ON ASSOCIATED GRADED MODULES HAVING A PURE RESOLUTION

TONY J. PUTHENPURAKAL

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ABSTRACT. Let $A = K[[X_1, \dots, X_n]]$ and let $\mathfrak{m} = (X_1, \dots, X_n)$. Let M be a Cohen-Macaulay A -module of codimension p . In this paper we give a necessary and sufficient condition for the associated graded module $G_{\mathfrak{m}}(M)$ to have a pure resolution over the polynomial ring $G_{\mathfrak{m}}(A) \cong K[X_1, \dots, X_n]$.

1. INTRODUCTION

Let K be a field. Let $R = K[X_1, \dots, X_n]$ and let M be a finitely generated graded R -module of projective dimension p . Recall that M has a *pure resolution* of type $(d_0, d_1, \dots, d_p)$ if the minimal resolution of M is of the following form:

$$0 \rightarrow R(-d_p)^{\beta_p} \rightarrow \dots \rightarrow R(-d_2)^{\beta_2} \rightarrow R(-d_1)^{\beta_1} \rightarrow R(-d_0)^{\beta_0} \rightarrow 0.$$

Herzog and Kühn showed in [4] that the Betti numbers of a pure resolution of a Cohen-Macaulay algebra are determined by its type and Huneke and Miller computed in [6] the multiplicity of such an algebra, also in terms of its type. These two results can be extended to Cohen-Macaulay R -modules; see [1, p. 88]. In the beautiful paper [1], Boij and Söderberg conjectured that the Betti diagram of any Cohen-Macaulay R -module is a non-negative linear combination of pure diagrams; furthermore, any pure diagram is a rational multiple of the Betti diagram of some Cohen-Macaulay R -module. One of their motivations was to prove the so-called multiplicity conjecture of Herzog, Huneke and Srinivasan (see [5], Conjecture 1 and 2), which they proved follows from their conjectures. These conjectures were proved in [2].

Let $A = K[[X_1, \dots, X_n]]$ and let $\mathfrak{m} = (X_1, \dots, X_n)$. Let $G_{\mathfrak{m}}(A)$ be the associated graded ring of A with respect to $\mathfrak{m}$, i.e., $G_{\mathfrak{m}}(A) = \bigoplus_{n \geq 0} \mathfrak{m}^n / \mathfrak{m}^{n+1}$. It is well known that $G_{\mathfrak{m}}(A) \cong K[X_1, \dots, X_n]$. Let M be a finitely generated A -module and let $G_{\mathfrak{m}}(M) = \bigoplus_{n \geq 0} \mathfrak{m}^n M / \mathfrak{m}^{n+1} M$ be the associated graded module of M with respect to $\mathfrak{m}$. A natural question is when does $G_{\mathfrak{m}}(M)$ have a pure resolution? To answer this question we construct a pure complex attached to M as follows:

Construction 1.1. (1) Let $\phi: A^n \rightarrow A^m$ be a non-zero A -linear map. Assume $\text{image } \phi \subseteq \mathfrak{m}^s A^m$ but $\text{image } \phi \not\subseteq \mathfrak{m}^{s+1} A^m$ for some $s \geq 0$. Let $\phi = (a_{ij})$ where $a_{ij} \in A$. By assumption $a_{ij} \in \mathfrak{m}^s$. It follows that

$$\phi = \sum_{j \geq s} \phi_j,$$

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where ϕ_j is a matrix with entries homogeneous forms of degree j . Set $\text{in}(\phi) = \phi_s$. We call $\text{in}(\phi)$ the *initial form* of ϕ . Set $v(\phi) = s$, the *order* of ϕ .

Example 1.2. Let

$$\phi = \begin{pmatrix} X_1^2 + X_2^2 X_3 & 0 & X_3^2 \\ 0 & X_2^4 & X_1 X_3 + X_2^3 \\ X_1^3 & 0 & X_1 X_2 X_3 + X_2^4 \end{pmatrix}.$$

Then

$$\text{in}(\phi) = \begin{pmatrix} X_1^2 & 0 & X_3^2 \\ 0 & 0 & X_1 X_3 \\ 0 & 0 & 0 \end{pmatrix}.$$

Also $v(\phi) = 2$.

(2) Let

$$\mathbb{F}: 0 \rightarrow F_p \xrightarrow{\phi_p} F_{p-1} \rightarrow \cdots \rightarrow F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \rightarrow 0$$

be a minimal resolution of M . Set $c_i = v(\phi_i)$ for $i = 1, 2, \dots, p$. Let $d_i = \sum_{j=1}^i c_j$ for $i = 1, 2, \dots, p$. Let $\beta_i = \text{rank } F_i$ be the i^{th} Betti number of M . It is easily shown (see 2.3) that we have a complex, $\text{in}(\mathbb{F})$,

$$0 \rightarrow R(-d_p)^{\beta_p} \xrightarrow{\text{in}(\phi_p)} R(-d_{p-1})^{\beta_{p-1}} \rightarrow \cdots \rightarrow R(-d_1)^{\beta_1} \xrightarrow{\text{in}(\phi_1)} R^{\beta_0} \rightarrow 0.$$

We also have an augmentation map $\epsilon: \text{in}(\mathbb{F}) \rightarrow G_{\mathfrak{m}}(M)$, i.e., an R -linear map $\epsilon: R^{\beta_0} \rightarrow G_{\mathfrak{m}}(M)$ such that $\epsilon \circ \text{in}(\phi_1) = 0$. Furthermore, ϵ is surjective. If $\mathbb{G}$ is another minimal resolution of M , then it can be easily shown that $\text{in}(\mathbb{F}) \cong \text{in}(\mathbb{G})$ as augmented complexes; see 2.3.

Our first result is:

Theorem 1.3. *Let M be a finitely generated A -module. Assume $G_{\mathfrak{m}}(M)$ has a pure resolution. Let $\mathbb{F}$ be a minimal free resolution of M . Then $\text{in}(\mathbb{F})$ is a minimal free resolution of $G_{\mathfrak{m}}(M)$.*

Theorem 1.3 gives a necessary condition for $G_{\mathfrak{m}}(M)$ to have a pure resolution. The following result gives a sufficient condition for $G_{\mathfrak{m}}(M)$ to have a pure resolution.

Theorem 1.4. *Let M be a Cohen-Macaulay A -module and let $p = \text{projdim } M$. Let $\beta_i = \beta_i(M)$. Let $\mathbb{F}$ be a minimal resolution of M . The following conditions are equivalent:*

- (i) $G_{\mathfrak{m}}(M)$ has a pure resolution.
- (ii) The following hold:
 - (a) $\text{in}(\mathbb{F})$ is acyclic.
 - (b) For $i \geq 1$,

$$\beta_i = (-1)^{i+1} \beta_0 \prod_{\substack{1 \leq j \leq p \\ j \neq i}} \frac{d_j}{d_j - d_i}.$$

- (c) The multiplicity of M ,

$$e_0(M) = \frac{\beta_0}{p!} \prod_{i=1}^p d_i.$$

If $M = A/I$, then using a computer algebra program one can find $G_{\mathfrak{m}}(M)$. However if M is not a cyclic module, there is no computer algebra program to find a presentation of $G_{\mathfrak{m}}(M)$ as an R -module. Our motivation for this work was to find non-trivial examples of a minimal resolution of $G_{\mathfrak{m}}(M)$. Even when $p = \text{projdim } M = 1, 2$ this is a non-trivial problem even when $G_{\mathfrak{m}}(M)$ is Cohen-Macaulay.

Here is an overview of the contents of the paper. In Section 2 we give our construction of the complex $\text{in}(\mathbb{F})$. In Section 3 we prove Theorem 1.3. In the next section we prove Theorem 1.4. Finally in Section 5 we give a non-trivial example of $G_{\mathfrak{m}}(M)$ having a pure resolution.

2. CONSTRUCTION OF THE COMPLEX $\text{in}(\mathbb{F})$

Throughout, $A = K[[X_1, \dots, X_n]]$ and let $\mathfrak{m}$ be the maximal ideal of A . All A -modules considered will be finitely generated. Let $R = G_{\mathfrak{m}}(A) = \bigoplus_{n \geq 0} \mathfrak{m}^n / \mathfrak{m}^{n+1}$ be the associated graded ring of A . It is well known that $R \cong K[\bar{X}_1, \dots, \bar{X}_n]$. Let M be an A -module and let $G_{\mathfrak{m}}(M) = \bigoplus_{n \geq 0} \mathfrak{m}^n M / \mathfrak{m}^{n+1} M$ be the associated graded module of M . Clearly $G_{\mathfrak{m}}(M)$ is a finitely generated R -module. Let $\mathbb{F}$ be a minimal resolution of M . In this section we construct our complex $\text{in}(\mathbb{F})$ of R -modules. We also show that there is an augmentation $\epsilon: \text{in}(\mathbb{F}) \rightarrow G_{\mathfrak{m}}(M)$ with ϵ surjective.

Proposition 2.1. *Let $F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0$ be a complex of free A -modules. Let $v(\phi_1) = c_1$ and $v(\phi_2) = c_2$. Set $d_1 = c_1$ and $d_2 = c_1 + c_2$. Then*

(1) *we have a complex of $R = G_{\mathfrak{m}}(A)$ -modules*

$$G_{\mathfrak{m}}(F_2)(-d_2) \xrightarrow{\text{in}(\phi_2)} G_{\mathfrak{m}}(F_1)(-d_1) \xrightarrow{\text{in}(\phi_1)} G_{\mathfrak{m}}(F_0).$$

(2) *Let $M = \text{coker } \phi_1$. Consider the natural map $\epsilon: G_{\mathfrak{m}}(F_0) \rightarrow G_{\mathfrak{m}}(M)$. Then*

(a) $\epsilon \circ \text{in}(\phi_1) = 0$.

(b) ϵ is surjective.

Proof. If E is an A -module, then set $\mathfrak{m}^i E = E$ for $i \leq 0$. Notice that for all $i \in \mathbb{Z}$ we have a complex

$$\mathfrak{m}^{i-d_2} F_2 \rightarrow \mathfrak{m}^{i-d_1} F_1 \rightarrow \mathfrak{m}^i F_0 \rightarrow \mathfrak{m}^i M \rightarrow 0.$$

After tensoring with $A/\mathfrak{m}$ and collecting terms we have a complex

$$G_{\mathfrak{m}}(F_2)(-d_2) \xrightarrow{\alpha} G_{\mathfrak{m}}(F_1)(-d_1) \xrightarrow{\beta} G_{\mathfrak{m}}(F_0) \xrightarrow{\epsilon} G_{\mathfrak{m}}(M) \rightarrow 0.$$

Observe that $\alpha = \text{in}(\phi_2)$ and $\beta = \text{in}(\phi_1)$. Also clearly ϵ is surjective. $\square$

Next we show

Proposition 2.2. *Let $\mathbb{F}: F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0$ and $\mathbb{G}: G_2 \xrightarrow{\psi_2} G_1 \xrightarrow{\psi_1} G_0$ be complexes of free A -modules. Suppose we have a commutative diagram*

$$\begin{array}{ccccc} F_2 & \xrightarrow{\phi_2} & F_1 & \xrightarrow{\phi_1} & F_0 \\ \downarrow \theta_2 & & \downarrow \theta_1 & & \downarrow \theta_0 \\ G_2 & \xrightarrow{\psi_2} & G_1 & \xrightarrow{\psi_1} & G_0 \end{array}$$

such that θ_i are isomorphisms for $i = 0, 1, 2$. Then

- (1) $v(\phi_i) = v(\psi_i)$ for $i = 1, 2$.
- (2) Let $v(\phi_1) = c_1$ and $v(\phi_2) = c_2$. Set $d_1 = c_1$ and $d_2 = c_1 + c_2$. Then we have a commutative diagram of $R = G_{\mathfrak{m}}(A)$ -modules

$$\begin{array}{ccccc} G_{\mathfrak{m}}(F_2)(-d_2) & \xrightarrow{\text{in}(\phi_2)} & G_{\mathfrak{m}}(F_1)(-d_1) & \xrightarrow{\text{in}(\phi_1)} & G_{\mathfrak{m}}(F_0) \\ \downarrow \text{in}(\theta_2) & & \downarrow \text{in}(\theta_1) & & \downarrow \text{in}(\theta_0) \\ G_{\mathfrak{m}}(G_2)(-d_2) & \xrightarrow{\text{in}(\psi_2)} & G_{\mathfrak{m}}(G_1)(-d_1) & \xrightarrow{\text{in}(\psi_1)} & G_{\mathfrak{m}}(G_0) \end{array}$$

such that

- (a) The rows are complexes of R -modules.
- (b) $\text{in}(\theta_i)$ is an isomorphism for $i = 0, 1, 2$.
- (3) Set $M = \text{coker } \phi_1$ and $M' = \text{coker } \phi_2$ and let $\xi: M \rightarrow M'$ be the isomorphism induced by the above commutative diagram. Then we have a commutative diagram

$$\begin{array}{ccccc} G_{\mathfrak{m}}(F_0) & \xrightarrow{\epsilon} & G_{\mathfrak{m}}(M) & \longrightarrow & 0 \\ \downarrow \text{in}(\theta_0) & & \downarrow G_{\mathfrak{m}}(\xi) & & \\ G_{\mathfrak{m}}(G_0) & \xrightarrow{\epsilon'} & G_{\mathfrak{m}}(M') & \longrightarrow & 0 \end{array}$$

Proof. (0) Let $\delta: A^m \rightarrow A^m$ be an invertible A -linear map. We write $\delta = \sum_{j \geq 0} \delta_j$ where δ_j is an $m \times m$ matrix of forms of degree j . As $\delta \otimes A/\mathfrak{m}: k^m \rightarrow k^m$ is an isomorphism, we get that δ_0 is an invertible matrix. Also notice $\delta_0 = \text{in}(\delta)$.

(1) and (2): Let $r = v(\phi_1)$. We write $\phi_1 = \sum_{j \geq r} \phi_{1,j}$ where $\phi_{1,j}$ is a matrix of forms of degree j . Let $s = v(\psi_1)$. Write $\psi_1 = \sum_{j \geq s} \psi_{1,j}$ where $\psi_{1,j}$ is a matrix of forms of degree j . For $i = 0, 1$ write $\theta_i = \sum_{j \geq 0} \theta_{i,j}$ as before. As $\theta_0 \circ \phi_1 = \psi_1 \circ \theta_1$, we get that $\theta_{0,0} \circ \phi_{1,r} = \psi_{1,s} \circ \theta_{1,0}$. By (0) we get that $\theta_{0,0}$ and $\theta_{1,0}$ are invertible matrices of constants. So $r = s$. We also get $\text{in}(\theta_0) \circ \text{in}(\phi_1) = \text{in}(\psi_1) \circ \text{in}(\theta_1)$. Similarly we get $v(\phi_2) = v(\psi_2)$ and $\text{in}(\theta_1) \circ \text{in}(\phi_2) = \text{in}(\psi_2) \circ \text{in}(\theta_2)$.

2(a) This follows from Proposition 2.1.

2(b) This follows from (0).

(3) This is obvious. □

Construction 2.3. Let

$$\mathbb{F}: 0 \rightarrow F_p \xrightarrow{\phi_p} F_{p-1} \rightarrow \cdots \rightarrow F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \rightarrow 0$$

be a minimal resolution of M . Set $c_i = v(\phi_i)$ for $i = 1, 2, \dots, p$. Let $d_i = \sum_{j=1}^i c_j$ for $i = 1, 2, \dots, p$. Let $\beta_i = \text{rank } F_i$ be the i^{th} Betti number of M . Note that $R_i^\beta = G_{\mathfrak{m}}(F_i)$. By Proposition 2.1 it follows that we have a complex, $\text{in}(\mathbb{F})$,

$$0 \rightarrow R(-d_p)^{\beta_p} \xrightarrow{\text{in}(\phi_p)} R(-d_{p-1})^{\beta_{p-1}} \rightarrow \cdots \rightarrow R(-d_1)^{\beta_1} \xrightarrow{\text{in}(\phi_1)} R^{\beta_0} \rightarrow 0.$$

By Proposition 2.1 we also have an augmentation map $\epsilon: \text{in}(\mathbb{F}) \rightarrow G_{\mathfrak{m}}(M)$, i.e., an R -linear map $\epsilon: R^{\beta_0} \rightarrow G_{\mathfrak{m}}(M)$ such that $\epsilon \circ \text{in}(\phi_1) = 0$. Furthermore, ϵ is clearly surjective. If $\mathbb{G}$ is another minimal resolution of M , then by Proposition 2.2 it follows that we have an isomorphism of augmented complexes $\text{in}(\mathbb{F})$ and $\text{in}(\mathbb{G})$.

Remark 2.4. We should note that our construction of $\text{in}(\mathbb{F})$ is related to the construction of the linear part of a resolution. This was first constructed by Serre. See [3] for a recent exposition.

3. PROOF OF THEOREM 1.3

In this section we give a proof of Theorem 1.3. We first prove:

Lemma 3.1. *Let $F_1 \xrightarrow{\phi} F_0 \rightarrow M \rightarrow 0$ be part of a minimal resolution of M . Assume that the minimal resolution of $G_{\mathfrak{m}}(M)$ has the following form:*

$$\cdots \rightarrow R^a(-s) \rightarrow G_{\mathfrak{m}}(F_0) \rightarrow G_{\mathfrak{m}}(M) \rightarrow 0.$$

Set $N = \text{image } \phi$ and $\mathcal{F} = \{N_i = \mathfrak{m}^i F_0 \cap N\}_{i \in \mathbb{Z}}$. Then

- (1) $N_i = \mathfrak{m}^{i-s} N$ for all $i \geq 0$.
- (2) $\text{rank } F_1 = a$.
- (3) $v(\phi) = s$.
- (4) The sequence $G_{\mathfrak{m}}(F_1)(-s) \xrightarrow{\text{in}(\phi)} G_{\mathfrak{m}}(F_0) \rightarrow G_{\mathfrak{m}}(M) \rightarrow 0$ can be extended to a minimal resolution of $G_{\mathfrak{m}}(M)$.
- (5) $\text{image } \text{in}(\phi) \cong G_{\mathfrak{m}}(N)(-s)$.

Proof. (1) It is well known that $\mathcal{F}$ is an $\mathfrak{m}$ -stable filtration on N . Furthermore, we have an exact sequence

$$(*) \quad 0 \rightarrow G_{\mathcal{F}}(N) \rightarrow G_{\mathfrak{m}}(F_0) \rightarrow G_{\mathfrak{m}}(M) \rightarrow 0.$$

By our assumption it follows that $G_{\mathcal{F}}(N)$ is generated in degree s . So we have $N_s = N$ and $N_{s+j} = \mathfrak{m}^j N + N_{s+j+1}$ for all $j \geq 1$. As $\mathcal{F}$ is $\mathfrak{m}$ -stable, there exists j_0 such that $N_{s+j+1} = \mathfrak{m} N_{s+j}$ for all $j \geq j_0$. Fix $j \geq j_0$. Then $N_{s+j} = \mathfrak{m}^j N + N_{s+j+1} = \mathfrak{m}^j N + \mathfrak{m} N_{s+j}$. So by Nakayama's Lemma $N_{s+j} = \mathfrak{m}^j N$. We now show by descending induction that $N_{s+j} = \mathfrak{m}^j N$ for all $j \leq j_0$. This is true for $j = j_0$ by the previous argument. Assume $N_{s+j+1} = \mathfrak{m}^{j+1} N$ for some $j \leq j_0 - 1$. Then notice

$$\mathfrak{m}^{j+1} N \subseteq \mathfrak{m} N_{s+j} \subseteq N_{s+j+1} = \mathfrak{m}^{j+1} N.$$

So we have $N_{s+j+1} = \mathfrak{m} N_{s+j}$. As $N_{s+j} = \mathfrak{m}^j N + N_{s+j+1} = \mathfrak{m}^j N + \mathfrak{m} N_{s+j}$, by Nakayama's Lemma we get that $N_{s+j} = \mathfrak{m}^j N$.

(2) As $R^a(-s) \rightarrow G_{\mathfrak{m}}(N)(-s) \rightarrow 0$ is minimal, we have $a = \mu(G_{\mathfrak{m}}(N)) = \mu(N) = \text{rank } F_1$.

(3) Set $r = v(\phi)$. By 2.1 we have a complex

$$G_{\mathfrak{m}}(F_1)(-r) \xrightarrow{\text{in}(\phi)} G_{\mathfrak{m}}(F_0) \xrightarrow{\epsilon} G_{\mathfrak{m}}(M) \rightarrow 0.$$

So $\ker \epsilon$ contains an element of degree r . So $s \leq r$. Furthermore, note that $N \subseteq \mathfrak{m}^r F_0$. So $N_j = \mathfrak{m}^j F_0 \cap N = N$ for $j \leq r$. It follows that $s \geq r$. Thus $s = r$.

(4) We first show that the complex is exact.

$$(**) \quad \mathfrak{m}^{i-s} F_1 \xrightarrow{\phi_{i-s}} \mathfrak{m}^i F_1 \xrightarrow{\epsilon_i} \mathfrak{m}^i M \rightarrow 0 \quad \text{is exact for all } i \geq 0.$$

Here ϕ_{i-s} is the restriction of ϕ to $\mathfrak{m}^{i-s} F_1$. Notice $\ker \epsilon_i = N \cap \mathfrak{m}^i F = N_i = \mathfrak{m}^{i-s} N$ and the map ϕ_{i-s} maps $\mathfrak{m}^{i-s} F_1$ surjectively to $\mathfrak{m}^{i-s} N$. We now tensor the exact sequence $(**)$ with $A/\mathfrak{m}$ to get the exact sequence

$$\frac{\mathfrak{m}^{i-s} F_1}{\mathfrak{m}^{i+1-s} F_1} \xrightarrow{\overline{\phi_{i-s}}} \frac{\mathfrak{m}^i F_0}{\mathfrak{m}^{i+1} F_0} \rightarrow \frac{\mathfrak{m}^i M}{\mathfrak{m}^{i+1} M} \rightarrow 0$$

for all $i \geq 0$. Thus we have an exact sequence

$$(\dagger) \quad G_{\mathfrak{m}}(F_1)(-s) \xrightarrow{\bar{\phi}} G_{\mathfrak{m}}(F_0) \rightarrow G_{\mathfrak{m}}(M) \rightarrow 0.$$

It can be easily verified that $\bar{\phi} = \text{in}(\phi)$. Furthermore, as $G_{\mathfrak{m}}(F_1) = R^a$, we get that $(\dagger)$ is part of a minimal resolution of $G_{\mathfrak{m}}(M)$.

(5) This follows from the exact sequence $(*)$ and (1). $\square$

We now give:

Proof of Theorem 1.3. We prove the result by induction on $p = \text{projdim } M$. The result clearly holds when $p = 0$. Furthermore, when $p = 1$ the result follows from Lemma 3.1. We assume the result when $p = r \geq 1$ and prove it when $p = r + 1$. Set $N = \text{Syz}_1^A(M)$ and let $v(\phi_1) = s_1$. Then by Lemma 3.1 we get

$$\text{Syz}_1^R(G_{\mathfrak{m}}(M)) = G_{\mathfrak{m}}(N)(-s_1).$$

It follows that $G_{\mathfrak{m}}(N)$ has a pure resolution. Truncate $\mathbb{F}$ to obtain a minimal resolution $\mathbb{F}'$ of N . By induction hypotheses $\text{in}(\mathbb{F}')$ is a minimal resolution of $G_{\mathfrak{m}}(N)$. After shifting in $(\mathbb{F}')$ suitably and by Lemma 3.1, we get that $\text{in}(\mathbb{F})$ is a minimal resolution of $G_{\mathfrak{m}}(M)$. $\square$

4. PROOF OF THEOREM 1.4

In this section we give a proof of our main result. We need the following result due to Herzog-Kühl (see [4, Theorem 1]) regarding modules having pure resolutions.

Lemma 4.1. *Let $R = K[X_1, X_2, \dots, X_n]$ and suppose E is a finitely generated graded R -module with pure resolution of type $(0, d_1, d_2, \dots, d_p)$. Set $\beta_i = \beta_i(E)$. If for $i \geq 1$*

$$\beta_i = (-1)^{i+1} \beta_0 \prod_{j \neq i} \frac{d_j}{d_j - d_i},$$

then E is a Cohen-Macaulay R -module.

We now give a proof of our main result.

Proof of Theorem 1.4. If $G_{\mathfrak{m}}(M)$ has a pure resolution, then by Theorem 1.3 the complex $\text{in}(\mathbb{F})$ is a minimal resolution of $G_{\mathfrak{m}}(M)$. The result now follows from [1, p. 88].

Conversely if $\text{in}(\mathbb{F})$ is acyclic and the Betti numbers satisfy the Herzog-Kühl conditions, then by Lemma 4.1, $E = \text{coker } \text{in}(\phi_1)$ is Cohen-Macaulay of dimension $n - p = \dim M$. Note we also have a surjective homomorphism $\epsilon: G_{\mathfrak{m}}(F_0) \rightarrow G_{\mathfrak{m}}(M)$ with $\epsilon \circ \text{in}(\phi_1) = 0$. So we have an exact sequence

$$0 \rightarrow C \rightarrow E \rightarrow G_{\mathfrak{m}}(M) \rightarrow 0.$$

Note $\dim C \leq \dim E = \dim G_{\mathfrak{m}}(M)$. As multiplicity of E equals multiplicity of $G_{\mathfrak{m}}(M)$, it follows that $\dim C < \dim E$. But E is Cohen-Macaulay. Therefore, $C = 0$. So $G_{\mathfrak{m}}(M) = E$ has a pure resolution. $\square$

5. AN EXAMPLE

All the computations in this section were done using the computer algebra package SINGULAR. Let $Q = K[x, y, z]_{\mathfrak{m}}$ where $\mathfrak{m} = (x, y, z)$. Also $A = \widehat{Q}$ and $R = G_{\mathfrak{m}}(A) = G_{\mathfrak{m}}(Q) = K[x, y, z]$. Using SINGULAR one can verify that we have an exact sequence

$$0 \rightarrow Q^2 \xrightarrow{\psi} Q^4 \xrightarrow{\phi} Q^2 \rightarrow M \rightarrow 0,$$

where

$$\phi = \begin{pmatrix} y^2 & x^2 & z^3 & 0 \\ 0 & 0 & z^2 & x^2 \end{pmatrix} \quad \text{and} \quad \psi = \begin{pmatrix} -x^2 & 0 \\ y^2 & z^3 \\ 0 & -x^2 \\ 0 & z^2 \end{pmatrix}.$$

Notice that the zeroth Fitting ideal of M is

$$\text{Fitt}_0(M) = I_2(\phi) = (y^2 z^2, y^2 x^2, x^2 z^2, x^4, z^3 x^2).$$

It is well known that $\text{Fitt}_0(M) \subseteq \text{ann}(M)$ and that $\sqrt{\text{Fitt}_0(M)} = \sqrt{\text{ann}(M)}$. If P is a prime containing $\text{Fitt}_0(M)$, then clearly $P \supseteq (x, y)$ or $P \supseteq (x, z)$. Thus $\dim M = 1$. Therefore, M is Cohen-Macaulay.

Notice

$$\text{in}(\phi) = \begin{pmatrix} y^2 & x^2 & 0 & 0 \\ 0 & 0 & z^2 & x^2 \end{pmatrix} \quad \text{and} \quad \text{in}(\psi) = \begin{pmatrix} -x^2 & 0 \\ y^2 & 0 \\ 0 & -x^2 \\ 0 & z^2 \end{pmatrix}.$$

Using SINGULAR or the Buchsbaum-Eisenbud criterion it follows that the complex

$$(*) \quad 0 \rightarrow R(-4)^2 \xrightarrow{\text{in}(\psi)} R(-2)^4 \xrightarrow{\text{in}(\phi)} R^2 \rightarrow 0$$

is acyclic. Also by an argument similar to that above shows that $\text{coker}(\text{in}(\phi))$ is Cohen-Macaulay of dimension one.

Set $T = Q/(x^4, y^2 z^2)$. Then M is a Cohen-Macaulay T -module. Also note that T is Cohen-Macaulay of dimension one with $e(T) = 16$. We have an exact sequence

$$T^4 \xrightarrow{\bar{\phi}} T^2 \rightarrow M \rightarrow 0.$$

Let $N = \text{Sy}_1^T(M)$. Then using SINGULAR one can check that $e(N) = 24$. Thus $e(M) = 8$. Therefore, by Theorem 1.4 we get that $(*)$ is a minimal resolution of $G_{\mathfrak{m}}(M)$.

Remark 5.1. Although in this example $p = 2$ and $\dim M = 1$, this is the first non-trivial example where we have an explicit resolution of $G_{\mathfrak{m}}(M)$.

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DEPARTMENT OF MATHEMATICS, INDIAN INSTITUTE OF TECHNOLOGY BOMBAY, POWAI, MUMBAI
400 076, INDIA

E-mail address: tputhen@math.iitb.ac.in