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THE TOPOLOGICAL COMPLEXITY AND THE HOMOTOPY COFIBER OF THE DIAGONAL MAP FOR NON-ORIENTABLE SURFACES

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Abstract. We show that the Lusternik-Schnirelmann category of the homotopy cofiber of the diagonal map of non-orientable surfaces equals three.

Also, we prove that the topological complexity of non-orientable surfaces of genus > 4 is four.

1. Introduction

The *topological complexity* $TC(X)$ of a space X was defined by Farber [F] as an invariant that measures the navigation complexity of X regarded as the configuration space for a robot motion planning. By a slightly modified definition $TC(X)$ is the minimal number k such that $X \times X$ admits a cover by $k+1$ open sets $U_0, \dots, U_k$ such that over each U_i there is a continuous motion planning algorithm $s_i : U_i \to PX$, i.e., a continuous map of U_i to the path space $PX = X^{[0,1]}$ with $s_i(x,y)(0) = x$ and $s_i(x,y)(1) = y$ for all $(x,y) \in U_i$. Here we have defined the reduced topological complexity. The original (non-reduced) topological complexity is by one larger.

The topological complexity of an orientable surface of genus g was computed in [F]:

$$TC(\Sigma_g) = \begin{cases} 2, & \text{if } g = 0, 1, \\ 4 & \text{if } g > 1. \end{cases}$$

For the non-orientable surfaces of genus $g > 1$ the complete answer is still unknown. What was known are the bounds: $3 \le TC(N_g) \le 4$ and the equality $TC(\mathbb{R}P^2) = 3$. In this paper we show that $TC(N_g) = 4$ for $g > 4$.

The topological complexity is a numeric invariant of topological spaces similar to the Lusternik-Schnirelmann category. It is unclear if TC can be completely reduced to the LS-category. One attempt of such reduction ([GV2], [Dr2]) deals with the problem of whether the topological complexity $TC(X)$ coincides with the Lusternik-Schnirelmann category $\mathrm{cat}(C_{\Delta X})$ of the homotopy cofiber of the diagonal map $\Delta : X \to X \times X$, $C_{\Delta X} = (X \times X)/\Delta X$. The coincidence of these two concepts was proven in [GV2] for a large class of spaces. Also in [GV1] the equality was proven for the weak in the sense of the Berstein and Hilton versions of TC and cat.

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In this paper we prove that $\operatorname{cat}(C_{\Delta N}) = 3$ for any non-orientable surface N. Thus, in view of the computation $TC(N_g) = 4$ for $g > 4$ we obtain counterexamples to the conjecture $TC(X) = \operatorname{cat}(C_{\Delta X})$.

Since both computations are rather technical, at the end of the paper we present a short counterexample: Higman's group. We show that $TC(BH) \neq \operatorname{cat}(C_{\Delta BH})$ where $BH = K(H, 1)$ is the classifying space for Higman's group H. The proof of that is short because the main difficulty there, the proof of the equality $TC(BH) = 4$, is hidden behind the reference [GLO].

2. Preliminaries

2.1. **Category of spaces.** By the definition the Lusternik-Schnirelmann category $\operatorname{cat} X \leq k$ for a topological space X if there is a cover $X = U_0 \cup \cdots \cup U_k$ by $k+1$ open subsets each of which is contractible in X.

Let $\pi = \pi_1(X)$. We recall that the cup product $\alpha \smile \beta$ of twisted cohomology classes $\alpha \in H^i(X; L)$ and $\beta \in H^j(X; M)$ takes value in $H^{i+j}(X; L \otimes M)$ where L and M are π-modules and $L \otimes M$ is the tensor product over $\mathbb{Z}$ [Bro]. Then the cup-length of X, denoted as $c.l.(X)$, is defined as the maximal integer k such that $\alpha_1 \smile \cdots \smile \alpha_k \neq 0$ for some $\alpha_i \in H^{n_i}(X; L_i)$ with $n_i > 0$. The following inequalities give estimates on the LS-category [CLOT]:

2.1.1. **Theorem.** $c.l.(X) \leq \operatorname{cat} X \leq \dim X$.
If X is k-connected, then $\operatorname{cat} X \leq \dim X/(k+1)$.

2.2. **Category of maps.** We recall that the LS-category of a map $f : Y \to X$ is the least integer k such that Y can be covered by $k+1$ open sets $U_0, \dots, U_k$ such that the restrictions $f|_{U_i}$ are null-homotopic for all i.

The following two facts are proven in [Dr3] (Proposition 4.3 and Theorem 4.4):

2.2.1. **Theorem.** *Let $u : X \to B\pi$ be a map classifying the universal covering of a CW complex X. Then the following are equivalent:*
(1) $\operatorname{cat}(u) \leq k$;
(2) u is homotopic to a map $f : X \to B\pi$ with $f(X) \subset B\pi^{(k)}$.

2.2.2. **Theorem.** *Let X be an n-dimensional CW complex whose universal covering $\tilde{X}$ is $(n-k)$-connected. Suppose that X admits a classifying map $u : X \to B\pi$ with* $\operatorname{cat} u \leq k$. *Then* $\operatorname{cat} X \leq k$.

2.3. **Inessential complexes.** One can extend Gromov's theory of inessential manifolds [Gr] to simplicial complexes and, in particular, to pseudo-manifolds. We call an n-dimensional complex X *inessential* if a map $u : X \to B\pi$ that classifies the universal covering of X can be deformed to the $(n-1)$-dimensional skeleton. Otherwise it is called *essential*.

2.3.1. **Proposition.** *An n-dimensional complex X is inessential if and only if* $\operatorname{cat} X \leq n-1$.

Proof. Suppose that $\operatorname{cat} X \leq n-1$. Then $\operatorname{cat}(u) \leq n-1$ where $u : X \to B\pi$ is a classifying map. By Theorem 2.2.1, X is inessential.

If X is inessential, by Theorem 2.2.1 $\operatorname{cat}(u) \leq n-1$. We apply Theorem 2.2.2 to X with $k = n-1$ to obtain that $\operatorname{cat} X \leq n-1$. □

Remark. Proposition 2.3.1 in the case when X is a closed manifold was proven in [KR].

2.4. **Pseudo-manifolds.** We recall that an n-dimensional *pseudo-manifold* is a simplicial complex X which is pure, non-branching and strongly connected. *Pure* means that X is the union of n-simplices. *Non-branching* means that there is a subpolyhedron $S \subset X$ of dimension $\leq n-2$ such that $X \setminus S$ is an n-manifold. *Strongly connected* means that every pair of n-simplices σ, σ' in X can be joined by a chain of simplices $\sigma_0, \dots, \sigma_m$ with $\sigma_0 = \sigma$, $\sigma_m = \sigma'$, and $\dim(\sigma_i \cap \sigma_{i-1}) = n-1$ for $i = 1, \dots, m$. Note that every n-dimensional pseudo-manifold X admits a CW complex structure with one vertex and one n-dimensional cell.

A sheaf $\mathcal{O}_X$ on an n-dimensional pseudo-manifold X generated by the presheaf $U \to H_n(X, X \setminus U)$ is called the *orientation sheaf.* We recall that in case of manifolds the orientation sheaf $\mathcal{O}_N$ on N is defined as the pull-back of the canonical $\mathbb{Z}$-bundle $\mathcal{O}$ on $\mathbb{R}P^\infty$ by the map $w_1 : N \to \mathbb{R}P^\infty$ that represents the first Stiefel-Whitney class.

A pseudo-manifold X is l*ocally orientable* if $\mathcal{O}_X$ is locally constant with the stalks isomorphic to $\mathbb{Z}$. For a locally orientable n-dimensional pseudo-manifold X, $H_n(X; \mathcal{O}_X) = \mathbb{Z}$, and the n-dimensional cell (we may assume that it is unique) defines a generator of $\mathbb{Z}$ called the fundamental class $[X]$ of X.

2.4.1. **Theorem.** *Let X be a locally orientable n-dimensional pseudo-manifold and let A be a $\pi_1(X)$-module. Then the cap product with $[X]$ defines an isomorphism*

$$[X]\cap : H^n(X; A) \to H_0(X; A \otimes \underline{Z})$$

where $\underline{Z}$ stands for the $\pi_1(X)$-module $\mathbb{Z}$ defined by the orientation sheaf $\mathcal{O}_X$.

Proof. We note that in these dimensions the proof of the classical Poincaré Duality for locally oriented manifolds ([Bre]) works for pseudo-manifolds as well. □

2.4.2. **Proposition.** *Suppose that a map $f : M \to B\pi$ of an n-dimensional locally orientable pseudo-manifold induces an epimorphism of the fundamental groups. Suppose that the orientation sheaf on M is the image under f^* of a sheaf on $B\pi$. Then f can be deformed to the $(n-1)$-skeleton $B\pi^{(n-1)}$ if and only if $f_*([M]) = 0$ where $[M]$ is the fundamental class.*

Proof. The 'only if' direction follows from the dimensional reason and the fact that f_* does not change under a homotopy.

Let $f_*([M]) = 0$. We show that the primary obstruction o_f for deformation of f to the $(n-1)$-skeleton is trivial. Since o_f is the image of the primary obstruction o' to deformation of $B\pi$ to $B\pi^{(n-1)}$, it suffices to prove the equality $f^*(o') = 0$. Note that

$$f_*([M] \cap f^*(o')) = f_*([M]) \cap o' = 0.$$

Since f induces an epimorphism of the fundamental groups, it induces an isomorphism of 0-dimensional homology groups with any local coefficients. Hence, $[M] \cap f^*(o') = 0$. Since in dimension 0 the Poincare Duality holds for locally orientable pseudo-manifolds, we obtain that $f^*(o') = 0$. □

2.5. **Homology of projective space.** We denote by $\mathcal{O}$ the canonical local coefficient system on the projective space $\mathbb{R}P^\infty$ with the fiber $\mathbb{Z}$.

2.5.1. **Proposition.**

$$H_i(\mathbb{R}P^\infty; \mathcal{O}) = \begin{cases} \mathbb{Z}_2, & \text{if } i \text{ is even,} \\ 0, & \text{if } i \text{ is odd.} \end{cases}$$

Proof. Let $\underline{Z}$ denote a $\mathbb{Z}_2$-module $\mathbb{Z}$ with the non-trivial $\mathbb{Z}_2$-action. We note that $H_i(\mathbb{R}P^\infty;\mathcal{O}) = H_i(\mathbb{Z}_2,\underline{Z})$. If $\mathbb{Z}_2 = \{1,t\}$, then the homology groups $H_*(\mathbb{Z}_2,\underline{Z})$ are the homology of the chain complex ([Bro])

$$\cdots \xrightarrow{1-t} \underline{Z} \xrightarrow{1+t} \underline{Z} \xrightarrow{1-t} \underline{Z} \xrightarrow{1+t} \underline{Z} \xrightarrow{1-t} \underline{Z}$$

which is the complex

$$\cdots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z}.$$

□

2.6. **Schwarz genus.** We recall that the *Schwarz genus* $g(f)$ of a fibration $f : E \to B$ is the minimal number k such that B can be covered by k open sets on which f admits a section [Sch]. Then $\operatorname{cat} X + 1 = g(c : * \to X)$ and $TC(X) + 1 = g(\Delta : X \to X \times X)$ where the constant map c and the diagonal map Δ are assumed to be represented by fibrations. Schwarz related the genus $g(f)$ with the existence of a section of a special fibration constructed out of f by means of an operation that generalizes the Whitney sum.

Here is the construction: Given two maps $f_1 : X_1 \to Y$ and $f_2 : X_2 \to Y$, we define the *fiberwise join* of spaces X_1 and X_2 as

$$X_1 *_Y X_2 = \{tx_1 + (1-t)x_2 \in X_1 * X_2 \mid f_1(x_1) = f_2(x_2)\}$$

and define the *fiberwise join* of f_1, f_2 as the map

$$f_1 *_Y f_2 : X_1 *_Y X_2 \to Y, \quad \text{with} \quad (f_1 *_Y f_2)(tx_1 + (1-t)x_2) = f_1(x_1) = f_2(x_2).$$

The iterated n times fiberwise join product of a map $f : E \to B$ is denoted as $*^n_B : *^n_B E \to B$.

2.6.1. **Theorem** ([Sch, Theorem 3])**.** *For a fibration $f : E \to B$ the Schwarz genus $g(f) \le n$ if and only if $*^n_B f : *^n_B \to E$ admits a section.*

Also Schwarz proved the following [Sch]:

2.6.2. **Theorem.** *Let β be the primary obstruction to a section of a fibration $f : E \to B$. Then β^n is the primary obstruction to a section of $*^n_b f$.*

Let $p^X : PX \to X \times X$ be the end points map: $p^X(\phi) = (\phi(0),\phi(1)) \in X \times X$. Here PX is the space of all paths $\phi : [0,1] \to X$ in X. Clearly, p^X is a Serre path fibration that represents the diagonal map $\Delta : X \to X \times X$. Let $P_0X \subset PX$ be the space of paths issued from a base point x_0 and let $\tilde{p}^X : P_0X \to X$ be defined as $\tilde{p}^X(\phi) = \phi(1)$. Then $\tilde{p}^X$ is a fibration representative for the map $x_0 \to X$.

For $n > 0$ we denote by $p^X_n = *^{n+1}_{X\times X} p^X$ and $\Delta_n(X) = *^{n+1}_{X\times X} PX$. Thus, elements of $\Delta_n(X)$ can be viewed as formal linear combinations $\sum_{i=0}^n t_i\phi_i$ where $\phi_i : [0,1] \to X$ with $\phi_1(0) = \cdots = \phi_n(0)$, $\phi_1(1) = \cdots = \phi_n(1)$, $t_i \ge 0$, and $\sum t_i = 1$.

Similarly we use the notation $\tilde{p}^X_n = *^{n+1}_X \tilde{p}^X$ and $G_n(X) = *^{n+1}_X P_0X$. The fibration $\tilde{p}^X_n$ is called the n-th Ganea fibration. For these fibrations Theorem 2.6.1 produces the following

2.6.3. **Theorem.** *For a CW-space X,*
(1) $TC(X) \le n$ if and only if there exists a section of $p^X_n : \Delta_n(X) \to X \times X$;
(2) $\operatorname{cat} X \le n$ if and only if there exists a section of $\tilde{p}^X_n : G_n(X) \to X$.

We note that the fiber of both fibrations p^X and $\tilde{p}^X$ is the loop space ΩX. The fiber $F_n = (p_n^X)^{-1}(x_0)$ of the fibration p_n^X is the join product $\Omega X * \cdots * \Omega X$ of $n+1$ copies of the loop space ΩX on X. So, F_n is $(n-1)$-connected.

A continuous map $f : X \to Y$ for any n defines the commutative diagram

$$\begin{array}{ccc} \Delta_n(X) & \longrightarrow & \Delta_n(Y) \\ {\scriptstyle p_n^X}\downarrow & & {\scriptstyle p_n^Y}\downarrow \\ X \times X & \xrightarrow{f\times f} & Y \times Y. \end{array}$$

2.6.4. **Corollary.** *If $TC(X) \le n$, then for any $f : X \to Y$ the map $f \times f$ admits a lift with respect to p_n^Y.*

2.7. **Berstein-Schwarz cohomology class.** The Berstein-Schwarz class of a discrete group π is the first obstruction β_π to a lift of $B\pi = K(\pi, 1)$ to the universal covering $E\pi$. Note that $\beta_\pi \in H^1(\pi, I(\pi))$ where $I(\pi)$ is the augmentation ideal of the group ring $\mathbb{Z}\pi$ [Be],[Sch].

2.7.1. **Theorem** (Universality [DR],[Sch]). *For any cohomology class $\alpha \in H^k(\pi, L)$ there is a homomorphism of π-modules $I(\pi)^k \to L$ such that the induced homomorphism for cohomology takes $(\beta_\pi)^k \in H^k(\pi; I(\pi)^k)$ to α where $I(\pi)^k = I(\pi) \otimes \cdots \otimes I(\pi)$ and $(\beta_\pi)^k = \beta_\pi \smile \cdots \smile \beta_\pi$.*

3. Computation of the LS-category of the cofiber

For $X = \mathbb{R}P^n$, $n > 1$, the equality $TC(X) = \mathrm{cat}(C_{\Delta X})$ was established in [GV2]. Together with the computation $TC(\mathbb{R}P^2) = 3$ from [FTY] it gives the following

3.0.2. **Theorem.**

$$\mathrm{cat}((\mathbb{R}P^2 \times \mathbb{R}P^2)/\Delta\mathbb{R}P^2) = 3.$$

The goal of this section is to extend this result to all non-orientable surfaces.

3.1. **Kunneth formula for twisted coefficients.** For sheafs $\mathcal{A}$ and $\mathcal{B}$ on X and Y we use the notation $\mathcal{A}\hat{\otimes}\mathcal{B}$ for $pr_X^*\mathcal{A} \otimes pr_Y^*\mathcal{B}$ where $pr_X : X \times Y \to X$ and $pr_Y : X \times Y \to Y$ are the projections. Similarly we use notation $\mathcal{A}\hat{*}\mathcal{B}$ for the periodic product $pr_X^*\mathcal{A} * pr_Y^*\mathcal{B}$. We recall that in the case when $\mathcal{A}\hat{*}\mathcal{B} = 0$ for locally nice spaces X and Y there is the Kunneth formula [Bre]

$$(*)\quad H_n(X\times Y; \mathcal{A}\hat{\otimes}\mathcal{B}) = \bigoplus_i H_i(X;\mathcal{A})\otimes H_{n-i}(Y;\mathcal{B}) \oplus \bigoplus_i H_{i-1}(X;\mathcal{A}) * H_{n-i}(Y;\mathcal{B}).$$

Note that for the stalk at $(x, y) \in X \times Y$ we have $(\mathcal{A}\hat{*}\mathcal{B})_{(x,y)} = \mathcal{A}_x * \mathcal{B}_y$. Thus, the Kunneth formula holds for local coefficients if one of the coefficient modules is torsion free as an abelian group.

3.2. **Free abelian topological groups.** Let $\mathbb{A}(N)$ denote the free abelian topological group generated by N (see [M],[G] or [Dr1]). Let $j : N \to \mathbb{A}(N)$ be the natural inclusion. By the Dold-Thom theorem [DT] (see also [Dr1]), $\pi_i(\mathbb{A}(N)) = H_i(N)$ and $j_* : \pi_i(N) \to \pi_i(\mathbb{A}(N))$ is the Hurewicz homomorphism. Therefore, $\mathbb{A}(\mathbb{R}P^2)$ is homotopy equivalent to $\mathbb{R}P^\infty$. Moreover, for a non-orientable surface N of genus g the space $\mathbb{A}(N)$ is homotopy equivalent to $\mathbb{R}P^\infty \times T^{g-1}$ where $T^m = S^1 \times \cdots \times S^1$ denotes the m-dimensional torus.

Let $\tilde{\mathcal{O}}$ be the twisted coefficient system on $\mathbb{A}(N)$ that comes from the canonical system $\mathcal{O}$ on $\mathbb{R}P^\infty$ as the pull-back under the projection $\mathbb{R}P^\infty \times T^{g-1} \to \mathbb{R}P^\infty$.

3.2.1. **Proposition.** *For any non-orientable surface N*

$$H_2(\mathbb{A}(N);\tilde{\mathcal{O}}) = \oplus\mathbb{Z}_2.$$

Proof. We replace $\mathbb{A}(N)$ by a homotopy equivalent space $T^{g-1}\times\mathbb{R}P^\infty$. Note that $\tilde{\mathcal{O}} = \mathbb{Z}\hat{\otimes}\mathcal{O}$ where $\mathbb{Z}$ is the trivial sheaf. Since $\mathbb{Z}*\mathbb{Z}=0$ we obtain that $\mathbb{Z}\hat{*}\mathcal{O}=0$ and we can apply the Kunneth formula. Then by the Kunneth formula for local coefficients,

$$H_2(\mathbb{A}(N);\tilde{\mathcal{O}}) = H_0(T^{g-1})\otimes H_2(\mathbb{R}P^\infty;\mathcal{O})$$
$$\oplus H_1(T^{g-1})\otimes H_1(\mathbb{R}P^\infty;\mathcal{O})$$
$$\oplus H_2(T^{g-1})\otimes H_0(\mathbb{R}P^\infty;\mathcal{O}).$$

The periodic product part of the Kunneth formula is trivial since the first factor has torsion free homology groups. Thus, taking into account Proposition 2.5.1 we obtain

$$H_2(A(N);\tilde{\mathcal{O}}) = H_2(\mathbb{R}P^\infty;\mathcal{O})\oplus(\mathbb{Z}_2\otimes H_2(T^{g-1})) = \mathbb{Z}_2\oplus H_2(T^{g-1};\mathbb{Z}_2) = \oplus\mathbb{Z}_2.$$

□

For a topological abelian group A we denote by $\mu_A=\mu : A\times A\to A$ the continuous group homomorphism defined by the formula $\mu(a,b)=a-b$.

3.2.2. **Proposition.** *Let $N=\mathbb{R}P^2$. Then the pull-back $(j\times j)^*\mu^*(\mathcal{O})$ is the $\mathbb{Z}$-orientation sheaf for the manifold $\mathbb{R}P^2\times\mathbb{R}P^2$ where $\mu=\mu_{\mathbb{A}(\mathbb{R}P^2)}$ and $\mathcal{O}$ is the canonical $\mathbb{Z}$-bundle over $\mathbb{A}(\mathbb{R}P^2)=\mathbb{R}P^\infty$.*

We make a forward reference to Corollary 3.3.3 for the proof.

3.2.3. **Proposition.** *Let $a\in H_2(\mathbb{R}P^\infty;\mathcal{O})$ be a generator. Then $\mu_*(a\otimes a)=0$ where $\mu=\mu_{A(\mathbb{R}P^2)}$.*

Proof. Note that $\pi_1((\mathbb{R}P^2\times\mathbb{R}P^2)/\Delta\mathbb{R}P^2)=\mathbb{Z}_2$ (see Proposition 3.4.1). Let

$$f:(\mathbb{R}P^2\times\mathbb{R}P^2)/\Delta\mathbb{R}P^2\to\mathbb{A}(\mathbb{R}P^2)$$

be a map that induces an isomorphism of the fundamental groups. We claim that the map $\mu\circ(j\times j)$ is homotopic to $f\circ q$ where $q:\mathbb{R}P^2\times\mathbb{R}P^2\to(\mathbb{R}P^2\times\mathbb{R}P^2)/\Delta\mathbb{R}P^2$ is the quotient map. This holds true since both maps induce the same homomorphism of the fundamental groups. In view of Theorem 3.0.2 and Proposition 2.3.1 the map f is homotopic to a map with the image in the 3-dimensional skeleton. Therefore, $f_*q_*(b\otimes b)=0$ where b is a generator of $H_2(\mathbb{R}P^2;\mathcal{O}_{\mathbb{R}P^2})=\mathbb{Z}$. Note that $j_*(b)=a$. Then $\mu_*(a\otimes a)=\mu_*(j\times j)_*(b\otimes b)=f_*q_*(b\otimes b)=0$. □

Let $N=N_g=\#_g\mathbb{R}P^2$ be a non-orientable surface of genus g. Let $\pi=\pi_1(N)$ and $G=Ab(\pi)=H_1(N)=\mathbb{Z}_2\oplus\mathbb{Z}^{g-1}$. We recall that by the Dold-Thom theorem [DT],[Dr1] the space $\mathbb{A}(N)$ is homotopy equivalent to $K(G,1)\sim\mathbb{R}P^\infty\times T^{g-1}$.

3.2.4. **Proposition.** *There is a homomorphism of topological abelian groups*

$$h:\mathbb{A}(N_g)\to\mathbb{A}(\mathbb{R}P^2)\times T^{g-1}$$

which is a homotopy equivalence.

Proof. Since the spaces are $K(G,1)$s, it suffices to construct a homomorphism that induces an isomorphism of the fundamental groups. For that it suffices to construct a map $f : N_g \to \mathbb{A}(\mathbb{R}P^2) \times T^{g-1}$ that induces an isomorphism of integral 1-dimensional homology groups. Then the homomorphism h is an extension to $\mathbb{A}(N_g)$ which exists by the universal property of free abelian topological groups. We consider two cases.

(1) If g is odd, then $N_g = M_{(g-1)/2}\#\mathbb{R}P^2$. We define f as the composition

$$M_{(g-1)/2}\#\mathbb{R}P^2 \xrightarrow{q} M_{(g-1)/2} \vee \mathbb{R}P^2 \xrightarrow{\phi\vee j} T^{g-1} \vee \mathbb{A}(\mathbb{R}P^2) \xrightarrow{i} T^{g-1} \times \mathbb{A}(\mathbb{R}P^2)$$

where q is collapsing of the separating circle in the connected sum, ϕ is a map that induces isomorphism of 1-dimensional homology, and i is the inclusion. It is easy to check that f induces an isomorphism $f_* : H_1(N_g) \to H_1(\mathbb{A}(\mathbb{R}P^2) \times T^{g-1})$.

(2) If g is even, then $N_g = M_{(g-2)/2}\#K$ where K is the Klein bottle. There is a homotopy equivalence $s : \mathbb{A}(K) \to S^1 \times \mathbb{A}(\mathbb{R}P^2)$. We define f as the composition

$$M_{(g-2)/2}\#K \xrightarrow{q} M_{(g-1)/2} \vee K \xrightarrow{\phi\vee s\circ j} T^{g-2} \vee (S^1 \times \mathbb{A}(\mathbb{R}P^2)) \xrightarrow{i} T^{g-2} \times S^1 \times \mathbb{A}(\mathbb{R}P^2)$$

where q is collapsing of the connecting circle, ϕ is a map that induces isomorphism of 1-dimensional homology, and i is the inclusion. One can check that f induces an isomorphism $f_* : H_1(N_g) \to H_1(\mathbb{A}(\mathbb{R}P^2) \times T^{g-1})$. □

3.2.5. **Proposition.** *For abelian topological groups A and B, $\mu_{A\times B} = \mu_A \times \mu_B$.*

Proof. For all $a, a' \in A$ and $b, b' \in B$ we have

$$\mu_{A\times B}(a \times b, a' \times b') = (a - a') \times (b - b') = \mu_A(a) \times \mu_B(b) = (\mu_A \times \mu_B)(a \times b).$$

□

3.3. **Twisted fundamental class.** The pull-back of the canonical $\mathbb{Z}$-bundle $\mathcal{O}$ over $\mathbb{R}P^\infty$ under the projection $\mathbb{R}P^\infty \times T^{g-1} \to \mathbb{R}P^\infty$ defines a local coefficient system $\tilde{\mathcal{O}}$ on $\mathbb{A}(N)$ with the fiber $\mathbb{Z}$. On the G-module level the action of the fundamental group on $\mathbb{Z}$ is generated by the projection homomorphism $p : \mathbb{Z}_2 \oplus \mathbb{Z}^{g-1} \to \mathbb{Z}_2$. We note that $\mathcal{O}_N = j^*(\tilde{\mathcal{O}})$.

For sheafs $\mathcal{A}$ and $\mathcal{B}$ on X and Y we use the notation $\mathcal{A}\hat{\otimes}\mathcal{B}$ for $pr_X^*\mathcal{A} \otimes pr_Y^*\mathcal{B}$ where $pr_X : X \times Y \to X$ and $pr_Y : X \times Y \to Y$ are the projections. We note that if $\mathcal{O}_X$ and $\mathcal{O}_Y$ are the orientation sheafs on manifolds X and Y, then $\mathcal{O}_X\hat{\otimes}\mathcal{O}_Y$ is the orientation sheaf on $X \times Y$.

3.3.1. **Proposition.** *The cross product $[N]_{\mathcal{O}_N} \times [N]_{\mathcal{O}_N}$ is a fundamental class for $N \times N$.*

Proof. Note that for the orientation sheaf $\mathcal{O}_{N\times N}$ on the manifold $N \times N$ we have $H_4(N \times N; \mathcal{O}_{N\times N}) = \mathbb{Z}$. We note that $\mathcal{O}_N \hat{*} \mathcal{O}_N = 0$. Then the Kunneth formula implies

$$\mathbb{Z} = H_4(N\times N; \mathcal{O}_{N\times N}) = H_4(N\times N; \mathcal{O}_N\hat{\otimes}\mathcal{O}_N) = H_2(N; \mathcal{O}_N)\otimes H_2(N; \mathcal{O}_N) = \mathbb{Z}\otimes\mathbb{Z}.$$

Thus $[N_{\mathcal{O}_N}] \otimes [N_{\mathcal{O}_N}]$ is a generator in $H_4(N \times N; \mathcal{O}_{N\times N})$. □

3.3.2. **Proposition.** *For $\mu = \mu_{\mathbb{A}(N)}$,*

$$\mu^*\tilde{\mathcal{O}} = \tilde{\mathcal{O}}\hat{\otimes}\tilde{\mathcal{O}}.$$

Proof. Let $p : \mathbb{Z}_2 \oplus \mathbb{Z}^{g-1} \to \mathbb{Z}_2$ be the projection. The sheaf on the left is defined by the representation $p\mu_* : \pi_1(\mathbb{A}(N) \times \mathbb{A}(N)) \to Aut(\mathbb{Z}) = \mathbb{Z}_2$. The sheaf on the right is defined by the representation

$$\alpha(p \times p) : \pi_1(\mathbb{A}(N)) \times \pi_1(\mathbb{A}(N)) \to \mathbb{Z}_2$$

where $\alpha : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}_2$, $\alpha(x, y) = x + y$, is the addition homomorphism. It is easy to check that these two coincide on the generating set

$$(\pi_1(\mathbb{A}(N)) \times 1) \cup (1 \times \pi_1(\mathbb{A}(N))) \subset \pi_1(\mathbb{A}(N) \times \mathbb{A}(N)).$$

Hence, they coincide on $\pi_1(\mathbb{A}(N) \times \mathbb{A}(N))$. Therefore, these sheafs are equal. □

3.3.3. **Corollary.** *The pull-back $(j \times j)^*\mu^*(\tilde{\mathcal{O}})$ is the orientation sheaf for the manifold $N \times N$.*

Proof. Since $\mathcal{O}_N = j^*\tilde{\mathcal{O}}$, in view of Proposition 3.3.2,

$$\mathcal{O}_{N\times N} = \mathcal{O}_n \hat{\otimes} \mathcal{O}_N = j^*\tilde{\mathcal{O}} \hat{\otimes} j^*\tilde{\mathcal{O}} = (j \times j)^*(\tilde{\mathcal{O}} \hat{\otimes} \tilde{\mathcal{O}}) = (j \times j)^*\mu^*(\tilde{\mathcal{O}}).$$

□

3.3.4. **Proposition.** *Let $I : \mathbb{A}(N) \to \mathbb{A}(N)$, $I(x) = -x$, be the taking inverse map. Then I fixes every local system $\mathcal{M}$ on $\mathbb{A}(N)$ and defines the identity homomorphism in homology $I_* : H_*(\mathbb{A}(N); \mathcal{M}) \to H_*(\mathbb{A}(N); \mathcal{M})$.*

Proof. In view of Proposition 3.2.4 it suffices to prove it for $\mathbb{A}(\mathbb{R}P^2) \times T^k$. Note that the inverse homomorphism $I : \mathbb{A}(\mathbb{R}P^2) \times T^k \to \mathbb{A}(\mathbb{R}P^2) \times T^k$ is the product of the inverse homomorphisms I^1 and I^2 for $\mathbb{A}(\mathbb{R}P^2)$ and T^k respectively. Also note that both $I^1 : \mathbb{A}(\mathbb{R}P^2) \to \mathbb{A}(\mathbb{R}P^2)$ and $I^2 : T^k \to T^k$ are homotopic to the identity. Thus, I defines the identity automorphism of the fundamental group and, hence, fixes $\mathcal{M}$. Then the homomorphism $I_* : H_*(\mathbb{A}(N); \mathcal{M}) \to H_*(\mathbb{A}(N); \mathcal{M})$ is defined and $I_* = 1$. □

3.3.5. **Proposition.** *For a non-orientable surface N the homomorphism*

$$(\mu_{\mathbb{A}(N)})_*(j \times j)_* : H_4(N \times N; \mathcal{O}_{N\times N}) \to H_4(\mathbb{A}(N); \tilde{\mathcal{O}})$$

is well defined and

$$(\mu_{\mathbb{A}(N)})_*(j \times j)_*([N \times N]_{\mathcal{O}_{N\times N}}) = 0$$

where $[N \times N]_{\mathcal{O}_{N\times N}} \in H_4(N \times N; \mathcal{O}_{N\times N})$ is the fundamental class.

Proof. By Corollary 3.3.3, $\mathcal{O}_{N\times N} = (j \times j)^*\mu^*(\tilde{\mathcal{O}})$ and, hence, the homomorphism

$$\mu_*(j \times j)_* : H_4(N \times N; \mathcal{O}_{N\times N}) \to H_4(\mathbb{A}(N); \tilde{\mathcal{O}})$$

is well defined.

As before, we replace $\mathbb{A}(N)$ by $\mathbb{A}(\mathbb{R}P^2) \times T^{g-1}$.

By Proposition 3.3.1 the cross product

$$[N]_{\mathcal{O}_N} \times [N]_{\mathcal{O}_N} \in H_4(N \times N; \mathcal{O}_N \hat{\otimes} \mathcal{O}_N)$$

is a fundamental class: $[N \times N]_{\mathcal{O}_{N\times N}} = \pm[N]_{\mathcal{O}_N} \times [N]_{\mathcal{O}_N}$.

Since $\mathcal{O}_N = j^*\tilde{\mathcal{O}}$, the homomorphism $j_* : H_2(N; \mathcal{O}_N) \to H_2(\mathbb{A}(N); \tilde{\mathcal{O}})$ is well defined. In view of the Kunneth formula (see formula (*) in subsection 3.1) we obtain

$$j_*([N]_{\mathcal{O}_N}) = a \otimes 1_B + 1_A \otimes b \in (H_2(A(\mathbb{R}P^2); \mathcal{O}_N) \otimes \mathbb{Z}) \oplus (\mathbb{Z}_2 \otimes H_2(T^{g-1})) = H_2(A(N); \tilde{\mathcal{O}})$$

with $a \in H_2(\mathbb{A}(\mathbb{R}P^2);\mathcal{O}) = H_2(\mathbb{R}P^\infty;\mathcal{O})$ being the generator, $b \in H_2(T^{g-1};\mathbb{Z})$, and generators $1_A \in H_0(\mathbb{A}(\mathbb{R}P^2);\mathcal{O}) = \mathbb{Z}_2$ and $1_B \in H_0(T^{g-1};\mathbb{Z}) = \mathbb{Z}$. Let $\bar{a} = a \otimes 1_B$ and $\bar{b} = 1_A \otimes b$.

We apply Proposition 3.2.5 with $\mu = \mu_{\mathbb{A}(\mathbb{R}P^2)\times T^{g-1}}$ and $[N] = [N]_{\mathcal{O}_N}$ to obtain:

$$\mu_*(j\times j)_*([N]\times[N]) = \mu_*(j_*([N])\times j_*([N])) = \mu_*((\bar{a}+\bar{b})\times(\bar{a}+\bar{b}))$$

$$= \mu_*(\bar{a}\times\bar{a}+\bar{a}\times\bar{b}+\bar{b}\times\bar{a}+\bar{b}\times\bar{b}) = \mu_*(\bar{a}\times\bar{a}) + \mu_*(\bar{a}\times\bar{b}+\bar{b}\times\bar{a}) + \mu_*(\bar{b}\times\bar{b})$$

$$= \mu^1_*(a\times a)\times 1_B + \mu^1_*(a\times 1_A)\times\mu^2_*(1_B\times b) + \mu^1_*(1_A\times a)\times\mu^2_*(b\times 1_B) + 1_A\times\mu^2_*(b\times b)$$

where $\mu^1 = \mu_{\mathbb{A}(\mathbb{R}P^2)}$ and $\mu^2 = \mu_{T^{g-1}}$. By Proposition 3.2.3, $\mu^1_*(a\times a)\times 1_B = 0$.

We recall that a $\mathbb{Z}$-twisted homology class in a space X with a local system $p : E \to X$ is defined by a cycle in X with coefficients in the sections of p on (singular) simplices in X. One can assume that the sections are taken in the ± 1-subbundle of the $\mathbb{Z}$-bundle p. This implies that every homology class is represented by a continuous map $f : M \to X$ of a pseudo-manifold that admits a lift $f' : M \to E$ with value in the ± 1-subbundle of p.

One can show that the homology class $a \in H_2(\mathbb{A}(\mathbb{R}P^2);\mathcal{O})$ is realized by a map $f : S^2 \to \mathbb{A}(\mathbb{R}P^2)$ that admits a lift to the ± 1-subbundle of $\mathcal{O}$. The homology class $1_A \in H_0(\mathbb{A}(\mathbb{R}P^2);\mathcal{O})$ can be realized by the point representing the unit $0 \in \mathbb{A}(\mathbb{R}P^2)$. Then the class $\mu^1_*(a\times 1_A)$ is realized by the same map

$$f = \mu^1(f,0) : S^2 \to \mathbb{A}(\mathbb{R}P^2),$$

i.e., $\mu^1_*(a\times 1_A) = a$, whereas the class $\mu^1_*(1_A\times a)$ is realized by the composition $I\circ f$. By Proposition 3.3.4, $\mu^1_*(1_A\times a) = I_*(a) = a$. Similarly, $\mu^2_*(b\times 1_B) = b$, and $\mu^2_*(1_B\times b) = I_*(b) = b$. Therefore, $\mu_*(\bar{a}\times\bar{b}+\bar{b}\times\bar{a}) = 2(a\times b)$ is divisible by 2. Then by Proposition 3.2.1, $\mu_*(\bar{a}\times\bar{b}+\bar{b}\times\bar{a}) = 0$.

Next we show that $\mu^2_*(b\times b) = 0$. In view of Proposition 3.2.1 it suffices to show that $\mu^2_*(b\times b) = 0 \bmod 2$. We recall that the Pontryagin product defines the structure of an exterior algebra on $H_*(T^r) = \Lambda[x_1,\dots,x_r]$. Thus, $b = \sum_{i<j}\lambda_{ij}x_i\wedge x_j$. Then

$$b\times b = \sum_{i<j,k<l}\lambda_{ij}\lambda_{kl}(x_i\wedge x_j)\times(x_k\wedge x_l).$$

Let $\{i,j\}\cap\{k,l\} = \emptyset$. By the argument similar to the above using Propositions 3.2.5 and 3.3.4 we obtain that

$$\mu^2_*((x_i\wedge x_j)\times(x_k\wedge x_l) + (x_k\wedge x_l)\times(x_i\wedge x_j))$$

is divisible by 2.

If $|\{i,j,k,l\}| \le 3$, then the problem can be reduced to a 3-torus. Then

$$\mu^2_*((x_i\wedge x_j)\times(x_k\wedge x_l)) = 0$$

by the dimensional reason. □

3.4. **Inessentiality of the cofiber.**

3.4.1. **Proposition.** *For a connected CW complex X the fundamental group $G = \pi_1(X\times X/\Delta X)$ is isomorphic to the abelianization of $\pi_1(X)$ and the induced homomorphism $\pi_1(X\times x_0) \to \pi_1((X\times X)/\Delta X)$ is surjective.*

Proof. Let $q : X \times X \to (X \times X)/\Delta X$ be the quotient map. Since q has connected point preimages, it induces an epimorphism of the fundamental groups. Suppose that $g = q_*(a, b)$, $g \in G$, $(a, b) \in \pi_1(X) \times \pi_1(X) = \pi_1(X \times X)$. Then

$$g = q_*(a, b) = q_*(b, b)q_*(b^{-1}a, 1) = eq_*(b^{-1}a, 1)$$

where $e \in G$ is the unit. This proves the second part.

Let $g, h \in G$. By the second part of the proposition, $g = q_*(a, 1)$ and $h = q_*(1, b)$. Since $(a, 1)(1, b) = (a, b) = (1, b)(a, 1)$, we obtain $gh = hg$. Thus, G is abelian. We show that $\pi_1(X \times x_0) \to \pi_1((X \times X)/\Delta X)$ is the abelianization homomorphism.

Note that the kernel of q_* is the normal closure of the diagonal subgroup $\Delta\pi_1(X)$ in $\pi_1(X) \times \pi_1(X)$. Thus every element

$$(x, 1) \in K = ker\{\pi_1(X \times x_0) \to \pi_1((X \times X)/\Delta X)\}$$

can be presented as the product

$$(x, 1) = (a_1^{y_1}, a_1^{z_1})(a_2^{y_2}, a_2^{z_2}) \cdots (a_n^{y_n}, a_n^{z_n})$$

for $a_i, y_i, z_i \in \pi_1(X)$ where $a^g = gag^{-1}$. This equality implies that

$$(a_n^{-1})^{z_n}(a_{n-1}^{-1})^{z_{n-1}} \cdots (a_1^{-1})^{z_1} = 1.$$

Then $x = (a_n^{-1})^{z_n} \cdots (a_1^{-1})^{z_1} a_1^{y_1} \cdots a_n^{y_n}$ lies in the kernel of the abelianization map. Therefore, $K \subset [\pi_1(X), \pi_1(X)]$. □

3.4.2. **Proposition.** *For any g the pseudo-manifold $(N_g \times N_g)/\Delta N_g$ is locally orientable and inessential.*

Proof. We use the notation $N = N_g$. To check the local orientability it suffices to show that $H_4(W, \partial W) = \mathbb{Z}$ for a regular neighborhood of the diagonal ΔN in $N \times N$. Since $H_4(W) = H_3(W) = 0$, the exact sequence of pair implies $H_4(W, \partial W) = H_3(\partial W)$. Note that the boundary ∂W is homeomorphic to the total space of the spherical bundle for the tangent bundle on N. The spectral sequence for this spherical bundle implies that

$$H_3(\partial W) = E^2_{2,1} = H_2(N; \underline{H_1(S^1)})$$

where the local system $\underline{H_1(S^1)}$ is the orientation sheaf on N. Thus, we obtain $H_3(\partial W) = \mathbb{Z}$.

Next, we show that the map $\mu \circ (j \times j)$ is homotopic to $f \circ q$ where $q : N \times N \to (N \times N)/\Delta N$ is the quotient map and f is a map classifying the universal covering for $(N \times N)/\Delta N$. Note that for the fundamental groups, $ker(q_*)$ is the normal closure of the diagonal $\Delta\pi$ in $\pi \times \pi$. Therefore, $(j \times j)_*(ker(q_*)) \subset \Delta(Ab(\pi)) = ker(\mu_*)$. Hence there is a homomorphism $\phi : Ab(\pi) \to Ab(\pi)$ such that $\mu_* \circ (j \times j)_* = \phi q_*$:

$$\begin{array}{ccc} \pi \times \pi & \xrightarrow{q_*} & \pi_1(N \times N/\Delta N) \\ {\scriptstyle (j\times j)_*}\big\downarrow & & \big\downarrow{\scriptstyle \phi} \\ Ab(\pi) \times Ab(\pi) & \xrightarrow{\mu_*} & Ab(\pi). \end{array}$$

By Proposition 3.4.1, $\pi_1(N \times N/\Delta N) = Ab(\pi) = \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2$. Since ϕ is surjective the homomorphism ϕ is an isomorphism. The homomorphism ϕ can be realized by a map $f : (N \times N)/\Delta N \to \mathbb{A}(N)$. Since f induces an isomorphism of the fundamental groups f is a classifying map. Since the maps $\mu \circ (j \times j)$ and $f \circ q$ with the target space $K(Ab(\pi), 1)$ induces isomorphisms of the fundamental groups, they are homotopic.

Finally, we note that the fundamental class of $(N \times N)/\Delta N$ is the image of that of $N \times N$. Then we apply Proposition 3.3.5 and Proposition 2.4.2. $\square$

3.4.3. **Corollary.** $\mathrm{cat}((N \times N)/\Delta N) \le 3$.

Proof. We apply Proposition 2.3.1. $\square$

3.4.4. **Theorem.** *For a non-orientable surface of genus g,*

$$\mathrm{cat}((N_g \times N_g)/\Delta N_g) = 3.$$

Proof. We take $x \in H^1(N_g; \mathbb{Z}_2)$ such that $x^2 \neq 0$. Note that $(x \times 1 + 1 \times x)^2 = x^2 \times 1 + 1 \times x^2$ in $H^*(N_g \times N_g; \mathbb{Z}_2)$. Then

$$(x \times 1 + 1 \times x)^3 = (x^2 \times 1 + 1 \times x^2)(x \times 1 + 1 \times x) = x \times x^2 + x^2 \times x \neq 0.$$

The restriction of $x \times 1 + 1 \times x$ to the diagonal $\Delta N_g \subset N_g \times N_g$ equals

$$x \smile 1 + 1 \smile x = 2x$$

where $\smile$ is the cup product. Since $2x = 0$ in $H^*(N_g; \mathbb{Z}_2)$, we obtain $x \times 1 + 1 \times x = q^*(y)$ for some $y \in H^1((N_g \times N_g)/\Delta N_g; \mathbb{Z}_2)$. Therefore, $y^3 \neq 0$ in $H^*((N_g \times N_g)/\Delta N_g; \mathbb{Z}_2)$. By the cup-length estimate (Theorem 2.1.1),

$$\mathrm{cat}((N_g \times N_g)/\Delta N_g) \ge 3.$$

This together with Corollary 3.4.3 implies the required equality. $\square$

4. Topological complexity of non-orientable surfaces

Let M be an orientable surface, $P = \mathbb{R}P^2$ be the projective plane, and let $q : M \vee P \to P$ denote the collapsing M map. We denote by $\mathcal{O}' = (q \times q)^* \mathcal{O}_{P \times P}$ the pull-back of the orientation sheaf on $P \times P$.

4.0.5. **Proposition.** *The 4-dimensional homology group of $(M \vee P)^2$ with coefficients in $\mathcal{O}'$ equals*

$$H_4((M \vee P)^2; \mathcal{O}') = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}.$$

Moreover, the inclusions of manifolds $\xi_i : W_i \to (M \vee P)^2$, $i = 1, \dots, 4$ induce isomorphisms $H_4(W_i; \xi_i^* \mathcal{O}') \to H_4((M \vee P)^2; \mathcal{O}')$ onto the summands where $W_1 = M^2$, $W_2 = P^2$, $W_3 = M \times P$, and $W_4 = P \times M$.

Proof. We note that from the Mayer-Vietoris exact sequence for the decomposition $(M \vee P)^2 = A \cup B$ with $A = M^2 \cup P^2$ and $B = (M \times P) \cup (P \times M)$,

$$\cdots \to H_4(A; \mathcal{O}'|_A) \oplus H_4(B; \mathcal{O}'|_B) \xrightarrow{\psi} H_4((M \vee P)^2; \mathcal{O}') \to H_3(M \vee M \vee P \vee P; \mathcal{O}'|_{\dots})$$

and dimensional reasons it follows that ψ is an isomorphism. Note that the intersections $M^2 \cap P^2$ and $(M \times P) \cap (P \times M)$ in $(M \vee P)^2$ are singletons. Hence, $A = M^2 \vee P^2$ and $B = M \times P \vee P \times M$. Thus, ψ defines the required isomorphism. $\square$

4.0.6. **Corollary.**

$$H_4((M \vee P)^2; \mathcal{O}') = H_4(M^2) \oplus H_4(P^2; \mathcal{O}_{P^2}) \oplus H_4(M \times P; \mathcal{O}_{M \times P}) \oplus H_4(P \times M; \mathcal{O}_{P \times M}).$$

Proof. The proof is a verification that the restriction of $\mathcal{O}'$ to each W_i, $i = 1, 2, 3, 4$, is the orientation sheaf. $\square$

Let the map $f : M\#P \to (M\#P)/S^1 = M\vee P$ be the collapsing of the connected sum circle. Note that the composition $q \circ f$ with the above q takes the orientation sheaf $\mathcal{O}_P$ to the orientation sheaf $\mathcal{O}_{M\#P}$.

4.0.7. **Proposition.** $(f \times f)_*([(M\#P)^2]) = [M^2] + [P^2] + [M \times P] + [P \times M]$.

Proof. Let $B \subset \xi(W_i)$ be a 4-ball. Then we claim that the homomorphism

$$H_4((M \vee P)^2; \mathcal{O}') \to H^4((M \vee P)^2; (M \vee P)^2 \setminus \overset{\circ}{B}; \mathcal{O}') = H_4(B, \partial B) = \mathbb{Z}$$

generated by the map of pairs and the excision is the projection of the direct sum $H_4((M \vee P)^2; \mathcal{O}') = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ onto the i-th summand. This follows from the commutative diagram

$$\begin{array}{ccccc}
H_4((M \vee P)^2; \mathcal{O}') & \longrightarrow & H_4((M \vee P)^2; (M \vee P)^2 \setminus \overset{\circ}{B}; \mathcal{O}') & \xleftarrow{=} & H_4(B, \partial B) = \mathbb{Z} \\
\xi_i \uparrow & & & & = \uparrow \\
H_4(W_i; \xi^* \mathcal{O}') & \xrightarrow{=} & H_n(W_i, W_i \setminus \xi_i^{-1}(\overset{\circ}{B}); \xi^* \mathcal{O}') & \xleftarrow{=} & H_4(B, \partial B) = \mathbb{Z}.
\end{array}$$

The commutative diagram

$$\begin{array}{ccccc}
H_4((M\#P)^2; \mathcal{O}_{(M\#P)^2}) & \xrightarrow{=} & H_4((M\#P)^2, (M\#P)^2 \setminus \overset{\circ}{B}; \mathcal{O}_{(M\#P)^2}) & \xleftarrow{=} & H_4(B, \partial B) = \mathbb{Z} \\
f \times f \downarrow & & & & = \downarrow \\
H_4((M \vee P)^2; \mathcal{O}') & \longrightarrow & H_4((M \vee P)^2; (M \vee P)^2 \setminus \overset{\circ}{B}; \mathcal{O}') & \xleftarrow{=} & H_4(B, \partial B) = \mathbb{Z}
\end{array}$$

shows that the projection of the image $f_*([M\#P])$ of the fundamental class onto the i-th summand, $i = 1, 2, 3, 4$, is a fundamental class. □

The proof of the following proposition is straightforward.

4.0.8. **Proposition.** *A retraction $r : X \to A$, $A \subset X$, defines a fiberwise retraction $\bar{r} : (p^X)^{-1}(A) \to PA$. Moreover, for each k it defines a fiberwise retraction $\bar{r}_k : (p_k^X)_X^{-1}(A) \to \Delta_k(A)$ of the fiberwise joins:*

$$\begin{array}{ccccc}
\Delta_k(X) & \xleftarrow{\supset} & (p_k^X)_X^{-1}(A) & \xrightarrow{\bar{r}_k} & \Delta_k(A) \\
p_k^X \downarrow & & p_k^X| \downarrow & & p_k^A \downarrow \\
X \times X & \xleftarrow{\supset} & A \times A & \xrightarrow{=} & A \times A.
\end{array}$$

We denote

$$g = (1 \vee j)^2 : (M \vee \mathbb{R}P^2)^2 \to (M \vee \mathbb{R}P^\infty)^2.$$

It is easy to see that the sheaf $\mathcal{O}'$ on $(M \vee \mathbb{R}P^2)^2$ is the pull-back under g of a sheaf $\tilde{\mathcal{O}}$ on $(M \vee \mathbb{R}P^\infty)^2$ which comes from the pull-back of the tensor product $\mathcal{O}\hat{\otimes}\mathcal{O}$ of the canonical Z-bundles on $\mathbb{R}P^\infty$.

4.0.9. **Proposition.** *Let $\kappa \in H^4((M \vee \mathbb{R}P^\infty)^2; \mathcal{F})$ be the primary obstruction to a section of*

$$\bar{p} = p_3^{M \vee \mathbb{R}P^\infty} : \Delta_3(M \vee \mathbb{R}P^\infty) \to (M \vee \mathbb{R}P^\infty)^2$$

where M is an orientable surface of genus ≥ 2. Then

(1) $[M^2] \cap g^*(\kappa) \neq 0$,
(2) $[(\mathbb{R}P^2)^2] \cap g^*(\kappa) = 0$, *and*
(3) $([M \times \mathbb{R}P^2] + [\mathbb{R}P^2 \times M]) \cap g^*(\kappa) = 0$.

Proof. (1) Assume that $[M^2]\cap g^*(\kappa)=0$. Then, $g_*([M^2])\cap\kappa=0$. This means that the map $\bar{p}$ admits a section over $M^2\subset(M\vee\mathbb{R}P^\infty)^2$. The collapsing $\mathbb{R}P^\infty$ to a point defines a retraction $r: M\vee\mathbb{R}P^\infty\to M$. By Proposition 4.0.8 the retraction r defines a fiberwise retraction of $\bar{p}^{-1}(M^2)$ onto $\Delta_3(M)$. This implies that $p_3^M:\Delta_3(M)\to M^2$ admits a section. Hence, by Theorem 2.6.3, $TC(M)\le 3$. This contradicts to the fact that $TC(M)=4$.

(2) Since $TC(\mathbb{R}P^2)=3$, by Corollary 2.6.4 the map g restricted to $(\mathbb{R}P^2)^2$ admits a lift with respect to $\bar{p}$. Hence the primary obstruction o' to such a lift is zero. Note that $o'=(g^*\kappa)|_{(\mathbb{R}P^2)^2}$ is the restriction to $(\mathbb{R}P^2)^2$ of the image of κ under g^*. Hence,

$$[\mathbb{R}P^2]\cap g^*(\kappa)=[\mathbb{R}P^2]\cap(g^*\kappa)|_{(\mathbb{R}P^2)^2}=0.$$

(3) Let $\sigma:(M\vee\mathbb{R}P^\infty)^2\to(M\vee\mathbb{R}P^\infty)^2$ be the natural involution: $\sigma(x,y)=(y,x)$. We may assume that the map σ is cellular. It defines an involution $\bar{\sigma}$ on the path space $P(M\vee\mathbb{R}P^\infty)$ and involutions $\bar{\sigma}_k$ on the iterated fiberwise joins $\Delta_k(M\vee\mathbb{R}P^\infty)$.

Let $K=(M\vee\mathbb{R}P^\infty)^2$ be a σ-invariant CW complex structure with an invariant subcomplex $Q=(M\times\mathbb{R}P^\infty)\vee(\mathbb{R}P^\infty\times M)$. We claim that there is a section $s:K^{(3)}\to\Delta_3(M\vee\mathbb{R}P^\infty)$ which is σ-equivariant on $Q^{(3)}$. First we fix an invariant section at the wedge point

$$s(x_0,x_0)=c_{x_0}+0+0+0\in\Delta_3(M\vee\mathbb{R}P^\infty)$$

where c_{x_0} is the constant path at x_0. Then we define our section s on $Q^{(3)}$ by induction on dimension of simplices. We note that $\sigma(e)\neq e$ for all cells in Q except the wedge vertex. Assume that an equivariant section s is defined on the i-skeleton $Q^{(i)}$, $i<3$. Then for all distinct pairs of i-cells $e,\sigma(e)$ we do an extension of s to e and define it on $\sigma(e)$ to be $\bar{\sigma}_3 s\sigma$. Note that an extension of s to e exists since the fiber of $\bar{p}$ is 2-connected. Also, in view of the 2-connectedness of the fiber the section s on $Q^{(3)}$ can be extended to $K^{(3)}$.

Thus, we may assume that the restriction of the obstruction cocycle to Q is symmetric. Hence, for the obstruction cohomology class we obtain $(\sigma^*\kappa)|_Q=\sigma_0^*(\kappa|_Q)=\kappa|_Q$ where $\sigma_0=\sigma|_Q$.

Let

$$q_0:(M\times\mathbb{R}P^\infty)\vee(\mathbb{R}P^\infty\times M)\to M\times\mathbb{R}P^\infty$$

be the projection to the orbit space of the σ-action, i.e., the folding map. Let $\kappa'=\kappa|_{M\times\mathbb{A}(\mathbb{R}P^2)}$. Then $\kappa|_Q=q_0^*(\kappa')$. Note that $\tilde{\mathcal{O}}$ restricted to $(M\times\mathbb{R}P^\infty)\vee(\mathbb{R}P^\infty\times M)$ equals $q_0^*\tilde{\mathcal{O}}|_{M\times\mathbb{R}P^\infty}$. Hence the homomorphism in homology $(q_0)_*$ is well defined. Since q_0 induces an epimorphism of the fundamental groups and takes both classes $g_*[M\times\mathbb{R}P^2]$ and $g_*[\mathbb{R}P^2\times M]$ to $g_*[M\times\mathbb{R}P^2]$, we obtain

$$(q_0)_*((g_*[M\times\mathbb{R}P^2]+g_*[\mathbb{R}P^2\times M])\cap\kappa)=2g_*[M\times\mathbb{R}P^2]\cap\kappa'=0.$$

The last equality follows from the fact that $[\mathbb{R}P^2]$ has order 2 in $\mathbb{R}P^\infty$ (see Proposition 3.2.1).

Since q_0 induces an epimorphism of the fundamental groups, we obtain

$$(g_*[M\times\mathbb{R}P^2]+g_*[\mathbb{R}P^2\times M])\cap\kappa=0.$$

Therefore, $([M\times\mathbb{R}P^2]+[\mathbb{R}P^2\times M])\cap g^*(\kappa)=0$. ◻

4.0.10. **Corollary.**

$$([M^2]+[(\mathbb{R}P^2)^2]+[M\times\mathbb{R}P^2]+[\mathbb{R}P^2\times M])\cap g^*(\kappa)\neq 0.$$

4.0.11. **Corollary.**

$$g_*([M^2]+[(\mathbb{R}P^2)^2]+[M\times\mathbb{R}P^2]+[\mathbb{R}P^2\times M])\cap\kappa\neq 0.$$

Proof. First we recall that g_* is well defined Note that $g_*(([M^2]+[(\mathbb{R}P^2)^2]+[M\times\mathbb{R}P^2]+[\mathbb{R}P^2\times M])\cap g^*(\kappa))=g_*([M^2]+[(\mathbb{R}P^2)^2]+[M\times\mathbb{R}P^2]+[\mathbb{R}P^2\times M])\cap\kappa$. Since g_* is an isomorphism in dimension 0, we derive the result from Corollary 4.0.10. ☐

4.0.12. **Theorem.** *For $g>4$, $TC(N_g)=4$.*

Proof. First we consider the case when g is odd. Then $N_g=M\#\mathbb{R}P^2$ for an orientable surface M of genus >1. Let $f:N_g=M\#\mathbb{R}P^2\to M\vee\mathbb{R}P^2$ be a map that collapses the connected sum circle. Clearly, f induces an epimorphism of the fundamental groups. Note that the orientation sheaf $\mathcal{O}_{N_g}$ is the pull-back $f^*q^*\mathcal{O}_{\mathbb{R}P^2}$ where $q:M\vee\mathbb{R}P^2\to\mathbb{R}P^2$ is the collapsing map.

We show that the map $g\circ(f\times f)=(1\vee j)f\times(1\vee j)f$ does not admit a lift with respect to

$$\bar{p}=p_3^{M\vee\mathbb{R}P^\infty}:\Delta_3(M\vee\mathbb{R}P^\infty)\to(M\vee\mathbb{R}P^\infty)^2.$$

Then, by Corollary 2.6.4 we obtain the inequality $TC(N_g)\geq 4$.

The primary obstruction o to such a lift is the image $(f\times f)^*g^*(\kappa)$ of the primary obstruction to a section. Note that by Proposition 4.0.7 and Corollary 4.0.11,

$$g_*(f\times f)_*([N_g^2]\cap o)=g_*(f\times f)_*([N_g^2])\cap\kappa=g_*([M^2]+[P^2]+[M\times P]+[P\times M])\cap\kappa\neq 0$$

where $P=\mathbb{R}P^2$. Therefore, $[N_g^2]\cap o\neq 0$. By the Poincaré duality (with local coefficients) we obtain that $o\neq 0$.

When $g>4$ is even, $N_g=M\#\mathbb{R}P^2\#\mathbb{R}P^2$ for an orientable surface M of genus >1. We consider the map $f:N_g\to M\vee\mathbb{R}P^2$ which is the composition of the quotient map $N_g\to M\vee\mathbb{R}P^2\vee\mathbb{R}P^2$ and union of the folding map $\mathbb{R}P^2\vee\mathbb{R}P^2\to\mathbb{R}P^2$ and the identity map on M. For such f the orientation sheaf on N_g can be pushed forward and the above argument works. ☐

1. *Remark.* Implicitly our proof of the inequality $TC(N_g)\geq 4$ is based on the zero-divisors cup-length estimate as in [F]. Indeed, by Schwarz' Theorem 2.6.2, $\kappa=\beta^4$ where $\beta\in H^1((M\vee\mathbb{R}P^\infty)^2;\mathcal{F}_0)$ is the primary obstruction for the section of the fibration

$$p^{M\vee\mathbb{R}P^\infty}:P(M\vee\mathbb{R}P^\infty)\to(M\vee\mathbb{R}P^\infty)^2.$$

Therefore, the restriction of β to the diagonal equals zero. We have proved that $\alpha^4\neq 0$ for the element $\alpha=(f\times f)^*g^*(\beta)\in H^1(N_g\times N_g;\mathcal{F}_0^*)$ which is trivial on the diagonal. The local coefficient system $\mathcal{F}_0^*$ as well as a cocycle a representing α can be presented explicitly (in terms of $\pi_1(N_g)$-modules and cross homomorphisms) as it was done in [C].

This is not surprising in view of Theorem 2.1, part (b) of Farber and Grant [FG].

2. *Remark.* The above technique can be pushed to get $TC(N_4)=4$ (see a version of this paper in ArXiv). The technique does not seem to be applicable to the Klein bottle nor to $N_3=\mathbb{R}P^2\#\mathbb{R}P^2\#\mathbb{R}P^2$.

5. Higman's group

Higman introduced [Hi] a group H generated by 4 elements a, b, c, d with the relations

$$a^{-1}ba = b^2,\ b^{-1}cb = c^2,\ c^{-1}dc = d^2,\ d^{-1}ad = a^2.$$

Among others the group H has the following properties: It is acyclic and it has finite 2-dimensional Eilenberg-Maclane complex $K(H,1)$.

5.0.13. **Theorem.** *Let $K = K(H,1)$ where H is Higman's group. Then*

$$2 = \mathrm{cat}(C_{\Delta K}) < TC(K) = 4.$$

Proof. By Proposition 3.4.1, $\pi_1(C_{\Delta K}) = H_1(H) = 0$. Then by Theorem 2.1.1

$$\mathrm{cat}(C_{\Delta K}) \leq (\dim C_{\Delta K})/2 = 2.$$

The equality $TC(K(H,1)) = 4$ is a computation by Grant, Lupton and Oprea [GLO]. □

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References

[Be] Israel Berstein, *On the Lusternik-Schnirelmann category of Grassmannians*, Math. Proc. Cambridge Philos. Soc. **79** (1976), no. 1, 129–134. MR0400212

[Bre] Glen E. Bredon, *Sheaf theory*, 2nd ed., Graduate Texts in Mathematics, vol. 170, Springer-Verlag, New York, 1997. MR1481706

[Bro] Kenneth S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics, vol. 87, Springer-Verlag, New York, 1994. Corrected reprint of the 1982 original. MR1324339

[CLOT] Octav Cornea, Gregory Lupton, John Oprea, and Daniel Tanré, *Lusternik-Schnirelmann category*, Mathematical Surveys and Monographs, vol. 103, American Mathematical Society, Providence, RI, 2003. MR1990857

[C] A. E. Costa, *Topological complexity of configuration spaces*, Ph.D. Thesis, Durham University, 2010.

[DT] Albrecht Dold and René Thom, *Quasifaserungen und unendliche symmetrische Produkte* (German), Ann. of Math. (2) **67** (1958), 239–281. MR0097062

[Dr1] Alexander Dranishnikov, *Free abelian topological groups and collapsing maps*, Topology Appl. **159** (2012), no. 9, 2353–2356, DOI 10.1016/j.topol.2011.03.019. MR2921824

[Dr2] Alexander Dranishnikov, *Topological complexity of wedges and covering maps*, Proc. Amer. Math. Soc. **142** (2014), no. 12, 4365–4376, DOI 10.1090/S0002-9939-2014-12146-0. MR3267004

[Dr3] Alexander N. Dranishnikov, *The LS category of the product of lens spaces*, Algebr. Geom. Topol. **15** (2015), no. 5, 2985–3010. MR3426700

[DR] Alexander N. Dranishnikov and Yuli B. Rudyak, *On the Berstein-Svarc theorem in dimension 2*, Math. Proc. Cambridge Philos. Soc. **146** (2009), no. 2, 407–413, DOI 10.1017/S0305004108001904. MR2475974

[F] Michael Farber, *Invitation to topological robotics*, Zurich Lectures in Advanced Mathematics, European Mathematical Society (EMS), Zürich, 2008. MR2455573

[FG] Michael Farber and Mark Grant, *Topological complexity of configuration spaces*, Proc. Amer. Math. Soc. **137** (2009), no. 5, 1841–1847, DOI 10.1090/S0002-9939-08-09808-0. MR2470845

[FTY] Michael Farber, Serge Tabachnikov, and Sergey Yuzvinsky, *Topological robotics: motion planning in projective spaces*, Int. Math. Res. Not. **34** (2003), 1853–1870, DOI 10.1155/S1073792803210035. MR1988783

[GV1] J. Garcia-Calcines and L. Vandembroucq, *Weak sectional category*, J. Lond. Math. Soc. (2) **82** (2010), no. 3, 621–642, DOI 10.1112/jlms/jdq048. MR2739059

[GV2] J. M. García Calcines and L. Vandembroucq, *Topological complexity and the homotopy cofibre of the diagonal map*, Math. Z. **274** (2013), no. 1-2, 145–165, DOI 10.1007/s00209-012-1061-5. MR3054321

[G] M. I. Graev, *Free topological groups* (Russian), Izvestiya Akad. Nauk SSSR. Ser. Mat. **12** (1948), 279–324. MR0025474

[GLO] Mark Grant, Gregory Lupton, and John Oprea, *New lower bounds for the topological complexity of aspherical spaces*, Topology Appl. **189** (2015), 78–91, DOI 10.1016/j.topol.2015.04.005. MR3342573

[Gr] Misha Gromov, *Metric structures for Riemannian and non-Riemannian spaces*, Progress in Mathematics, vol. 152, Birkhäuser Boston, Inc., Boston, MA, 1999. Based on the 1981 French original [MR0682063 (85e:53051)]; With appendices by M. Katz, P. Pansu and S. Semmes; Translated from the French by Sean Michael Bates. MR1699320

[Hi] Graham Higman, *A finitely generated infinite simple group*, J. London Math. Soc. **26** (1951), 61–64. MR0038348

[KR] Mikhail G. Katz and Yuli B. Rudyak, *Lusternik-Schnirelmann category and systolic category of low-dimensional manifolds*, Comm. Pure Appl. Math. **59** (2006), no. 10, 1433–1456, DOI 10.1002/cpa.20146. MR2248895

[M] A. Markoff, *On free topological groups* (Russian, with English summary), Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **9** (1945), 3–64. MR0012301

[Sch] A. S. Švarc, *The genus of a fibered space* (Russian), Trudy Moskov. Mat. Obšč. **10** (1961), 217–272. MR0154284

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