

## LAMPLIGHTER GROUPS AND VON NEUMANN'S CONTINUOUS REGULAR RING

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**ABSTRACT.** Let  $\Gamma$  be a discrete group. Following Linnell and Schick one can define a continuous ring  $c(\Gamma)$  associated with  $\Gamma$ . They proved that if the Atiyah Conjecture holds for a torsion-free group  $\Gamma$ , then  $c(\Gamma)$  is a skew field. Also, if  $\Gamma$  has torsion and the Strong Atiyah Conjecture holds for  $\Gamma$ , then  $c(\Gamma)$  is a matrix ring over a skew field. The simplest example when the Strong Atiyah Conjecture fails is the lamplighter group  $\Gamma = \mathbb{Z}_2 \wr \mathbb{Z}$ . It is known that  $\mathbb{C}(\mathbb{Z}_2 \wr \mathbb{Z})$  does not even have a classical ring of quotients. Our main result is that if  $H$  is amenable, then  $c(\mathbb{Z}_2 \wr H)$  is isomorphic to a continuous ring constructed by John von Neumann in the 1930s.

### 1. INTRODUCTION

Let us consider  $\text{Mat}_{k \times k}(\mathbb{C})$  the algebra of  $k$  by  $k$  matrices over the complex field. This ring is a unital  $*$ -algebra with respect to the conjugate transposes. For each element  $A \in \text{Mat}_{k \times k}(\mathbb{C})$  one can define  $A^*$  satisfying the following properties:

- $(\lambda A)^* = \bar{\lambda} A^*$ ,
- $(A + B)^* = A^* + B^*$ ,
- $(AB)^* = B^* A^*$ ,
- $0^* = 0, 1^* = 1$ .

Also, each element has a normalized rank  $\text{rk}(A) = \text{Rank}(A)/k$  with the following properties:

- $\text{rk}(0) = 0, \text{rk}(1) = 1$ ,
- $\text{rk}(A + B) \leq \text{rk}(A) + \text{rk}(B)$ ,
- $\text{rk}(AB) \leq \min\{\text{rk}(A), \text{rk}(B)\}$ ,
- $\text{rk}(A^*) = \text{rk}(A)$ ,
- if  $e$  and  $f$  are orthogonal idempotents, then  $\text{rk}(e + f) = \text{rk}(e) + \text{rk}(f)$ .

The ring  $\text{Mat}_{k \times k}(\mathbb{C})$  has an algebraic property; namely, von Neumann called regularity: Any principal left-(or right) ideal can be generated by an idempotent. Furthermore, among these generating idempotents there is a unique projection (that is,  $\text{Mat}_{k \times k}(\mathbb{C})$  is a  $*$ -regular ring). In a von Neumann regular ring any non-zerodivisor is necessarily invertible. One can also observe that the algebra of matrices is proper, that is,  $\sum_{i=1}^n a_i a_i^* = 0$  implies that all the matrices  $a_i$  are zero.

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One should note that if  $R$  is a  $*$ -regular ring with a rank function, then the rank extends to  $\text{Mat}_{k \times k}(R)$  [6], where the extended rank has the same property as  $\text{rk}$  except that the rank of the identity is  $k$ .

One can immediately see that the rank function defines a metric  $d(A, B) := \text{rk}(A - B)$  on any algebra with a rank, and the matrix algebra is complete with respect to this metric. These complete  $*$ -regular algebras are called continuous  $*$ -algebras (see [5] for an extensive study of continuous rings). Note that for the matrix algebras the possible values of the rank functions are  $0, 1/k, 2/k, \dots, 1$ . John von Neumann observed that there are some interesting examples of infinite dimensional continuous  $*$ -algebras, where the rank function can take any real values in between 0 and 1. His first example was purely algebraic.

**Example 1.** Let us consider the following sequence of diagonal embeddings:

$$\mathbb{C} \rightarrow \text{Mat}_{2 \times 2}(\mathbb{C}) \rightarrow \text{Mat}_{4 \times 4}(\mathbb{C}) \rightarrow \text{Mat}_{8 \times 8}(\mathbb{C}) \rightarrow \dots$$

One can observe that all the embeddings are preserving the rank and the  $*$ -operation. Hence the direct limit  $\varinjlim \text{Mat}_{2^k \times 2^k}(\mathbb{C})$  is a  $*$ -regular ring with a proper rank function. The addition, multiplication, the  $*$ -operation and the rank function can be extended to the metric completion  $\mathcal{M}$  of the direct limit ring. The resulting algebra  $\mathcal{M}$  is a simple, proper, continuous  $*$ -algebra, where the rank function can take all the values on the unit interval.

**Example 2.** Consider a finite, tracial von Neumann algebra  $\mathcal{N}$  with trace function  $\text{tr}_{\mathcal{N}}$ . Then  $\mathcal{N}$  is a  $*$ -algebra equipped with a rank function. If  $P$  is a projection, then  $\text{rk}_{\mathcal{N}}(P) = \text{tr}_{\mathcal{N}}(P)$ . For a general element  $A \in \mathcal{N}$ ,  $\text{rk}_{\mathcal{N}}(A) = 1 - \lim_{t \rightarrow \infty} \int_0^t \text{tr}_{\mathcal{N}}(E_{\lambda}) d\lambda$ , where  $\int_0^{\infty} E_{\lambda} d\lambda$  is the spectral decomposition of  $A^*A$ . In general,  $\mathcal{N}$  is not regular, but it has the Ore property with respect to its zero divisors. The Ore localization of  $\mathcal{N}$  with respect to its non-zero divisors is called the algebra of affiliated operators and denoted by  $U(\mathcal{N})$ . These algebras are also proper continuous  $*$ -algebras [1]. The rank of an element  $A \in U(\mathcal{N})$  is given by the trace of the projection generating the principal ideal  $U(\mathcal{N})A$ . It is important to note that  $U(\mathcal{N})$  is the rank completion of  $\mathcal{N}$  (Lemma 2.2, [12]).

Linnell and Schick observed [9] that if  $X$  is a subset of a proper  $*$ -regular algebra  $R$ , then there exists a smallest  $*$ -regular subalgebra containing  $X$ , the  $*$ -regular closure. Now let  $\Gamma$  be a countable group and  $\mathbb{C}\Gamma$  be its complex group algebra. Then one can consider the natural embedding of the group algebra to its group von Neumann algebra  $\mathbb{C}\Gamma \rightarrow \mathcal{N}\Gamma$ . Let  $U(\Gamma)$  denote the Ore localization of  $\mathcal{N}(\Gamma)$  and the embedding  $\mathbb{C}\Gamma \rightarrow U(\Gamma)$ . Since  $U(\Gamma)$  is a proper  $*$ -regular ring, one can consider the smallest  $*$ -algebra  $\mathcal{A}(\Gamma)$  in  $U(\Gamma)$  containing  $\mathbb{C}(\Gamma)$ . Let  $c(\Gamma)$  be the completion of the algebra  $\mathcal{A}$  above. It is a continuous  $*$ -algebra [5]. Of course, if the rank function has only finitely many values in  $\mathcal{A}$ , then  $c(\Gamma)$  equals  $\mathcal{A}(\Gamma)$ . Note that if  $\mathbb{C}\Gamma$  is embedded into a continuous  $*$ -algebra  $T$ , then one can still define  $c_T(\Gamma)$  as the smallest continuous ring containing  $\mathbb{C}\Gamma$ . In [3] we proved that if  $\Gamma$  is amenable,  $c(\Gamma) = c_T(\Gamma)$  for any embedding  $\mathbb{C}\Gamma \rightarrow T$  associated to sofic representations of  $\Gamma$ , hence  $c(\Gamma)$  can be viewed as a canonical object. Linnell and Schick calculated the algebra  $c(\Gamma)$  for several groups, where the rank function has only finitely many values on  $\mathcal{A}$ . They proved the following results:

- If  $\Gamma$  is torsion-free and the Atiyah Conjecture holds for  $\Gamma$ , then  $c(\Gamma)$  is a skew-field. This is the case when  $\Gamma$  is amenable and  $\mathbb{C}\Gamma$  is a domain. Then

$c(\Gamma)$  is the Ore localization of  $\mathbb{C}\Gamma$ . If  $\Gamma$  is the free group of  $k$  generators, then  $c(\Gamma)$  is the Cohen-Amitsur free skew field of  $k$  generators. The Atiyah Conjecture for a torsion-free group means that the rank of an element in  $\text{Mat}_{k \times k}(\mathbb{C}\Gamma) \subset \text{Mat}_{k \times k}(U(\mathcal{N}(\Gamma)))$  is an integer.

- If the orders of the finite subgroups of  $\Gamma$  are bounded and the Strong Atiyah Conjecture holds for  $\Gamma$ , then  $c(\Gamma)$  is a finite dimensional matrix ring over some skew-field. In this case the Strong Atiyah Conjecture means that the ranks of an element in  $\text{Mat}_{k \times k}(\mathbb{C}\Gamma) \subset \text{Mat}_{k \times k}(U(\mathcal{N}(\Gamma)))$  is in the abelian group  $\frac{1}{\text{lcm}(\Gamma)}\mathbb{Z}$ , where  $\text{lcm}(\Gamma)$  indicates the least common multiple of the orders of the finite subgroups of  $\Gamma$ .

The lamplighter group  $\Gamma = \mathbb{Z}_2 \wr \mathbb{Z}$  has finite subgroups of arbitrarily large orders. Also, although  $\Gamma$  is amenable,  $\mathbb{C}\Gamma$  does not satisfy the Ore condition with respect to its non-zero divisors [8]. In other words, it has no classical ring of quotients. The goal of this paper is to calculate  $c(\mathbb{Z}_2 \wr \mathbb{Z})$  and even  $c(\mathbb{Z}_2 \wr H)$ , where  $H$  is a countably infinite amenable group.

**Theorem 1.** *If  $H$  is a countably infinite amenable group, then  $c(\mathbb{Z}_2 \wr H)$  is the simple continuous ring  $\mathcal{M}$  of von Neumann.*

## 2. CROSSED PRODUCT ALGEBRAS

In this section we recall the notion of crossed product algebras and the group-measure space construction of Murray and von Neumann. Let  $\mathcal{A}$  be a unital, commutative  $*$ -algebra and  $\phi : \Gamma \rightarrow \text{Aut}(\mathcal{A})$  be a representation of the countable group  $\Gamma$  by  $*$ -automorphisms. The associated crossed product algebra  $\mathcal{A} \rtimes \Gamma$  is defined the following way. The elements of  $\mathcal{A} \rtimes \Gamma$  are the finite formal sums

$$\sum_{\gamma \in \Gamma} a_\gamma \cdot \gamma,$$

where  $a_\gamma \in \mathcal{A}$ . The multiplicative structure is given by

$$\delta \cdot a_\gamma = \phi(\delta)(a_\gamma) \cdot \delta.$$

The  $*$ -structure is defined by  $\gamma^* = \gamma^{-1}$  and  $(\gamma \cdot a)^* = a^* \cdot \gamma^{-1}$ . Note that

$$(\delta \cdot a_\gamma)^* = (\phi(\delta)a_\gamma \cdot \delta)^* = \delta^* \cdot \phi(\delta)a_\gamma^* = \phi(\delta^{-1})\phi(\delta)a_\gamma^* \cdot \delta^{-1} = a_\gamma^* \cdot \delta^*.$$

Now let  $(X, \mu)$  be a probability measure space and  $\tau : \Gamma \curvearrowright X$  be a measure preserving action of a countable group  $\Gamma$  on  $X$ . Then we have a  $*$ -representation  $\hat{\tau}$  of  $\Gamma$  in  $\text{Aut}(L^\infty(X, \mu))$ , where  $L^\infty(X, \mu)$  is the commutative  $*$ -algebra of bounded measurable functions on  $X$  (modulo zero measure perturbations)

$$\hat{\tau}(\gamma)(f)(x) = f(\tau(\gamma^{-1})(x)).$$

Let  $\mathcal{H} = l^2(\Gamma, L^2(X, \mu))$  be the Hilbert space of  $L^2(X, \mu)$ -valued functions on  $\Gamma$ . That is, each element of  $\mathcal{H}$  can be written in the form of

$$\sum_{\gamma \in \Gamma} b_\gamma \cdot \gamma,$$

where  $\sum_{\gamma \in \Gamma} \|b_\gamma\|^2 < \infty$ . Then we have a representation  $L$  of  $L^\infty(X, \mu) \rtimes \Gamma$  on  $l^2(\Gamma, L^2(X, \mu))$  by

$$L\left(\sum_{\gamma \in \Gamma} a_\gamma \cdot \gamma\right)\left(\sum_{\delta \in \Gamma} b_\delta \cdot \delta\right) = \sum_{\delta \in \Gamma} \left(\sum_{\gamma \in \Gamma} a_\gamma(\hat{\tau}(\gamma)(\beta_\delta)) \cdot \gamma\delta\right).$$

Note that  $L(\sum_{\gamma \in \Gamma} a_\gamma \cdot \gamma)$  is always a bounded operator. A trace is given on  $L^\infty(X, \mu) \rtimes \Gamma$  by

$$\text{Tr}(S) = \int_X a_1(x) d\mu(x).$$

The weak operator closure of  $L(L_c^\infty(X, \mu) \rtimes \Gamma)$  in  $B(l^2(\Gamma, L^2(X, \mu)))$  is the von Neumann algebra  $\mathcal{N}(\tau)$  associated to the action. Here  $L_c^\infty(X, \mu)$  denotes the subspace of functions in  $L^\infty(X, \mu)$  having only countable many values.

Note that one can extend  $\text{Tr}$  to  $\text{Tr}_{\mathcal{N}(\tau)}$  on the von Neumann algebra to make it a tracial von Neumann algebra.

We will denote by  $c(\tau)$  the smallest continuous algebra in  $U(\mathcal{N}(\tau))$  containing  $L_c^\infty(X, \mu) \rtimes \Gamma$ . One should note that the weak closure of  $L_c^\infty(X, \mu) \rtimes \Gamma$  in  $B(l^2(\Gamma, L^2(X, \mu)))$  is the same as the weak closure of  $L^\infty(X, \mu) \rtimes \Gamma$ . Hence our definition for the von Neumann algebra of an action coincides with the classical definition. On the other hand,  $c(L_c^\infty(X, \mu) \rtimes \Gamma)$  is smaller than  $c(L^\infty(X, \mu) \rtimes \Gamma)$ .

### 3. THE BERNOULLI ALGEBRA

Let  $H$  be a countable group. Consider the Bernoulli shift space  $B_H := \prod_{h \in H} \{0, 1\}$  with the usual product measure  $\nu_H$ . The probability measure preserving action  $\tau_H : H \curvearrowright (B_H, \nu_H)$  is defined by

$$\tau_H(\delta)(x)(h) = x(\delta^{-1}h),$$

where  $x \in B_H$ ,  $\delta, h \in H$ . Let  $\mathcal{A}_H$  be the commutative  $*$ -algebra of functions that depend only on finitely many coordinates of the shift space. It is well known that the Rademacher functions  $\{R_S\}_{S \subset H, |S| < \infty}$  form a basis in  $\mathcal{A}_H$ , where

$$R_S(x) = \prod_{\delta \in S} \exp(i\pi x(\delta)).$$

The Rademacher functions with respect to the pointwise multiplication form an abelian group isomorphic to  $\oplus_{h \in H} \mathbb{Z}_2$  the Pontrjagin dual of the compact group  $B_H$  satisfying

- $R_S R_{S'} = R_{S \Delta S'}$ ,
- $\int_{B_H} R_S d\nu = 0$ , if  $|S| > 0$ ,
- $R_\emptyset = 1$ .

The group  $H$  acts on  $\mathcal{A}_H$  by

$$\hat{\tau}_H(\delta)(f)(x) = f(\tau_H(\delta^{-1})(x)).$$

Hence,

$$\hat{\tau}_H(\delta)R_S = R_{\delta S}.$$

Therefore, the elements of  $\mathcal{A}_H \rtimes H$  can be uniquely written as in the form of the finite sums

$$\sum_{\delta} \sum_S c_{\delta,S} R_S \cdot \delta,$$

where  $\delta \cdot R_S = R_{\delta S} \cdot \delta$ .

Now let us turn our attention to the group algebra  $\mathbb{C}(\mathbb{Z}_2 \wr H)$ . For  $\delta \in H$ , let  $t_{\delta}$  be the generator in  $\sum_{h \in H} \mathbb{Z}_2$  belonging to the  $\delta$ -component. Any element of  $\mathbb{C}(\mathbb{Z}_2 \wr H)$  can be written in a unique way as a finite sum

$$\sum_{\delta} \sum_S c_{\delta,S} t_S \cdot \delta,$$

where  $t_S = \prod_{s \in S} t_s$ ,  $\delta \cdot t_S = t_{\delta S}$ ,  $t_S t_{S'} = t_{S \triangle S'}$ . Also note that

$$\text{Tr}(\sum_{\delta} \sum_S c_{\delta,S} t_S \cdot \delta) = c_{1, \emptyset}.$$

Hence we have the following proposition.

**Proposition 3.1.** *There exists a trace preserving \*-isomorphism*

$$\kappa : \mathbb{C}(\mathbb{Z}_2 \wr H) \rightarrow \mathcal{A}_H \rtimes H$$

such that

$$\kappa(\sum_{\delta} \sum_S c_{\delta,S} t_S \cdot \delta) = \sum_{\delta} \sum_S c_{\delta,S} R_S \cdot \delta.$$

Recall that if  $A \subset \mathcal{N}_1$ ,  $B \subset \mathcal{N}_1$  are weakly dense \*-subalgebras in finite tracial von Neumann algebras  $\mathcal{N}_1$  and  $\mathcal{N}_2$  and  $\kappa : A \rightarrow B$  is a trace preserving \*-homomorphism, then  $\kappa$  extends to a trace preserving isomorphism between the von Neumann algebras themselves (see e.g. [7, Corollary 7.1.9]). Therefore,  $\kappa : \mathbb{C}(\mathbb{Z}_2 \wr H) \rightarrow \mathcal{A}_H \rtimes H$  extends to a trace (and hence rank) preserving isomorphism between the von Neumann algebras  $\mathcal{N}(\mathbb{Z}_2 \wr H)$  and  $\mathcal{N}(\tau_H)$ .

**Proposition 3.2.** *For any countable group  $H$ ,*

$$c(\mathbb{Z}_2 \wr H) \cong c(\tau_H).$$

*Proof.* The rank preserving isomorphism  $\kappa : \mathcal{N}(\mathbb{Z}_2 \wr H) \rightarrow \mathcal{N}(\tau_H)$  extends to a rank preserving isomorphism between the rank completions, that is, the algebras of affiliated operators. It is enough to prove that the rank closure of  $\mathcal{A}_H \rtimes H$  is  $L_c^\infty(B_H, \nu_H) \rtimes H$ .  $\square$

**Lemma 3.1.** *Let  $f \in L_c^\infty(B_H, \nu_H)$ . Then  $\text{rk}_{\mathcal{N}(\tau_H)}(f) = \nu_H(\text{supp}(f))$ .*

*Proof.* By definition,

$$\text{rk}_{\mathcal{N}(\tau_H)}(f) = 1 - \lim_{\lambda \rightarrow 0} \text{tr}_{\mathcal{N}(\tau_H)} E_\lambda,$$

where  $E_\lambda$  is the spectral projection of  $f^* f$  corresponding to  $\lambda$  and

$$\text{tr}_{\mathcal{N}(\tau_H)} E_\lambda = \nu_H(\{x \mid |f^2(x)| \leq \lambda\}).$$

Hence,  $\text{rk}_{\mathcal{N}(\tau_H)}(f) = 1 - \nu_H(\{x \mid f^2(x) = 0\}) = \nu_H(\text{supp}(f))$ .  $\square$

Let  $\{m_n\}_{n=1}^\infty \subset \mathcal{A}_H$ ,  $m_n \xrightarrow{\text{rk}} m \in L_c^\infty(B_H, \nu_H)$ . Then  $m_n \cdot \gamma \xrightarrow{\text{rk}} m \cdot \gamma$ . Therefore our proposition follows from the lemma below.

**Lemma 3.2.**  *$\mathcal{A}_H$  is dense in  $L_c^\infty(B_H, \nu_H)$  with respect to the rank metric.*

*Proof.* By Lemma 3.1,  $L_{fin}^\infty(B_H, \nu_H)$  is dense in  $L_c^\infty(B_H, \nu_H)$ , where  $L_{fin}^\infty(B_H, \nu_H)$  is the  $*$ -algebra of functions taking only finitely many values. Recall that  $V \subset B_H$  is a basic set if  $1_V \in \mathcal{A}_H$ . It is well known that any measurable set in  $B_H$  can be approximated by basic sets, that is, for any  $U \subset B_H$ , there exists a sequence of basic sets  $\{V_n\}_{n=1}^\infty$  such that

$$(1) \quad \lim_{n \rightarrow \infty} \nu_H(V_n \triangle U) = 0.$$

By (1) and Lemma 3.1

$$\lim_{n \rightarrow \infty} \text{rk}_{\mathcal{N}(\tau_n)}(1_{V_n} - 1_U) = 0.$$

Let  $f = \sum_{m=1}^l c_m 1_{U_m}$ , where  $U_m$  are disjoint measurable sets. Let

$$\lim_{n \rightarrow \infty} \nu_H(V_n^m \triangle U_m) = 0,$$

where  $\{V_n^m\}_{n=1}^\infty$  are basic sets. Then

$$\lim_{n \rightarrow \infty} \text{rk}_{\mathcal{N}(\tau_n)}\left(\sum_{m=1}^l c_m 1_{V_n^m} - f\right) = 0.$$

Therefore,  $\mathcal{A}_H$  is dense in  $L_{fin}^\infty(B_H, \nu_H)$ .  $\square$

#### 4. THE ODOMETER ALGEBRA

The odometer algebra is constructed via the odometer action using the algebraic crossed product construction. Let us consider the compact group of 2-adic integers  $\hat{\mathbb{Z}}_{(2)}$ . Recall that  $\hat{\mathbb{Z}}_{(2)}$  is the completion of the integers with respect to the dyadic metric

$$d_{(2)}(n, m) = 2^{-k},$$

where  $k$  is the power of two in the prime factor decomposition of  $|m - n|$ . The group  $\hat{\mathbb{Z}}_{(2)}$  can be identified with the compact group of one way infinite sequences with respect to the binary addition.

The Haar measure  $\mu_{\text{haar}}$  on  $\hat{\mathbb{Z}}_{(2)}$  is defined by  $\mu_{\text{haar}}(U_n^l) = 1/2^n$ , where  $0 \leq l \leq 2^n - 1$  and  $U_n^l$  is the clopen subset of elements in  $\hat{\mathbb{Z}}_{(2)}$  having residue  $l$  modulo  $2^n$ . Let  $T$  be the addition map  $x \rightarrow x + 1$  in  $\hat{\mathbb{Z}}_{(2)}$ . The map  $T$  defines an action  $\rho: \mathbb{Z} \curvearrowright (\hat{\mathbb{Z}}_{(2)}, \mu_{\text{haar}})$ . The dynamical system  $(T, \hat{\mathbb{Z}}_{(2)}, \mu_{\text{haar}})$  is called the odometer action. As in Section 3, we consider the  $*$ -subalgebra of function  $\mathcal{A}_M$  in  $L^\infty(\hat{\mathbb{Z}}_{(2)}, \mu_{\text{haar}})$  that depends only on finitely many coordinates of  $\hat{\mathbb{Z}}_{(2)}$ . We consider a basis for  $\mathcal{A}_M$ . For  $n \geq 0$  and  $0 \leq l \leq 2^n - 1$  let

$$F_n^l(x) = \exp\left(\frac{2\pi i x(\text{mod } 2^n)}{2^n} l\right).$$

Notice that  $F_{n+1}^{2l} = F_n^l$ . Then the functions  $\{F_n^l\}_{n, l | (l, n)=1}$  form the Prüfer 2-group

$$\mathbb{Z}_{(2)} = \mathbb{Z}_1 \subset \mathbb{Z}_2 \subset \mathbb{Z}_4 \subset \mathbb{Z}_8 \subset \dots$$

with respect to the pointwise multiplication. The discrete group  $\mathbb{Z}_{(2)}$  is the Pontrjagin dual of the compact abelian group  $\hat{\mathbb{Z}}_{(2)}$ . The element  $F_n^1$  is the generator of the cyclic subgroup  $\mathbb{Z}_{2^n}$ . Note that

$$\int_{\hat{\mathbb{Z}}_{(2)}} F_n^l d\mu_{\text{haar}} = 0$$

except if  $l = 0, n = 0$ , when  $F_n^l \equiv 1$ . Observe that if  $k \in \mathbb{Z}$ , then

$$(2) \quad \rho(k)F_n^l = F_n^{l+k(\bmod 2^n)}$$

since  $F_n^l(x - k) = F_n^{l+k(\bmod 2^n)}(x)$ . Hence we have the following lemma.

**Lemma 4.1.** *The elements of  $\mathcal{A}_M \rtimes \mathbb{Z}$  can be uniquely written as finite sums in the form*

$$\sum_k \sum_{n \geq 0} \sum_{l|(l,n)=1} c_{n,l,k} F_n^l \cdot k,$$

where  $k \cdot F_n^l = F_n^{l+k(\bmod 2^n)}$  and  $F_0^0 = 1$ .

## 5. PERIODIC OPERATORS

**Definition 5.1.** A function  $\mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{C}$  is a periodic operator if there exists some  $n \geq 1$  such that

- $A(x, y) = 0$ , if  $|x - y| > 2^n$ ,
- $A(x, y) = A(x + 2^n, y + 2^n)$ .

Observe that the periodic operators form a  $*$ -algebra, where

- $(A + B)(x, y) = A(x, y) + B(x, y)$ ,
- $AB(x, y) = \sum_{z \in \mathbb{Z}} A(x, z)B(z, y)$ ,
- $A^*(x, y) = A(y, x)$ .

**Proposition 5.1.** *The algebra of periodic operators  $\mathcal{P}$  is  $*$ -isomorphic to a dense subalgebra of  $\mathcal{M}$ .*

*Proof.* We call  $A \in \mathcal{P}$  an element of type- $n$  if

- $A(x, y) = A(x + 2^n, y + 2^n)$ ,
- $A(x, y) = 0$  if  $0 \leq x \leq 2^n - 1, y > 2^n - 1$ ,
- $A(x, y) = 0$  if  $0 \leq x \leq 2^n - 1, y < 0$ .

Clearly, the elements of type- $n$  form an algebra  $\mathcal{P}_n$  isomorphic to  $\text{Mat}_{2^n \times 2^n}(\mathbb{C})$  and  $\mathcal{P}_n \rightarrow \mathcal{P}_{n+1}$  is the diagonal embedding. Hence, we can identify the algebra of finite type elements  $\mathcal{P}_f = \bigcup_{n=1}^{\infty} \mathcal{P}_n$  with  $\varinjlim \text{Mat}_{2^n \times 2^n}(\mathbb{C})$ .

For  $A \in \mathcal{P}$ , if  $n \geq 1$  is large enough, let  $A_n \in \mathcal{P}_n$  be defined the following way:

- $A_n(x, y) = A(x, y)$  if  $2^nl \leq x, y \leq 2^nl + 2^n - 1$  for some  $l \in \mathbb{Z}$ .
- Otherwise,  $A(x, y) = 0$ .

**Lemma 5.1.** (i):  $\{A_n\}_{n=1}^{\infty}$  is a Cauchy-sequence in  $\mathcal{M}$ .

(ii):  $(A + B)_n = A_n + B_n$ .

(iii):  $\text{rk}_{\mathcal{M}}(A_n^* - (A^*)_n) = 0$ .

(iv):  $\text{rk}_{\mathcal{M}}((AB)_n - A_n B_n) = 0$ .

(v):  $\lim_{n \rightarrow \infty} A_n = 0$  if and only if  $A = 0$ .

*Proof.* First observe that for any  $Q \in \mathcal{P}_n$

$$\text{rk}_{\mathcal{M}}(Q) \leq \frac{|\{0 \leq x \leq 2^n - 1 \mid \exists 0 \leq y \leq 2^n - 1 \text{ such that } A_n(x, y) \neq 0\}|}{2^n}.$$

Suppose that  $A(x, y) = A(x + 2^k, y + 2^k)$  and  $k < n < m$ . Then

$$|\{0 \leq x \leq 2^n - 1 \mid A_n(x, y) \neq A_m(x, y) \text{ for some } 0 \leq y \leq 2^n - 1\}| \leq 2^k 2^{m-n}.$$

Hence by the previous observation,  $\{A_n\}_{n=1}^{\infty}$  is a Cauchy sequence. Note that (iii) and (iv) can be proved similarly; the proof of (ii) is straightforward. In order to

prove (v) let us suppose that  $A(x, y) = 0$  whenever  $|x - y| \geq 2^k$ . Let  $n > k$  and  $0 \leq y \leq 2^k - 1$  such that  $A(x, y) \neq 0$  for some  $-2^k \leq x \leq 2^k - 1$ . Therefore  $\text{rk}_{\mathcal{M}} A_n \geq \frac{2^{n-k}-1}{2^n}$ . Thus (v) follows.  $\square$

Let us define  $\phi : \mathcal{P} \rightarrow \mathcal{M}$  by  $\phi(A) = \lim_{n \rightarrow \infty} A_n$ . By the previous lemma,  $\phi$  is an injective  $*$ -homomorphism.  $\square$

**Definition 5.2.** A periodic operator  $A$  is diagonal if  $A(x, y) = 0$ , whenever  $x \neq y$ . The diagonal operators form the abelian  $*$ -algebra  $\mathcal{D} \subset \mathcal{P}$ .

**Lemma 5.2.** We have the isomorphism  $\mathcal{D} \cong \mathbb{C}(\mathbb{Z}_{(2)})$ , where  $\mathbb{Z}_{(2)}$  is the Prüfer 2-group.

*Proof.* For  $n \geq 1$  and  $0 \leq l \leq 2^n - 1$  let  $E_n^l \in \mathcal{D}$  be defined by

$$E_n^l(x, x) := \exp\left(\frac{2\pi i x (\text{mod } 2^n)}{2^n} l\right).$$

It is easy to see that  $E_{n+1}^{2l} = E_n^l$  and the multiplicative group generated by  $E_n^1$  is isomorphic to  $\mathbb{Z}_{2^n}$ . Observe that the set  $\{E_n^l\}_{n,l,(l,n)=1}$  forms a basis in the space of  $n$ -type diagonal operators. Therefore,  $\mathcal{D} \cong \bigcup_{n=1}^{\infty} \mathbb{C}(Z_{2^n}) = \mathbb{C}(\mathbb{Z}_{(2)})$ .  $\square$

Let  $J \in \mathcal{P}$  be the following element:

- $J(x, y) = 1$ , if  $y = x + 1$ .
- Otherwise,  $J(x, y) = 0$ .

Then

$$(3) \quad J \cdot E_n^l = E_n^{l+1 (\text{mod } 2^n)}.$$

Also, any periodic operator  $A$  can be written in a unique way as a finite sum

$$\sum_{k \in \mathbb{Z}} D_k \cdot J^k,$$

where  $D_k$  is a diagonal operator in the form

$$D_k = \sum_{n=0}^{\infty} \sum_{l|(l,n)=1} c_{l,n,k} E_n^l.$$

Thus, by (2) and (3), we have the following corollary.

**Corollary 5.1.** The map  $\psi : \mathcal{P} \rightarrow \mathcal{A}_M \rtimes \mathbb{Z}$  defined by

$$\psi\left(\sum_k \sum_{n \geq 0} \sum_{l|(l,n)=1} c_{l,n,k} E_n^l \cdot k\right) = \sum_k \sum_{n \geq 0} \sum_{l|(l,n)=1} c_{l,n,k} F_n^l \cdot k$$

is a  $*$ -isomorphism of algebras.

## 6. LÜCK'S APPROXIMATION THEOREM REVISITED

The goal of this section is to prove the following proposition.

**Proposition 6.1.** We have  $c(\rho) \cong \mathcal{M}$  where  $\rho$  is the odometer action.

*Proof.* Let us define the linear map  $t : \mathcal{P} \rightarrow \mathbb{C}$  by

$$t(A) := \frac{\sum_{i=0}^{2^n-1} A(i, i)}{2^n},$$

where  $A \in \mathcal{P}$  and  $A(x + 2^n, y + 2^n)$  for all  $x, y \in \mathbb{Z}$ .  $\square$

**Lemma 6.1.**  $\text{Tr}_{\mathcal{N}(\rho)}(\psi(A)) = t(A)$ , where  $\psi$  is the  $*$ -isomorphism of Corollary 5.1.

*Proof.* Recall that  $\text{Tr}_{\mathcal{N}(\rho)}(F_n^l) = 0$ , except, when  $l = 0, n = 0, F_n^l = 1$ . If  $n \neq 0$  and  $l \neq 0$ , then  $t(E_n^l)$  is the sum of all  $k$ -th roots of unity for a certain  $k$ , hence  $t(E_n^l) = 0$ . Also,  $t(1) = 1$ . Thus, the lemma follows.  $\square$

It is enough to prove that

$$(4) \quad \text{rk}_{\mathcal{M}}(A) = \text{rk}_{\mathcal{N}(\rho)}(\psi(A)).$$

Indeed by (4),  $\psi$  is a rank-preserving  $*$ -isomorphism between  $\mathcal{P}$  and  $\mathcal{A}_M \rtimes \mathbb{Z}$ . Hence the isomorphism  $\psi$  extends to a metric isomorphism

$$\hat{\psi} : \overline{\mathcal{P}} \rightarrow \overline{\mathcal{A}_M \rtimes \mathbb{Z}},$$

where  $\overline{\mathcal{P}}$  is the closure of  $\mathcal{P}$  in  $\mathcal{M}$  and  $\overline{\mathcal{A}_M \rtimes \mathbb{Z}}$  is the closure of  $\mathcal{A}_M \rtimes \mathbb{Z}$  in  $U(\mathcal{N}(\rho))$ . Since  $\mathcal{P}$  is dense in  $\mathcal{M}$ ,  $\overline{\mathcal{P}} \cong \mathcal{M}$ . Also,  $\overline{\mathcal{A}_M \rtimes \mathbb{Z}}$  is a  $*$ -subalgebra of  $U(\mathcal{N}(\rho))$ , since the  $*$ -ring operations are continuous with respect to the rank metric. Therefore  $\overline{\mathcal{A}_M \rtimes \mathbb{Z}}$  is a continuous algebra isomorphic to  $\mathcal{M}$ . Observe that the rank closure  $\overline{\mathcal{A}_M \rtimes \mathbb{Z}}$  is isomorphic to the rank closure of  $L_c^\infty(\hat{\mathbb{Z}}_{(2)}, \mu_{\text{haar}}) \rtimes \mathbb{Z}$  by the argument of Lemma 3.2. Therefore,  $c(\rho) \cong \mathcal{M}$ . Thus from now on, our only goal is to prove (4).

**Lemma 6.2.** Let  $A \in \mathcal{P}$  and  $A_n \in \text{Mat}_{2^n \times 2^n}(\mathbb{C})$  as in Section 5. Then the matrices  $\{A_n\}_{n=1}^\infty$  have uniformly bounded norms.

*Proof.* Let  $M, N$  be chosen in such a way that

- $|A_n(x, y)| \leq M$  for any  $x, y \in \mathbb{Z}, n \geq 1$ .
- $|A_n(x, y)| = 0$  if  $|x - y| \geq \frac{N}{2}$ .

Now let  $v = (v(1), v(2), \dots, v(2^n)) \in \mathbb{C}^{2^n}$ ,  $\|v\|^2 = 1$ . Then

$$\begin{aligned} \|A_n v\|^2 &= \sum_{x=1}^{2^n} \left| \sum_{y: |x-y| < N/2} A_n(x, y) v(y) \right|^2 \leq M^2 \sum_{x=1}^{2^n} \left| \sum_{y: |x-y| < N/2} v(y) \right|^2 \\ &\leq M^2 N \sum_{x=1}^{2^n} \sum_{y: |x-y| < N/2} |v(y)|^2 \leq M^2 \sum_{y=1}^{2^n} N |v(y)|^2 = M^2 N^2. \end{aligned}$$

Therefore, for any  $n \geq 1$ ,  $\|A_n\| \leq MN$ .  $\square$

**Lemma 6.3.** Let  $A \in \mathcal{P}$ . Then for any  $k \geq 1$

$$\lim_{k \rightarrow \infty} t((A_n^* A_n)^k) = t((A^* A)^k) = \text{Tr}_{\mathcal{N}(\rho)}(\psi(A^* A)^k).$$

*Proof.* Let  $m \geq 1, l \geq 1, q \geq 1$  be integers such that

- $A(x, y) = A(x + 2^m, y + 2^m)$  for any  $x, y \in \mathbb{Z}$ .
- $A(x, y) = 0$ , if  $|x - y| \geq l$ .
- $|(A^* A)^k(x, x)| \leq q$  and  $|(A_n^* A_n)^k(x, x)| \leq q$  for any  $x \in \mathbb{Z}$ .

By definition,

$$\begin{aligned} t((A_n^* A_n)^k) &= \frac{\sum_{x=1}^{2^n} (A_n^* A_n)^k(x, x)}{2^n} \\ t((A^* A)^k) &= \frac{\sum_{x=1}^{2^n} (A^* A)^k(x, x)}{2^n}. \end{aligned}$$

Observe that if  $2lk < x, 2^n - 2lk$ , then

$$(A^*A)^k(x, x) = (A_n^*A_n)(x, x).$$

Hence,

$$|t((A^*A)^k) - t((A_n^*A_n)^k)| \leq \frac{4klq}{2^n}.$$

Thus our lemma follows.  $\square$

Now, we follow the idea of Lück [10]. Let  $\mu$  be the spectral measure of  $\psi(A) \in \mathcal{N}(\rho)$ . That is

$$\mathrm{Tr}_{\mathcal{N}(\rho)} f(A^*A) = \int_0^K f(x) d\mu(x),$$

for all  $f \in C[0, K]$ , where  $K > 0$  is chosen in such a way that  $\mathrm{Spec} \psi(A^*A) \subset [0, K]$  and  $\|A_n^*A_n\| \leq K$  for all  $n \geq 1$ . Also, let  $\mu_n$  be the spectral measure of  $A_n^*A_n$ , that is,

$$t(f(A_n^*A_n)) = \int_0^K f(x) d\mu_n(x),$$

or all  $f \in C[0, K]$ . As in [10], we can see that the measures  $\{\mu_n\}_{n=1}^\infty$  converge weakly to  $\mu$ . Indeed by Lemma 6.3,

$$\lim_{n \rightarrow \infty} t(P(A_n^*A_n)) = \mathrm{Tr}_{\mathcal{N}(\rho)} P(A^*A)$$

for any real polynomial  $P$ . Therefore,

$$\lim_{n \rightarrow \infty} t(f(A_n^*A_n)) = \mathrm{Tr}_{\mathcal{N}(\rho)} f(A^*A)$$

for all  $f \in C[0, K]$ .

Since  $\mathrm{rk}_{\mathcal{M}}(A_n) = \mathrm{rk}_{\mathcal{M}}(A_n^*A_n)$  and  $\mathrm{rk}_{\mathcal{N}(\rho)}(\psi(A)) = \mathrm{rk}_{\mathcal{N}(\rho)}(\psi(A^*A))$ , in order to prove (4) it is enough to see that

$$\lim_{n \rightarrow \infty} \mathrm{rk}_{\mathcal{M}}(A_n^*A_n) = \mathrm{rk}_{\mathcal{N}(\rho)}(\psi(A^*A)).$$

Observe that  $\mathrm{rk}_{\mathcal{M}}(A_n^*A_n) = 1 - \mu_n(0)$  and

$$\mathrm{rk}_{\mathcal{N}(\rho)}(\psi(A^*A)) = 1 - \lim_{\lambda \rightarrow 0} \mathrm{Tr}_{\mathcal{N}(\rho)} E_\lambda = \mu(0).$$

Hence, our proposition follows from the lemma below (an analogue of Lück's Approximation Theorem).

**Lemma 6.4.**  $\lim_{n \rightarrow \infty} \mu_n(0) = \mu(0)$ .

*Proof.* Let  $F_n(\lambda) = \int_0^\lambda \mu_n(t) dt$  and  $F(\lambda) = \int_0^\lambda \mu(t) dt$  be the distribution functions of our spectral measures. Since  $\{\mu_n\}_{n=1}^\infty$  weakly converges to the measure  $\mu$ , it is enough to show that  $\{F_n\}_{n=1}^\infty$  converges uniformly. Let  $n \leq m$  and  $D_m^n : \mathrm{Mat}_{2^n \times 2^n}(\mathbb{C}) \rightarrow \mathrm{Mat}_{2^m \times 2^m}(\mathbb{C})$  be the diagonal operator. Let  $\varepsilon > 0$ . By Lemma 5.1, if  $n, m$  are large enough,

$$\mathrm{Rank}(D_m^n(A_n) - A_m) \leq \varepsilon 2^m.$$

Hence, by Lemma 3.5 in [2],

$$\|F_n - F_m\|_\infty \leq \varepsilon.$$

Therefore,  $\{F_n\}_{n=1}^\infty$  converges uniformly.  $\square$

## 7. ORBIT EQUIVALENCE

First let us recall the notion of orbit equivalence. Let  $\tau_1 : \Gamma_1 \curvearrowright (X, \mu)$  resp.  $\tau_2 : \Gamma_2 \curvearrowright (Y, \nu)$  be essentially free probability measure preserving actions of the countably infinite groups  $\Gamma_1$  resp.  $\Gamma_2$ . The two actions are called orbit equivalent if there exists a measure preserving bijection  $\Psi : (X, \mu) \rightarrow (Y, \nu)$  such that for almost all  $x \in X$  and  $\gamma \in \Gamma_1$  there exists  $\gamma_x \in \Gamma_2$  such that

$$\tau_2(\gamma_x)(\Psi(x)) = \Psi(\tau_1(\gamma)(x)).$$

Feldman and Moore [4] proved that if  $\tau_1$  and  $\tau_2$  are orbit equivalent, then  $\mathcal{N}(\tau_1) \cong \mathcal{N}(\tau_2)$ . The goal of this section is to prove the following proposition.

**Proposition 7.1.** *If  $\tau_1$  and  $\tau_2$  are orbit equivalent actions, then  $c(\tau_1) \cong c(\tau_2)$ .*

Our Theorem 1 follows from the proposition. Indeed, by Proposition 3.2 and Proposition 6.1

$$\mathcal{M} \cong c(\rho) \quad \text{and} \quad c(\mathbb{Z}_2 \wr H) \cong c(\tau_H).$$

By the famous theorem of Ornstein and Weiss [11], the odometer action and the Bernoulli shift action of a countably infinite amenable group are orbit equivalent. Hence  $\mathcal{M} \cong c(\mathbb{Z}_2 \wr H)$ .  $\square$

*Proof.* We build the proof of our proposition on the original proof of Feldman and Moore. Let  $\gamma \in \Gamma_1$ ,  $\delta \in \Gamma_2$ . Let

$$M(\delta, \gamma) = \{y \in Y \mid \tau_2(\delta)(y) = \Psi(\tau_1(\gamma)\Psi^{-1}(y))\},$$

$$N(\gamma, \delta) = \{x \in X \mid \tau_1(\gamma)(x) = \Psi^{-1}(\tau_2(\delta)\Psi(x))\}.$$

Observe that  $\Psi(N(\delta, \gamma)) = M(\gamma, \delta)$ . Following Feldman and Moore ([4, Proposition 2.1]) for any  $\gamma \in \Gamma_1$ ,  $\delta \in \Gamma_2$

$$\kappa(\gamma) = \sum_{h \in \Gamma_2} h \cdot 1_{M(h, \gamma)}$$

and

$$\lambda(\delta) = \sum_{g \in \Gamma_1} g \cdot 1_{N(g, \delta)}$$

are well defined. That is,  $\sum_{n=1}^k h_n \cdot 1_{M(h_n, \gamma)}$  converges weakly to  $\kappa(\gamma) \in \mathcal{N}(\tau_2)$  as  $k \rightarrow \infty$  and  $\sum_{n=1}^k g_n \cdot 1_{N(g_n, \delta)}$  converges weakly to  $\lambda(\delta) \in \mathcal{N}(\tau_1)$  as  $k \rightarrow \infty$ , where  $\{\gamma_n\}_{n=1}^\infty$  resp.  $\{\delta_n\}_{n=1}^\infty$  are enumerations of the elements of  $\Gamma_1$  resp.  $\Gamma_2$ .

Furthermore, one can extend  $\kappa$  resp.  $\lambda$  to maps

$$\kappa' : L^\infty((X, \mu) \rtimes \Gamma_1) \rightarrow \mathcal{N}(\tau_2)$$

resp.

$$\lambda' : L^\infty((Y, \nu) \rtimes \Gamma_2) \rightarrow \mathcal{N}(\tau_1)$$

by

$$\kappa' \left( \sum_{\gamma \in \Gamma_1} a_\gamma \cdot \gamma \right) = \sum_{\gamma \in \Gamma_1} (a_\gamma \circ \Psi^{-1}) \cdot \kappa(\gamma) = \sum_{\gamma \in \Gamma_1} (a_\gamma \circ \Psi^{-1}) \cdot \sum_{n=1}^\infty h_n \cdot 1_{M(h_n, \gamma)}$$

and

$$\lambda' \left( \sum_{\delta \in \Gamma_2} b_\delta \cdot \delta \right) = \sum_{\delta \in \Gamma_2} (b_\delta \circ \Psi) \cdot \lambda(\delta) = \sum_{\delta \in \Gamma_2} (b_\delta \circ \Psi) \cdot \sum_{n=1}^\infty g_n \cdot 1_{N(g_n, \delta)}.$$

The maps  $\kappa'$  resp.  $\lambda'$  are injective trace-preserving  $*$ -homomorphisms with weakly dense ranges. Hence they extend to isomorphisms of von Neumann algebras

$$\hat{\kappa} : \mathcal{N}(\tau_1) \rightarrow \mathcal{N}(\tau_2), \hat{\lambda} : \mathcal{N}(\tau_2) \rightarrow \mathcal{N}(\tau_1),$$

where  $\hat{\kappa}$  and  $\hat{\lambda}$  are, in fact, the inverses of each other.

**Lemma 7.1.**

$$(5) \quad \lim_{k \rightarrow \infty} \operatorname{rk}_{\mathcal{N}(\tau_2)} \left( \sum_{\gamma \in \Gamma_1} (a_\gamma \circ \Psi^{-1}) \cdot \sum_{n=1}^k h_n \cdot 1_{M(h_n, \gamma)} - \hat{\kappa} \left( \sum_{\gamma \in \Gamma_1} a_\gamma \cdot \gamma \right) \right) = 0,$$

$$(6) \quad \lim_{k \rightarrow \infty} \operatorname{rk}_{\mathcal{N}(\tau_1)} \left( \sum_{\delta \in \Gamma_2} (b_\delta \circ \Psi) \cdot \sum_{n=1}^k g_n \cdot 1_{N(g_n, \delta)} - \hat{\lambda} \left( \sum_{\delta \in \Gamma_2} b_\delta \cdot \delta \right) \right) = 0.$$

*Proof.* By definition, the disjoint union  $\bigcup_{n=1}^\infty M(h_n, \gamma)$  equals  $Y$  (modulo a set of measure zero). We need to show that if  $\{\sum_{n=1}^k T_n \cdot 1_{M(h_n, \gamma)}\}_{k=1}^\infty$  weakly converges to an element  $S \in \mathcal{N}(\tau_2)$ , then  $\{\sum_{n=1}^k T_n \cdot 1_{M(h_n, \gamma)}\}_{k=1}^\infty$  converges to  $S$  in the rank metric as well, where  $T_n \in L_c^\infty(Y, \nu) \rtimes \Gamma_2$ . Let  $P_k = \sum_{n=1}^k 1_{M(h_n, \gamma)} \in l^2(\Gamma, L^2(Y, \nu))$ . We denote by  $\hat{P}_k$  the element  $\sum_{n=1}^k 1_{M(h_n, \gamma)}$  in  $L_c^\infty(Y, \nu) \rtimes \Gamma_2$ . By definition, if  $L(A)(P_k) = 0$ , then  $A\hat{P}_k = 0$ . Now, by weak convergence,

$$L(S)(P_k) = \lim_{l \rightarrow \infty} \sum_{n=1}^l T_n \cdot 1_{M(h_n, \gamma)}(P_k).$$

That is,

$$L(S - \sum_{n=1}^k T_n \cdot 1_{M(h_n, \gamma)})(P_k) = 0.$$

Therefore,

$$(S - \sum_{n=1}^k T_n \cdot 1_{M(h_n, \gamma)})\hat{P}_k = 0.$$

Thus,

$$(S - \sum_{n=1}^k T_n \cdot 1_{M(h_n, \gamma)}) = (S - \sum_{n=1}^k T_n \cdot 1_{M(h_n, \gamma)})(1 - \hat{P}_k).$$

By Lemma 3.1,  $\operatorname{rk}_{\mathcal{N}(\tau_2)}(1 - \hat{P}_k) = 1 - \sum_{n=1}^k \nu(M(h_n, \gamma))$ , hence

$$\lim_{k \rightarrow \infty} \operatorname{rk}_{\mathcal{N}(\tau_2)}(S - \sum_{n=1}^k T_n \cdot 1_{M(h_n, \gamma)}) = 0.$$

□

Now let us turn back to the proof of our proposition. By (5),  $\hat{\kappa}$  maps the algebra  $L_c^\infty(X, \mu) \rtimes \Gamma_1$  into the rank closure of  $L_c^\infty(Y, \nu) \rtimes \Gamma_2$ . Since  $\hat{\kappa}$  preserves the rank,  $\hat{\kappa}$  maps the rank closure of  $L_c^\infty(X, \mu) \rtimes \Gamma_1$  into the rank closure of  $L_c^\infty(Y, \nu) \rtimes \Gamma_2$ . Similarly,  $\hat{\lambda}$  maps the rank closure of  $L_c^\infty(Y, \nu) \rtimes \Gamma_2$  into the rank closure of  $L_c^\infty(X, \mu) \rtimes \Gamma_1$ . That is,  $\hat{\kappa}$  provides an isomorphism between the rank closures of  $L_c^\infty(X, \mu) \rtimes \Gamma_1$  and  $L_c^\infty(Y, \nu) \rtimes \Gamma_2$ . Therefore, the smallest continuous ring containing  $L_c^\infty(X, \mu) \rtimes \Gamma_1$  in  $U(\mathcal{N}(\tau_1))$  is mapped to the smallest continuous ring containing  $L_c^\infty(Y, \nu) \rtimes \Gamma_2$  in  $U(\mathcal{N}(\tau_2))$ . □

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