbf{M}), L_q(\mathbf{M})\right) \asymp n^{-\frac{r}{s}+\frac{1}{p}-\frac{1}{q}},$$

(3) *if* $2 \le p \le q \le \infty$, *then*

$$\delta_n\left(B_p^r(\mathbf{M}), L_q(\mathbf{M})\right) \asymp n^{-\frac{r}{s}+\frac{1}{p}-\frac{1}{q}}.$$

Our results could be carried over to Besov spaces on manifolds using general results about interpolation of compact operators [19].

Our results generalize some of the known estimates for the particular case in which $\mathbf{M}$ is a compact symmetric space of rank one which were obtained in papers [4] and [2]. They, in turn, generalized and extended results from [1], [10], [14], [16], [12], [13], [14].

2. Kernels on compact Riemannian manifolds

We consider $(\mathbf{M}, g)$ to be a smooth, connected, compact Riemannian manifold without boundary with Riemannian measure dx. Let L be the Laplace-Beltrami operator of the metric g which is well defined on $C^\infty(\mathbf{M})$. We will use the same notation L for the closure of L from $C^\infty(\mathbf{M})$ in $L_2(\mathbf{M})$. This closure is a self-adjoint non-negative operator on the space $L_2(\mathbf{M})$. The spectrum of this operator, say $0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \dots$, is discrete and approaches infinity. Let $u_0, u_1, u_2, \dots$ be a corresponding complete system of real-valued orthonormal eigenfunctions, and let

$\mathbf{E}_t(L)$, $t > 0$, be the span of all eigenfunctions of L, whose corresponding eigenvalues are not greater than t. Since the operator L is of order two, the dimension $\mathcal{N}_t$ of the space $\mathbf{E}_t(L)$ is given asymptotically by Weyl's formula, [11], which says, in sharp form that for some $c > 0$,

$$\mathcal{N}_t(L) = \frac{vol(\mathbf{M})\sigma_s}{(2\pi)^s} t^{s/2} + O(t^{(s-1)/2}), \quad \sigma_s = \frac{2\pi^{s/2}}{s\Gamma(s/2)}, \quad s = dim\ \mathbf{M}, \tag{2.1}$$

where $s = dim\ \mathbf{M}$. Because $\mathcal{N}_{\lambda_l} = l + 1$, we conclude that, for some constants $c_1, c_2 > 0$,

$$c_1 l^{2/s} \le \lambda_l \le c_2 l^{2/s} \tag{2.2}$$

for all l. Since $L^m u_l = \lambda_l^m u_l$, and L^m is an elliptic differential operator of degree $2m$, Sobolev's lemma, combined with the last fact, implies that for any integer $k \ge 0$, there exist $C_k, \nu_k > 0$, such that

$$\|u_l\|_{C^k(\mathbf{M})} \le C_k (l+1)^{\nu_k}. \tag{2.3}$$

For a $t > 0$ let's consider the function

$$K_t(x,y) = \sum_{\lambda_l \le t} u_l(x) u_l(y) \tag{2.4}$$

which is known as the spectral function associated to L. In [11] one can find the following estimate:

$$K_t(x,x) = \frac{\sigma_s}{(2\pi)^s} t^{s/2} + O(t^{s-1}). \tag{2.5}$$

Since

$$K_t(x,x) = \sum_{0<\lambda_l \le t} (u_l(x))^2 = \|K_t(x,\cdot)\|_2^2$$

estimates (2.1) and (2.5) imply that there exists $0 < C_1 < C_2$ such that

$$C_1 t^{s/2} \le \sum_{0<\lambda_l \le t} (u_l(x))^2 \le C_2 t^{s/2}. \tag{2.6}$$

We also note that

$$dim\ \mathbf{E}_t(L) \asymp t^{s/2}, \quad s = dim\ \mathbf{M}.$$

Definition 2.1. For each positive integer J, we let

$$\mathcal{S}_J(\mathbf{R}^+) = \left\{ F \in C^J([0,\infty)) : \|F\|_{\mathcal{S}_J} := \sum_{i+j \le J} \left\| \lambda^i \frac{\partial^j}{\partial \lambda^j} F \right\|_\infty < \infty \right\}. \tag{2.7}$$

For a fixed $t > 0$ if J is sufficiently large, one can use (2.1), (2.2) and (2.3) to show that the right side of

$$K_t^F(x,y) := \sum_l F(t^2 \lambda_l) u_l(x) u_l(y) \tag{2.8}$$

converges uniformly to a continuous function on $\mathbf{M} \times \mathbf{M}$, and in fact that for some $C_t > 0$,

$$\|K_t^F\|_\infty \le C_t \|F\|_{\mathcal{S}_J}. \tag{2.9}$$

Using the spectral theorem, one can define the bounded operator $F(t^2L)$ on $L_2(\mathbf{M})$. In fact, for $f \in L_2(\mathbf{M})$,

$$[F(t^2L)f](x) = \int K_t^F(x,y)f(y)dy. \tag{2.10}$$

We call K_t^F the kernel of $F(t^2L)$. $F(t^2L)$ maps $C^\infty(\mathbf{M})$ to itself continuously, and may thus be extended to be a map on distributions. In particular we may apply $F(t^2L)$ to any $f \in L_p(\mathbf{M}) \subseteq L_1(\mathbf{M})$ (where $1 \le p \le \infty$), and by Fubini's theorem $F(t^2L)f$ is still given by (2.10).

For $x, y \in \mathbf{M}$, let $d(x,y)$ denote the geodesic distance from x to y. We will frequently need the following fact.

Lemma 2.2. *If $\mathcal{N} > s$, $x \in \mathbf{M}$, and $t > 0$, then*

$$\int_{\mathbf{M}} \frac{1}{[1 + (d(x,y)/t)]^{\mathcal{N}}} dy \le Ct^s, \quad s = dim\mathbf{M}, \tag{2.11}$$

with C independent of x or t.

Proof. Note that there exist $c_1, c_2 > 0$, such that for all $x \in \mathbf{M}$ and all sufficiently small $r \le \delta$ one has

$$c_1 r^n \le |B(x,r)| \le c_2 r^n,$$

and if $r > \delta$,

$$c_3 \delta^n \le |B(x,r)| \le |\mathbf{M}| \le c_4 r^n.$$

Fix x, t and let $A_j = B(x, 2^j t) \setminus B(x, 2^{j-1}t)$, so that, $|A_j| \le c_4 2^{nj} t^n$. Now break the integral into integrals over $B(x,t), A_1, \dots$ and note that $\sum_{j=0}^{\infty} 2^{(n-N)j} < \infty$. $\square$

The following statements can be found in [5]-[8].

Lemma 2.3. *Assume $F \in \mathcal{S}_J(\mathbf{R}^+)$ for a sufficiently large $J \in \mathbb{N}$. For $t > 0$, let $K_t^F(x,y)$ be the kernel of $F(t^2L)$. Suppose that $0 < t \le 1$. Then for some $C > 0$,*

$$|K_t^F(x,y)| \le \frac{Ct^{-s}}{\left[1 + \frac{d(x,y)}{t}\right]^{s+1}}, \quad s = dim\,\mathbf{M}, \tag{2.12}$$

for all t and all $x, y \in \mathbf{M}$.

Lemma 2.4. *Assume $F \in \mathcal{S}_J(\mathbf{R}^+)$ for a sufficiently large $J \in \mathbb{N}$. Consider $1 \le \alpha \le \infty$, with conjugate index α'. There exists a constant $C > 0$ such that for all $0 < t \le 1$,*

$$\left(\int_{\mathbf{M}} |K_t^F(x,y)|^\alpha dy\right)^{1/\alpha} \le Ct^{-s/\alpha'} \qquad \textit{for all } x, \tag{2.13}$$

and

$$\left(\int_{\mathbf{M}} |K_t^F(x,y)|^\alpha dx\right)^{1/\alpha} \le Ct^{-s/\alpha'} \qquad \textit{for all } y. \tag{2.14}$$

Proof. We need only prove (2.13), since $K_t^F(y,x) = K_t^F(x,y)$.

If $\alpha < \infty$, (2.13) follows from Lemma 2.3, which tells us that

$$\int_{\mathbf{M}} |K_t^F(x,y)|^\alpha dy \le C \int_{\mathbf{M}} \frac{t^{-s\alpha}}{[1+(d(x,y)/t)]^{\alpha(s+1)}} dy \le C t^{s(1-\alpha)}$$

with C independent of x or t, by (2.11).

If $\alpha = \infty$, the left side of (2.13) is, as usual, to be interpreted as the L_∞ norm of $h_{t,x}(y) = K_t^F(x,y)$. But in this case the conclusion is immediate from Lemma 2.3.

This completes the proof. □

Lemma 2.5. *If C_1, C_2 are the same as in (2.6) and if*

$$b/a > (C_2/C_1)^{2/s}, \quad s = dim\ \mathbf{M}, \tag{2.15}$$

then

$$\sum_{a/t^2 < \lambda_l \le b/t^2} |u_l(x)|^2 \ge \left(C_1 b^{s/2} - C_2 a^{s/2}\right) t^{-s} > 0, \quad s = dim\ \mathbf{M}. \tag{2.16}$$

Proof. By the inequalities (2.6) we have

$$\sum_{a/t^2 < \lambda_l \le b/t^2} |u_l(x)|^2 = \sum_{0 < \lambda_l \le b/t^2} |u_l(x)|^2 - \sum_{0 < \lambda_l \le a/t^2} |u_l(x)|^2 \ge \left(C_1 b^{s/2} - C_2 a^{s/2}\right) t^{-s}.$$

The lemma is proven. □

Lemma 2.6. *For any $0 < a < b$ and sufficiently large $J \in \mathbb{N}$ there exists an even function F in $\mathcal{S}_J(\mathbf{R})$ such that $\widehat{F}$ is supported in a $(-\Lambda, \Lambda)$ for some $\Lambda > 0$ and the inequality*

$$0 < c_1 \le |F(\lambda)| \le c_2$$

holds for all $a \le \lambda \le b$ for some $c_1,\ c_2 > 0$.

Proof. For a sufficiently large J consider an even function $G \in \mathcal{S}_J(\mathbf{R})$ which is identical to the one on $(a,\ b)$. Since Fourier transform maps continuously $\mathcal{S}_J(\mathbf{R})$ into itself one can find an even smooth function $\widehat{F} \in \mathcal{S}_J(\mathbf{R})$ which is supported in a $(-\Lambda,\ \Lambda)$ for some $\Lambda > 0$ and which is sufficiently close to the Fourier transform $\widehat{G} \in \mathcal{S}_J(\mathbf{R})$ in the topology of $\mathcal{S}_J(\mathbf{R})$. Clearly, the function F will have all the desired properties. It proves the lemma. □

Theorem 2.7. *For a sufficiently large $J \in \mathbb{N}$ there exists an even function F in the space $\mathcal{S}_J(\mathbf{R})$ such that $\widehat{F}$ has support in a $(-\Lambda, \Lambda)$ and for which*

$$\left(\int_{\mathbf{M}} \left|K_t^F(x,y)\right|^\alpha dy\right)^{1/\alpha} \asymp t^{-s/\alpha'}, \quad 1/\alpha + 1/\alpha' = 1, \quad 0 < t \le 1, \tag{2.17}$$

for all $1 \le \alpha \le \infty$.

Proof. Due to Lemma 2.4 we have to prove only the lower estimate. Assume that $0 < a < b$ and satisfy (2.15). Let F be a function whose existence is proved in the previous lemma for this (a,b). For $\alpha = 2$ one has

$$\int_{\mathbf{M}} |K_t^F(x,y)|^2 dy = \sum_l |F(t^2\lambda_l)|^2 |u_l(x)|^2 \ge \sum_{l: a/t^2 \le \lambda_l \le b/t^2} |F(t^2\lambda_l)|^2 |u_l(x)|^2$$

$$\ge c_1^2 \sum_{l: a/t^2 \le \lambda_l \le b/t^2} |u_l(x)|^2 \ge c_1^2 \left(C_1 b^{s/2} - C_2 a^{s/2}\right) t^{-s} > 0.$$

Using the inequality

$$\|K_t^F(x,\cdot)\|_2^2 \leq \|K_t^F(x,\cdot)\|_1 \|K_t^F(x,\cdot)\|_\infty, \tag{2.18}$$

and Lemma 2.4 for $\alpha = 1$ we obtain for $\alpha = 1$

$$\|K_t^F(x,\cdot)\|_1 \geq \frac{\|K_t^F(x,\cdot)\|_2^2}{\|K_t^F(x,\cdot)\|_\infty} \geq C_3 > 0,$$

and similarly for $\alpha = \infty$.

Note that if $q < 2 < r$, and $0 < \theta < 1$ is such that $\theta/q + (1-\theta)/r = 1/2$, then by the Hölder inequality

$$\|K_t^F(x,\cdot)\|_2 \leq \|K_t^F(x,\cdot)\|_q^\theta \|K_t^F(x,\cdot)\|_r^{1-\theta}. \tag{2.19}$$

Assume now that $2 < \alpha \leq \infty$. Then for $q = 1,\ r = \alpha$, we have $\alpha' < 2(1-\theta)$ and using lower and upper estimates for $p = 2$ and $p = 1$ respectively we obtain

$$\|K_t^F(x,\cdot)\|_\alpha \geq \frac{\|K_t^F(x,\cdot)\|_2^{1/(1-\theta)}}{\|K_t^F(x,\cdot)\|_1^{\theta/(1-\theta)}} \geq C_4 t^{-s/\alpha'}$$

for some $C_4 > 0$.

The case $0 \leq \alpha < 2$ is handled in a similar way by setting in (2.19) $q = \alpha$, $r = \infty$. The lemma is proved. ☐

The next theorem plays an important role in this paper (see also [5], [6]).

Theorem 2.8. *Suppose that for a sufficiently large $J \in \mathbb{N}$ a function $\psi(\xi) = F(\xi^2)$ belongs to $\mathcal{S}_J(\mathbf{R})$, is even, and satisfies* $\operatorname{supp} \hat{\psi} \subseteq (-1,1)$. *For $t > 0$, let $K_t^F(x,y)$ be the kernel of $\psi(t\sqrt{L}) = F(t^2 L)$. Then for some $C_0 > 0$, if $d(x,y) > C_0 t$, then $K_t^F(x,y) = 0$.*

Proof. First, let us formulate the finite speed of propagation property for the wave equation (we closely follow Theorem 4.5 (iii) in Ch. IV of [18]).

Suppose that $L_{\mathbf{R}^s}$ is a second-order differential operator on an open set $\mathbf{R}^s$ in $\mathbf{R}^s$, that L_1 is elliptic, and in fact that, for some $c > 0$, its principal symbol $\sigma_2(L_{\mathbf{R}^s})(x,\xi) \geq c^2|\xi|^2$, for all $(x,\xi) \in \mathbf{R}^s \times \mathbf{R}^s$. Suppose that $U \subseteq \mathbf{R}^s$ is open, and that $\overline{U} \subseteq \mathbf{R}^s$. Then if $\operatorname{supp} h,\ g \subseteq Q \subseteq U$, where Q is compact, then any solution u of

$$\left(\frac{\partial^2}{\partial t^2} + L_{\mathbf{R}^s}\right)\phi = 0, \tag{2.20}$$

$$\phi(0,x) = h(x), \tag{2.21}$$

$$\phi_t(0,x) = g(x), \tag{2.22}$$

on U satisfies $\operatorname{supp}\phi(t,\cdot) \subseteq \{x : \text{dist}\ (x,Q) \leq |t|/c\}$.

It is an easy consequence of this that a similar result holds on manifolds (see explanations in [6] and [8]). Let L be a smooth elliptic second-order non-negative operator on a manifold $\mathbf{M}$ and consider the problem

$$\left(\frac{\partial^2}{\partial t^2} + L\right)\phi = 0, \tag{2.23}$$

$$\phi(0,x) = h(x), \tag{2.24}$$

$$\phi_t(0,x) = 0, \tag{2.25}$$

on $\mathbf{M}$. It is easy to verify that if u_l form an orthonormal basis of eigenfunctions of L, with corresponding eigenvalues λ_l and

$$h(x)=\sum_l a_l u_l(x), \quad a_l=\int_{\mathbf{M}} h(y)u_l(y)dy,$$

then the solution to (2.23)-(2.25) is

$$\phi(x,t)=\sum_l \left[a_l \cos\left(t\sqrt{L}\right) u_l\right](x)=\left[\cos\left(t\sqrt{L}\right) h\right](x), \tag{2.26}$$

or

$$\phi(x,t)=\sum_l a_l \cos\left(t\sqrt{\lambda_l}\right) u_l(x).$$

To prove the theorem it suffices to note that for some c

$$\left[\psi(t\sqrt{L})h\right](x)=c\int_{-1}^{1}\widehat{\psi}(s)\left[\cos\left(st\sqrt{L}\right) h\right](x)ds \tag{2.27}$$

for any $h\in C^{\infty}(\mathbf{M})$. This formula follows from the eigenfunction expansion of h and the Fourier inversion formula. Indeed, since $\widehat{\psi}$ is even and $\operatorname{supp}\widehat{\psi}\subset(-1,1)$ we have

$$\begin{aligned}\int_{-1}^{1}\widehat{\psi}(s)\left[\cos(st\sqrt{L})h\right](x)ds&=\int_{-1}^{1}\widehat{\psi}(s)\left[\cos(st\sqrt{L})h+i\ \sin(st\sqrt{L})h\right](x)ds\\&=\int_{-\infty}^{\infty}\widehat{\psi}(s)\int_{\mathbf{M}}\sum_l e^{ist\sqrt{\lambda_l}}u_l(x)u_l(y)h(y)dyds\\&=\int_{\mathbf{M}}\sum_l\left(\int_{-\infty}^{\infty}\widehat{\psi}(s)e^{ist\sqrt{\lambda_l}}ds\right)u_l(x)u_l(y)h(y)dy\\&=\int_{\mathbf{M}}\sum_l\psi(t\sqrt{\lambda_l})u_l(x)u_l(y)h(y)dy=\left[\psi(t\sqrt{L})h\right](x).\end{aligned}$$

We also note

$$\left[\psi(t\sqrt{L})h\right](x)=\int_{\mathbf{M}}\sum_l\psi(t\sqrt{\lambda_l})u_l(x)u_l(y)h(y)dy$$

$$=\int_{\mathbf{M}}\sum_l F\left(t^2\lambda_l\right)u_l(x)u_l(y)h(y)dy=\int_{\mathbf{M}}K_t^F(x,y)h(y)dy=F\left(t^2L\right)h(x), \tag{2.28}$$

where

$$\sum_l F\left(t^2\lambda_l\right)u_l(x)u_l(y)=K_t^F(x,y).$$

Let us summarize. Since according to (2.26) the function $\phi(x,t)=\cos\left(t\sqrt{L}\right)h(x)$ is the solution to (2.23)-(2.25) the finite speed of propagation principle implies that if h has support in a set $Q\subset\mathbf{M}$, then for every $t>0$ the function $\cos\left(t\sqrt{L}\right)h(x)$ has support in the set $\{x:\text{dist }(x,Q)\leq C|t|\}$ where C is independent on Q.

Consider a function $h\in C^{\infty}(\mathbf{M})$ which is supported in a ball $B_{\epsilon}(y)$ whose center is a $y\in\mathbf{M}$ and radius is a small $\varepsilon>0$. By (2.27) and (2.28) the function

$\psi\left(t\sqrt{L}\right)h(x) = F\left(t^2L\right)h(x)$ and the kernel $K_t^F(x,y)$ (as a function in x) both have support in the same set

$$\{x : \operatorname{dist}(x, B_\varepsilon(y)) \le C|t|\}.$$

Since $\operatorname{dist}(x,y) = \operatorname{dist}(x, B_\varepsilon(y)) + \varepsilon$ we obtain that for any $\varepsilon > 0$ the support of $K_t^F(x,y)$ is in the set

$$\{x : \operatorname{dist}(x,y) \le C|t| + \varepsilon\}.$$

The theorem is proven. □

3. Discretization and reduction to finite-dimensional spaces

Lemma 3.1. *Let* $\mathbf{M}$ *be a compact Riemannian manifold. For each positive integer* N *with* $2N^{-1/s} <$ *diam* $\mathbf{M}$, *there exists a collection of disjoint balls* $\mathcal{A}^N = \left\{B\left(x_i^N, N^{-1/s}\right)\right\}$, *such that the balls with the same centers and three times the radii cover* $\mathbf{M}$, *and such that* $P_N := \#\mathcal{A}^N \asymp N$.

Proof. We need only let $\mathcal{A}^N$ be a maximal disjoint collection of balls of radius $N^{-1/s}$. Then surely the balls with the same centers and three times the radii cover $\mathbf{M}$. Thus by disjointness

$$\mu(\mathbf{M}) \ge \sum_{i=1}^{P_N} \mu\left(B\left(x_i^N, N^{-1/s}\right)\right) \gg \sum_{i=1}^{P_N} 1/N = P_N/N,$$

while by the covering property,

$$P_N/(3^sN) \gg \sum_{i=1}^{P_N} \mu\left(B\left(x_i^N, 3N^{-1/s}\right)\right) \ge \mu(\mathbf{M})$$

so that $P_N \asymp N$ as claimed. □

Now we formulate and sketch the proof of the following Lemma 3.2. See [8] for more details.

In what follows we consider collections of balls $\mathcal{A}^N$ as in Lemma 3.1.

Lemma 3.2. *Let* $\mathbf{M}$ *be a compact Riemannian manifold. Then there are smooth functions* φ_i^N $(2N^{-1/s} <$ *diam* $\mathbf{M}$, $1 \le i \le P_N)$, *as follows:*

(1) $\operatorname{supp}\varphi_i^N \subseteq B_i^N := B(x_i^N, N^{-1/s})$;
(2) *for* $1 \le p \le \infty$, $\|\varphi_i^N\|_p \asymp N^{-1/p}$, *with constants independent of* i *or* N.

Proof. For a sufficiently large $J \in \mathbb{N}$ let $h_0(\xi) = F_0(\xi^2)$ be an even element of $\mathcal{S}_J(\mathrm{R})$ with supp $\hat{h}_0 \subseteq (-1,1)$. For a positive integer Q yet to be chosen, let $F(\lambda) = \lambda^Q F_0(\lambda)$, and set

$$h(\xi) = F(\xi^2) = \xi^{2Q}F_0(\xi^2), \tag{3.1}$$

so that $\hat{h} = c\partial^{2Q}\hat{h}_0$ still has support contained in $(-1,1)$. Thus, by Theorem 2.8, there is a $C_0 > 0$ such that for $t > 0$, the kernel $K_t^F(x,y)$ of $h(t\sqrt{\mathcal{L}}) = F(t^2\mathcal{L})$ has the property that $K_t^F(x,y) = 0$ whenever $d(x,y) > C_0t$. Thus if $t = N^{-1/s}/2C_0$,

$$\varphi_i^N(x) := \frac{1}{N}K_t^F(x_i^N, x) \tag{3.2}$$

satisfies (1). By Theorem 2.7, $\|\varphi_i^N\|_p \asymp N^{-1}(N^{-1/s})^{-s/p'} = N^{-1/p}$, so (2) holds. The lemma is proven. □

Let φ_i^N be the same as above. We consider their span

$$\mathcal{H}_p^N=\left\{\sum_{i=1}^{P_N}a_i\varphi_i^N : a=(a_1,\dots,a_{P_N})\in\mathbf{R}^{P_N}\right\} \tag{3.3}$$

as a finite-dimensional Banach space $\mathcal{H}_p^N$ with the norm

$$\left\|\sum_{i=1}^{P_N}a_i\varphi_i^N\right\|_{\mathcal{H}_p^N}=\left\|\sum_{i=1}^{P_N}a_i\varphi_i^N\right\|_{L_p(\mathbf{M})}\asymp CN^{-1/p}\|a\|_p, \tag{3.4}$$

where C is independent on N. Clearly, for any $r>0$ the operator $L^{r/2}$ maps $\mathcal{H}_p^N$ onto the span

$$\mathcal{M}_p^N=\left\{\sum_{i=1}^{P_N}a_iL^{r/2}\varphi_i^N : a=(a_1,\dots,a_{P_N})\in\mathbf{R}^{P_N}\right\},$$

which we will consider with the norm

$$\left\|\sum_{i=1}^{P_N}a_iL^{r/2}\varphi_i^N\right\|_{L_p(\mathbf{M})},$$

and will denote as $\mathcal{M}_p^N\subset L_p(\mathbf{M})$. Our next goal is to estimate the norm of $L^{r/2}$ as an operator from the Banach space $\mathcal{H}_p^N$ onto Banach space $\mathcal{M}_p^N$.

Lemma 3.3. *If φ_i^N are the same as in Lemma* 3.2*, then for $1\le p\le\infty$, and $r>0$,*

$$\left\|\sum_{i=1}^{P_N}a_iL^{r/2}\varphi_i^N\right\|_{L_p(\mathbf{M})}\le CN^{\frac{r}{s}-\frac{1}{p}}\|a\|_p, \tag{3.5}$$

with C independent of $a=(a_1,\dots,a_{P_N})\in\mathbf{R}^{P_N}$, p or N.

Proof. By the Riesz-Thorin interpolation theorem, we need only to verify the estimates for $p=1$ and $p=\infty$. For $t=N^{-1/s}/2C_0$ and φ_i^N defined in (3.2) we have

$$L^{r/2}\varphi_i^N=N^{-1}t^{-r}\sum_l(t^2\lambda_l)^{r/2}F(t^2\lambda_l)u_l(x_i^N)u_l(x)=CN^{\frac{r}{s}-1}K_t^G(x_i^N,x), \tag{3.6}$$

where $G(\lambda)=\lambda^{r/2}F(\lambda)$ and F is defined in (3.1). Clearly, for a fixed $r>0$ function G belongs to a certain $\mathcal{S}_{J_0}(\mathbf{R}^+)$ for some $J_0\in\mathbb{N}$ if Q in (3.1) is sufficiently large. Note that C in (3.6) is independent of N,i or t. Thus, by (3.6) and Lemma 2.4, for $p=1$ we have $\|L^{r/2}\varphi_i^N\|_1\le CN^{\frac{r}{s}-1}$, with C independent of i,N. It proves (3.5) for $p=1$. As for $p=\infty$, we again set $t=N^{-1/s}/2C_0$. By Lemma 2.3 and (3.2), we have that for any x,

$$\left|\sum_{i=1}^{P_N}a_iL^{r/2}\varphi_i^N(x)\right|\le CN^{\frac{r}{s}-1}\|a\|_\infty\sum_{i=1}^{P_N}\frac{t^{-s}}{(1+d(x_i^N,x)/t)^{s+1}}. \tag{3.7}$$

Since $t^{-s}=(2C_0)^s\,N\asymp\mu\left(B_i^N\right)N^2$, we obtain

$$\left|\sum_{i=1}^{P_N}a_iL^{r/2}\varphi_i^N(x)\right|\le CN^{\frac{r}{s}+1}\|a\|_\infty\sum_{i=1}^{P_N}\frac{\mu(B_i^N)}{(1+d(x_i^N,x)/t)^{s+1}}. \tag{3.8}$$

The triangle inequality shows that for all $x \in \mathbf{M}$, all $t > 0$, all i and N, and all $y \in B_i^N$, one has $(1 + d(y,x)/t) \le C(1 + d(x_i^N, x)/t)$ with C independent of x, y, t, i, N. Combining this with (2.11) we finally obtain

$$(3.9) \qquad \left| \sum_{i=1}^{P_N} a_i L^{r/2} \varphi_i^N(x) \right| \le C N^{\frac{r}{s}+1} \|a\|_\infty \int_{\mathbf{M}} \frac{dy}{(1 + d(y,x)/t)^{s+1}} \le C N^{\frac{r}{s}} \|a\|_\infty.$$

Lemma 3.3 is proved. □

The next step is to reduce our main problem to a finite-dimensional situation.

Let us remark that we are using the following notation. S_n will stay for either Kolmogorov n-width d_n or linear n-width δ_n; the notation s_n will be used for either d_n or Gelfand n-width d^n; S^n will be used for either d_n, d^n, or δ_n.

Below we will need the following relations (see [15, pp. 400–403]):

$$(3.10) \qquad S^n(H_1, Y) \le S^n(H, Y),$$

if $H_1 \subset H$, and

$$(3.11) \qquad d^n(H, Y) = d^n(H, Y_1), \quad S_n(H, Y) \le S_n(H, Y_1), \quad H \subset Y_1 \subset Y,$$

where Y_1 is a subspace of Y. Moreover, the following inequality holds:

$$(3.12) \qquad \delta_n(H, Y) \ge max(d_n(H, Y), d^n(H, Y)).$$

In what follows we are using the notation of Lemmas 3.1-3.3.

Lemma 3.4. *For $1 \le p, q \le \infty$, if $s_n = d_n$ or d^n, then*

$$(3.13) \qquad s_n \left(B_p^r(\mathbf{M}), L_q(\mathbf{M}) \right) \ge C N^{-\frac{r}{s} + \frac{1}{p} - \frac{1}{q}} s_n(b_p^{P_N}, \ell_q^{P_N}),$$

for any sufficiently large n, N, with C independent of n, N.

Proof. With the φ_i^N as in Lemma 3.2, we consider the space of functions of the form

$$(3.14) \qquad g_a = \sum_{i=1}^{P_N} a_i \varphi_i^N,$$

for $a = (a_1, \ldots, a_{P_N}) \in \mathbf{R}^{P_N}$. By Lemma 3.2 and the disjointness of the B_i^N,

$$(3.15) \qquad \|g_a\|_q \asymp N^{-1/q} \|a\|_q,$$

with constants independent of N or a. By Lemma 3.3 for some $c > 0$, if we set $\epsilon = \epsilon_N = c N^{-\frac{r}{s} + \frac{1}{p}}$, and if $a \in \epsilon b_p^{P_N}$, then $g_a \in B_p^r$. Thus,

$$(3.16) \qquad \mathcal{G}_p^N := \{ g_a \in \mathcal{H}_p^N : a \in \epsilon b_p^{P_N} \} \subseteq B_p^r.$$

For the Gelfand widths, it is a consequence of the Hahn-Banach theorem, that if $K \subseteq X \subseteq Y$, where X is a subspace of the normed space Y, then $d^n(K, X) = d^n(K, Y)$ for all n. Thus, using (3.4) we obtain

$$d^n \left(B_p^r(\mathbf{M}), L_q(\mathbf{M}) \right) \ge d^n(\mathcal{G}_p^N, L_q) = d^n(\mathcal{G}_p^N, \mathcal{H}_q^N)$$
$$\ge C N^{-1/q} d^n(\epsilon_N b_p^{P_N}, \ell_q^{P_N}) = C N^{-r/s + 1/p - 1/q} d^n(b_p^{P_N}, \ell_q^{P_N})$$

for some C independent of n, N. This proves the lemma for the Gelfand widths.

For the Kolmogorov widths, for the same reason, we need only show that

$$(3.17) \qquad d_n(B_p^r, L_q) \ge C d_n(\mathcal{G}_p^N, \mathcal{H}_q^N)$$

with C independent of n, N.

To this end we define the projection operator $\Pi_N : L_q \to \mathcal{H}_q^N$ by

$$\Pi_N h = g_a, \qquad\qquad \text{where } a_i = \frac{\int h\varphi_i^N}{\|\varphi\|_2^2}.$$

By Lemma 3.2 and Hölder's inequality, we have that each $|a_i| \le C\|h\chi_i^N\|_q N^{1-1/q'}$, where χ_i^N is the characteristic function of B_i^N. By (3.15) and the disjointness of the B_i^N, we have that

$$\|\Pi_N h\|_q = \|g_a\|_q \asymp N^{-1/q}\|a\|_q \le cN^{1-1/q-1/q'}\|h\|_q = c\|h\|_q, \tag{3.18}$$

with C independent of n, N.

Accordingly, for any $g \in \mathcal{H}_q^N$ and $h \in L_q$, we have that

$$\|g - \Pi_N h\|_q = \|\Pi_N g - \Pi_N h\|_q \le c\|g-h\|_q.$$

Thus, if K is any subset of $\mathcal{H}_q^N$, $d_n(K, L_q) \ge c^{-1}d_n(K, \mathcal{H}_q^N)$. In particular

$$d_n\left(B_p^r(\mathbf{M}), L_q(\mathbf{M})\right) \ge d_n(\mathcal{G}_p^N, L_q) \ge c^{-1}d_n(\mathcal{G}_p^N, \mathcal{H}_q^N).$$

This establishes (3.17), and completes the proof. □

4. Proof of the main result

In this section we will prove Theorem 1.1.

We will need several facts about widths. First, say $p \ge p_1$, $q \le q_1$, and $S^n = d_n, d^n$ or δ_n. One then has the following two evident facts:

$$S^n\left(B_p^r(\mathbf{M}), L_q(\mathbf{M})\right) \le CS^n\left(B_{p_1}^r(\mathbf{M}), L_{q_1}(\mathbf{M})\right) \tag{4.1}$$

with C independent of n, while

$$S^n(b_p^Q, \ell_q^Q) \ge CS^n(b_{p_1}^Q, \ell_{q_1}^Q) \tag{4.2}$$

with C independent of n, Q.

By Lemma 3.1, we may choose $\nu > 0$ such that $P_{\nu n} \ge 2n$ for all sufficiently large n. In this proof we will always take $N = \nu n$. We consider the various ranges of p, q separately:

(1) $1 \le q \le p \le \infty$.
In this case, we note that if $S^n = d_n, d^n$ or δ_n, then by (4.1),

$$S^n\left(B_p^r(\mathbf{M}), L_q(\mathbf{M})\right) \ge CS^n\left(B_\infty^r(\mathbf{M}), L_1(\mathbf{M})\right). \tag{4.3}$$

On the other hand, if $s_n = d_n$ or d^n, then by (3.1) on page 410 of [15], $s_n(b_\infty^{P_N}, \ell_1^{P_N}) = P_N - n \ge n$. By this, (4.3) and Lemma 3.4, we find that

$$s_n\left(B_p^r(\mathbf{M}), L_q(\mathbf{M})\right) \gg n^{-\frac{r}{s}-1}n = n^{-\frac{r}{s}}$$

first for $s_n = d_n$ or d^n and then for δ_n, by (3.12). This completes the proof in this case.

(2) $1 \le p \le q \le 2$.
In this case, for the Gelfand widths we just observe, by (4.1), that

$$d^n\left(B_p^r(\mathbf{M}), L_q(\mathbf{M})\right) \ge Cd^n\left(B_p^r(\mathbf{M}), L_p(\mathbf{M})\right) \gg n^{-\frac{r}{s}} \tag{4.4}$$

by case 1. For the Kolmogorov widths we observe, by Lemma 3.4 and (4.2), that

$$\begin{aligned} d_n\left(B_p^r(\mathbf{M}), L_q(\mathbf{M})\right) &\gg n^{-\frac{r}{s}+\frac{1}{p}-\frac{1}{q}}d_n(b_p^{P_N}, \ell_q^{P_N}) \\ &\gg n^{-\frac{r}{s}+\frac{1}{p}-\frac{1}{q}}d_n(b_1^{P_N}, \ell_2^{P_N}) \gg n^{-\frac{r}{s}+\frac{1}{p}-\frac{1}{q}}, \end{aligned} \tag{4.5}$$

since, by (3.3) on page 411 of [15], $d_n(b_1^{P_N}, \ell_2^{P_N}) = \sqrt{1-n/P_N} \geq 1/\sqrt{2}$. Finally, for the linear widths, we have by (3.12), that

$$\delta_n\left(B_p^r(\mathbf{M}), L_q(\mathbf{M})\right) \gg n^{-\frac{r}{s}+\frac{1}{p}-\frac{1}{q}}.$$

This completes the proof in this case.

(3) $2 \leq p \leq q \leq \infty$.

In this case, for the Kolmogorov widths we just observe, by (3.10), that

$$d_n\left(B_p^r(\mathbf{M}), L_q(\mathbf{M})\right) \geq C d_n\left(B_p^r(\mathbf{M}), L_p(\mathbf{M})\right) \gg n^{-\frac{r}{s}} \tag{4.6}$$

by case 1. For the Gelfand widths we observe, by Lemma 3.4 and (4.2), that

$$\begin{aligned} d^n\left(B_p^r(\mathbf{M}), L_q(\mathbf{M})\right) &\gg n^{-\frac{r}{s}+\frac{1}{p}-\frac{1}{q}} d^n(b_p^{P_N}, \ell_q^{P_N}) \\ &\gg n^{-\frac{r}{s}+\frac{1}{p}-\frac{1}{q}} d^n(b_2^{P_N}, \ell_\infty^{P_N}) \gg n^{-\frac{r}{s}+\frac{1}{p}-\frac{1}{q}}, \end{aligned} \tag{4.7}$$

since, by (3.5) on page 412 of [15],

$$d^n(b_2^{P_N}, \ell_\infty^{P_N}) = \sqrt{1-n/P_N} \geq 1/\sqrt{2}.$$

Finally, for the linear widths, we have by (3.12), that

$$\delta_n\left(B_p^r(\mathbf{M}), L_q(\mathbf{M})\right) \gg n^{-\frac{r}{s}+\frac{1}{p}-\frac{1}{q}}.$$

This completes the proof in this case.

(4) $1 \leq p \leq 2 \leq q \leq \infty$.

If $1 \leq \alpha \leq \alpha_1 \leq \infty$, then by Hölder's inequality, for every $a = (a_1, \ldots, a_{P_N})$

$$\|a\|_\alpha \leq P_N^{\frac{1}{\alpha}-\frac{1}{\alpha_1}} \|a\|_{\alpha_1}. \tag{4.8}$$

This implies that

$$b_{\alpha_1}^{P_N} \subseteq P_N^{\frac{1}{\alpha_1}-\frac{1}{\alpha}} b_\alpha^{P_N}. \tag{4.9}$$

From Lemma 3.4, (4.2) and (4.8), we find that

$$\begin{aligned} d_n\left(B_p^r(\mathbf{M}), L_q(\mathbf{M})\right) &\gg n^{-\frac{r}{s}+\frac{1}{p}-\frac{1}{q}} d_n(b_p^{P_N}, \ell_q^{P_N}) \\ &\gg n^{-\frac{r}{s}+\frac{1}{p}-\frac{1}{q}} d_n(b_1^{P_N}, \ell_q^{P_N}) \gg n^{-\frac{r}{s}+\frac{1}{p}-\frac{1}{2}} d_n(b_1^{P_N}, \ell_2^{P_N}) \gg n^{-\frac{r}{s}+\frac{1}{p}-\frac{1}{2}}. \end{aligned} \tag{4.10}$$

From Lemma 3.4, (4.2) and (4.9), we find that

$$\begin{aligned} d^n\left(B_p^r(\mathbf{M}), L_q(\mathbf{M})\right) &\gg n^{-\frac{r}{s}+\frac{1}{p}-\frac{1}{q}} d^n(b_p^{P_N}, \ell_q^{P_N}) \\ &\gg n^{-\frac{r}{s}+\frac{1}{p}-\frac{1}{q}} d^n(b_p^{P_N}, \ell_\infty^{P_N}) \gg n^{-\frac{r}{s}+\frac{1}{2}-\frac{1}{q}} d^n(b_2^{P_N}, \ell_\infty^{P_N}) \gg n^{-\frac{r}{s}+\frac{1}{2}-\frac{1}{q}}. \end{aligned} \tag{4.11}$$

Finally, from (4.10), (4.11) and (3.12),

$$\delta_n\left(B_p^r(\mathbf{M}), L_q(\mathbf{M})\right) \gg max\left(n^{-\frac{r}{s}+\frac{1}{p}-\frac{1}{2}}, n^{-\frac{r}{s}+\frac{1}{2}-\frac{1}{q}}\right). \tag{4.12}$$

This completes the proof of our main Theorem 1.1.

Acknowledgment

In July of 2015 at a BIRS-CMO meeting in Oaxaca Professor B. Kashin asked the author if his results with D. Geller [8] about n-widths on homogeneous compact manifolds could be extended to general compact Riemannian manifolds. This paper gives a partial answer to that question. The author is grateful to Professor B. Kashin for stimulating his interest in this problem.

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