EQUIDISTRIBUTION OF EXPANDING CURVES IN HOMOGENEOUS SPACES AND DIOPHANTINE APPROXIMATION ON SQUARE MATRICES

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(Communicated by Nimish Shah)

ABSTRACT. In this paper, we study an analytic curve $\varphi : I = [a, b] \rightarrow M(n \times n, \mathbb{R})$ in the space of n by n real matrices. There is a natural map $u : M(n \times n, \mathbb{R}) \rightarrow H = \mathrm{SL}(2n, \mathbb{R})$. Let G be a Lie group containing H and $\Gamma < G$ be a lattice of G . Let $X = G/\Gamma$. Then given a dense H -orbit in X , one could embed $u(\varphi(I))$ into X . We consider the expanding translates of the curve by some diagonal subgroup $A = \{a(t) : t \in \mathbb{R}\} \subset H$. We will prove that if φ satisfies certain geometric conditions, then the expanding translates will tend to be equidistributed in G/Γ , as $t \rightarrow +\infty$. As an application, we show that for almost every point on $\varphi(I)$, the Diophantine approximation given by Dirichlet's Theorem is not improvable.

1. INTRODUCTION

1.1. Dirichlet's Theorem on Diophantine approximation. For any real vector space $V = \mathbb{R}^k$, let $\|\cdot\|$ denote the supreme norm on V , i.e., for $\mathbf{x} = (x_1, x_2, \dots, x_k) \in \mathbb{R}^k$, $\|\mathbf{x}\| := \max_{1 \leq i \leq k} |x_i|$. Given $m, n \in \mathbb{N}$, let $M(m \times n, \mathbb{R})$ denote the space of m by n real matrices. The Dirichlet's Theorem on Diophantine properties of m by n matrices says the following:

Theorem 1.1 (Dirichlet's Theorem). *Given any $\Phi \in M(m \times n, \mathbb{R})$ and any $N \geq 1$, there exists a nonzero integer vector $\mathbf{p} \in \mathbb{Z}^n$ with $\|\mathbf{p}\| \leq N^m$ and an integer vector $\mathbf{q} \in \mathbb{Z}^m$ such that $\|\Phi\mathbf{p} - \mathbf{q}\| \leq N^{-n}$.*

Now we consider the following finer question: for a particular $\Phi \in M(m \times n, \mathbb{R})$, could we improve Dirichlet's Theorem? By improving Dirichlet's Theorem, we mean that there exists a constant $0 < \mu < 1$, such that for all large $N > 0$, there exists a nonzero integer vector $\mathbf{p} \in \mathbb{Z}^n$ with $\|\mathbf{p}\| \leq \mu N^m$, and an integer vector $\mathbf{q} \in \mathbb{Z}^m$ such that $\|\Phi\mathbf{p} - \mathbf{q}\| \leq \mu N^{-n}$. If such constant μ exists, then we say that Φ is DT_μ -improvable. We say that Φ is DT -improvable if Φ is DT_μ -improvable for some $0 < \mu < 1$ (here DT stands for Dirichlet's Theorem).

In 1970, Davenport and Schmidt [DS70] considered this question and answered it to some extent: they proved that almost every matrix $\Phi \in M(m \times n, \mathbb{R})$ is

Received by the editors August 11, 2015 and, in revised form, February 19, 2016, February 23, 2016 and February 24, 2016.

2010 *Mathematics Subject Classification.* Primary 37A17; Secondary 22F30, 11J13.

Key words and phrases. Equidistribution, homogeneous spaces, Ratner's Theorem, Diophantine approximation, Dirichlet's Theorem.

The author was supported in part by a Postdoctoral Fellowship at MSRI.

not DT-improvable. In [DS70], it was also proved that for $m = 1$ and $n = 2$, $M(1 \times 2, \mathbb{R}) = \mathbb{R}^2$, almost every point on the curve

$$\phi(s) = (s, s^2) : s \in \mathbb{R}$$

is not $DT_{1/4}$ -improvable. Their result on the curve ϕ was generalized by Baker [Bak78]: it was proved that for any smooth curve in $\mathbb{R}^2$ satisfying some curvature condition, almost every point on the curve is not DT_μ -improvable for some $0 < \mu < 1$ depending on the curve. In 2002, Bugeaud [Bug02] generalized the result of Davenport and Schmidt in the following sense: for $m = 1$, and general n , almost every point on the curve $\varphi(s) = (s, s^2, \dots, s^n)$ is not DT_μ -improvable for some small constant $0 < \mu < 1$. Their proofs are based on the technique of **regular systems** introduced by Davenport and Schmidt [DS70].

Based on the work of Dani [Dan84], as well as Kleinbock and Margulis [KM98], Kleinbock and Weiss [KW08] studied this Diophantine approximation problem in the language of homogeneous dynamics, and proved the following result: for $m = 1$ and arbitrary n , if an analytic curve in $M(1 \times n, \mathbb{R}) \cong \mathbb{R}^n$ satisfies some nondegeneracy condition, then almost every point on the curve is not DT_μ -improvable for some small constant $0 < \mu < 1$ depending on the curve. In 2009, Nimish Shah [Sha09b] proved the following stronger result: for $m = 1$ and general n , if an analytic curve $\varphi : I = [a, b] \rightarrow \mathbb{R}^n$ is not contained in any proper affine subspace, then almost every point on the curve is not DT-improvable.

In this paper, we focus on the case of square matrices. We will prove the following theorem.

Theorem 1.2. *For $m = n$, suppose an analytic curve*

$$\varphi : I = [a, b] \rightarrow M(n \times n, \mathbb{R})$$

satisfies the following conditions:

- A.1** *For some $s_0 \in I$ the derivative $\varphi^{(1)}(s_0)$ is invertible (in particular, this implies that there exists a closed subinterval $J_{s_0} \subset I$ such that $\varphi(s) - \varphi(s_0) \in GL(n, \mathbb{R})$ for $s \in J_{s_0}$).*
- A.2** *There exists a $s_0 \in I$ satisfying the condition above such that $\{(\varphi(s) - \varphi(s_0))^{-1} : s \in J_{s_0}\}$ is not contained in any proper affine subspace of $M(n \times n, \mathbb{R})$.*

Then almost every point on the curve is not DT-improvable.

Remark 1.3. Because φ is analytic, $\varphi^{(1)}(s)$ is invertible at some $s_0 \in I$ implies that it is invertible for all $s \in I$ except finitely many points.

1.2. Equidistribution of expanding curves on homogeneous spaces. Now we describe the correspondence between Diophantine approximation and homogeneous dynamics.

Let $G = SL(m + n, \mathbb{R})$, and let $\Gamma = SL(m + n, \mathbb{Z})$. The homogeneous space $X = G/\Gamma$ may be identified with the space of unimodular lattices of $\mathbb{R}^{m+n}$. In this identification, $g\Gamma$ corresponds to the unimodular lattice $g\mathbb{Z}^{m+n}$. For $r > 0$, let B_r denote the closed ball in $\mathbb{R}^{m+n}$ centered at the origin and of radius r . For any $0 < \mu < 1$, the subset

$$K_\mu := \{\Lambda \in G/\Gamma : \Lambda \cap B_\mu = \{\mathbf{0}\}\}$$

contains an open neighborhood of $\mathbb{Z}^{m+n} \in X$. Define

$$A := \left\{ a(t) := \begin{bmatrix} e^{nt} \mathbf{I}_m & \\ & e^{-mt} \mathbf{I}_n \end{bmatrix} : t \in \mathbb{R} \right\}.$$

Let us define the embedding $u : M(m \times n, \mathbb{R}) \hookrightarrow \mathrm{SL}(m+n, \mathbb{R})$ by

$$\Phi \in M(m \times n, \mathbb{R}) \mapsto u(\Phi) := \begin{bmatrix} \mathbf{I}_m & \Phi \\ & \mathbf{I}_n \end{bmatrix}.$$

Suppose $\Phi \in M(m \times n, \mathbb{R})$ is DT_μ -improvable for some $0 < \mu < 1$; then for any $N > 0$ large enough, there exist a nonzero integer vector $\mathbf{p} \in \mathbb{Z}^n$ and an integer vector $\mathbf{q} \in \mathbb{Z}^m$, such that $\|\mathbf{p}\| \leq \mu N^m$ and $\|\Phi \mathbf{p} - \mathbf{q}\| \leq \mu N^{-n}$. Then we have that $\|a(\log N)u(\Phi)(-\mathbf{q}, \mathbf{p})\| \leq \mu$. This implies that $a(\log N)u(\Phi)\mathbb{Z}^{m+n} \notin K_\mu$ for all $N > 0$ large enough. Thus, to prove that $\Phi \in M(m \times n, \mathbb{R})$ is not DT_μ -improvable, it suffices to show that the forward trajectory $\{a(t)u(\Phi)[e] : t > 0\}$ meets K_μ infinitely many times. In particular, for an analytic curve

$$\varphi : I = [a, b] \rightarrow M(m \times n, \mathbb{R}),$$

if we could show that the expanding translates $\{a(t)u(\varphi(I))[e] : t > 0\}$ tend to be equidistributed in X , then we could deduce that for almost every point $\varphi(s)$ on the curve, the forward trajectory $\{a(t)u(\varphi(s))[e] : t > 0\}$ is dense (see [Sha09b, §2] for the proof). This implies that almost every $\varphi(s)$ is not DT -improvable.

It turns out that when $m = n$, we could prove the following much stronger result:

Theorem 1.4. *Let G be a Lie group containing $H = \mathrm{SL}(2n, \mathbb{R})$, and $\Gamma < G$ be a lattice in G . Let μ_G denote the unique G -invariant probability measure on the homogeneous space G/Γ . Take $x = g\Gamma \in G/\Gamma$ such that its H -orbit Hx is dense in G/Γ . Let us fix the diagonal group*

$$(1.1) \quad A = \left\{ a(t) = \begin{bmatrix} e^t \mathbf{I}_n & \\ & e^{-t} \mathbf{I}_n \end{bmatrix} \right\}.$$

Let $\varphi : I = [a, b] \rightarrow M(n \times n, \mathbb{R})$ be an analytic curve, and embed the curve into H via

$$u : X \in M(n \times n, \mathbb{R}) \mapsto u(X) = \begin{bmatrix} \mathbf{I}_n & X \\ & \mathbf{I}_n \end{bmatrix}.$$

Let μ_t denote the normalized Lebesgue measure on the curve $a(t)u(\varphi(I))x \subset G/\Gamma$, i.e., for a compactly supported continuous function $f \in C_c(G/\Gamma)$,

$$\int f d\mu_t := \frac{1}{|I|} \int_{s \in I} f(a(t)u(\varphi(s))x) ds.$$

If the curve φ satisfies the conditions **A.1** and **A.2** given in Theorem 1.2, then $\mu_t \rightarrow \mu_G$ as $t \rightarrow +\infty$ in weak-* topology, i.e., for any function $f \in C_c(G/\Gamma)$,

$$\lim_{t \rightarrow +\infty} \frac{1}{|I|} \int_{s \in I} f(a(t)u(\varphi(s))x) ds = \int_{G/\Gamma} f d\mu_G.$$

Remark 1.5.

1. The assumption that Hx is dense in G/Γ does not reduce the generality of the theorem. In fact, since H is generated by unipotent subgroups contained in H , by Ratner's Theorem (cf. [Rat91b]), the closure of Hx must be some homogeneous subspace Fx where F is some Lie subgroup of G containing H such that the orbit Fx is closed and of finite volume. Then we can make the assumption hold by replacing G , Γ and $x = g\Gamma$ by F , $g\Gamma g^{-1} \cap F$ and $[e]$ respectively.

2. To ensure the equidistribution, in condition **A.2** given in Theorem 1.2, we only need $\{(\varphi(s) - \varphi(s_0))^{-1} : s \in J_{s_0}\}$ to avoid a family of special proper affine subspaces coming from a particular representation of H . We will discuss it in Remark 4.4.

3. Even in the case $G = H = \mathrm{SL}(2n, \mathbb{R})$, Theorem 1.4 is much stronger than Theorem 1.2, since it applies to arbitrary lattice $\Gamma \subset G$.

1.3. Relation to extremity of submanifolds in matrix spaces. Another direction to study Diophantine properties of a real matrix $\Phi \in \mathrm{M}(m \times n, \mathbb{R})$ is to determine whether Φ is very well approximable. We say $\Phi \in \mathrm{M}(m \times n, \mathbb{R})$ is very well approximable if there exists some constant $\delta > 0$ such that there exist infinitely many nonzero integer vectors $\mathbf{p} \in \mathbb{Z}^n$ and integer vectors $\mathbf{q} \in \mathbb{Z}^m$ such that

$$\|\Phi \mathbf{p} - \mathbf{q}\| \leq \|\mathbf{p}\|^{-n/m-\delta}.$$

A submanifold $\mathcal{U} \subset \mathrm{M}(m \times n, \mathbb{R})$ is called extremal if with respect to the Lebesgue measure on $\mathcal{U}$, almost every point is not very well approximable. Given a submanifold in $\mathrm{M}(m \times n, \mathbb{R})$, it is natural to ask whether it is extremal. Because of the same correspondence as above, this problem can also be studied through homogeneous dynamics. Kleinbock and Margulis [KM98] proved that if a submanifold $\mathcal{U} \subset \mathrm{M}(1 \times n, \mathbb{R}) = \mathbb{R}^n$ is nondegenerate, then $\mathcal{U}$ is extremal. Kleinbock, Margulis and Wang [KMW10] gave a necessary and sufficient condition of a submanifold in $\mathrm{M}(m \times n, \mathbb{R})$ being extremal. The condition is stated in terms of a particular representation of $H = \mathrm{SL}(m+n, \mathbb{R})$ and could not be translated to a geometric condition. Recently, Aka, Breuillard, Rosenzweig and de Saxcé [ABRdS14] defined a family of proper subvarieties in $\mathrm{M}(m \times n, \mathbb{R})$ called pencils, and announced a theorem stating that if a submanifold $\mathcal{U} \subset \mathrm{M}(m \times n, \mathbb{R})$ is not contained in a pencil, then $\mathcal{U}$ is extremal. It turns out that condition **A.1** in Theorem 1.2 implies the sufficient condition given in [ABRdS14]. We will discuss it in detail in Appendix A.

1.4. Organization of the paper. The paper is organized as follows. In §2, we will follow the argument developed in [Sha09c] to modify μ_t to another probability measure λ_t for each $t > 0$. We will show that $\lambda_t \rightarrow \mu_G$ will imply $\mu_t \rightarrow \mu_G$ and that any limit measure μ_∞ of $\{\lambda_t : t > 0\}$ is a probability measure and is invariant under the action of a unipotent subgroup. In §3, we will apply Ratner's Theorem and the linearization technique to show that if the limit measure μ_∞ is not μ_G , then the curve $\varphi(I)$ must satisfy some linear algebraic condition with respect to a particular finite dimensional representation $\mathcal{V}$ of $H = \mathrm{SL}(2n, \mathbb{R})$. In §4, we will complete the proof of Theorem 1.4. The proof is based on the linear algebraic condition we get in §3, and a technical lemma proved in [Yan13] on $\mathrm{SL}(2, \mathbb{R})$ representations. In Appendix A, we will discuss the condition given in [ABRdS14] and its relation to condition **A.1** in Theorem 1.2.

1.5. Notation. Throughout this paper, we will use the following notation.

Let $\pi : G \rightarrow X = G/\Gamma$ denote the natural projection mapping $g \in G$ to $g\Gamma \in X$.

For $\epsilon > 0$ small, and two quantities A and B , $A \stackrel{\epsilon}{\approx} B$ means that $|A - B| \leq \epsilon$. Fix a right G -invariant metric $d(\cdot, \cdot)$ on G ; then for $x_1, x_2 \in G/\Gamma$, and $\epsilon > 0$, $x_1 \stackrel{\epsilon}{\approx} x_2$ means $x_2 = gx_1$ such that $d(g, e) < \epsilon$. Given some quantity $A > 0$, we denote by $O(A)$ some quantity B such that $|B| \leq CA$ for some constant $C > 0$.

Let F be a Lie group, and V be a finite dimensional linear representation of F . Then for a one-parameter diagonal subgroup $A = \{a(t) : t \in \mathbb{R}\}$ of F , we could decompose V as a direct sum of eigenspaces of A , i.e.,

$$V = \bigoplus_{\lambda \in \mathbb{R}} V^\lambda(A),$$

where $V^\lambda(A) = \{v \in V : a(t)v = e^{\lambda t}v\}$.

We define

$$V^+(A) = \bigoplus_{\lambda > 0} V^\lambda(A),$$

$$V^-(A) = \bigoplus_{\lambda < 0} V^\lambda(A),$$

and similarly,

$$V^{+0}(A) = V^+(A) + V^0(A),$$

$$V^{-0}(A) = V^-(A) + V^0(A).$$

For a vector $v \in V$, we denote by $v^+(A)$ ($v^-(A)$, $v^0(A)$, $v^{+0}(A)$ and $v^{-0}(A)$ respectively) the projection of v onto $V^+(A)$ ($V^-(A)$, $V^0(A)$, $V^{+0}(A)$ and $V^{-0}(A)$ respectively).

2. NONDIVERGENCE OF LIMIT MEASURES AND UNIPOTENT INVARIANCE

2.1. Preliminaries on Lie group structures. We first recall some basic facts on $H = \mathrm{SL}(2n, \mathbb{R})$.

Let $A \subset H$ denote the one-parameter diagonal subgroup defined in (1.1), and let $Z_H(A)$ denote the centralizer of A in H . It is easy to see that

$$Z_H(A) = \left\{ \begin{bmatrix} B & \\ & C \end{bmatrix} : B, C \in \mathrm{GL}(n, \mathbb{R}), \text{ and } \det B \det C = 1 \right\}.$$

Define

$$U^+(A) := \{h \in H : a(-t)ha(t) \rightarrow e \text{ as } t \rightarrow +\infty\},$$

and

$$U^-(A) := \{h \in H : a(t)ha(-t) \rightarrow e \text{ as } t \rightarrow +\infty\}.$$

Then it is easy to see that $U^+(A)$ and $U^-(A)$ can be parametrized as follows:

$$U^+(A) = \left\{ u(X) = \begin{bmatrix} \mathrm{I}_n & X \\ & \mathrm{I}_n \end{bmatrix} : X \in \mathrm{M}(n \times n, \mathbb{R}) \right\},$$

and

$$U^-(A) = \left\{ u^-(X) = \begin{bmatrix} \mathrm{I}_n & \\ X & \mathrm{I}_n \end{bmatrix} : X \in \mathrm{M}(n \times n, \mathbb{R}) \right\}.$$

The Lie algebras $\mathfrak{u}^\pm(A)$ of $U^\pm(A)$ can be parametrized as follows:

$$\mathfrak{u}^+(A) = \left\{ \mathfrak{n}^+(X) := \log u(X) = \begin{bmatrix} \mathbf{0} & X \\ \mathbf{0} & \mathbf{0} \end{bmatrix} : X \in \mathrm{M}(n \times n, \mathbb{R}) \right\},$$

and

$$\mathfrak{u}^-(A) = \left\{ \mathfrak{n}^-(X) := \log u^-(X) = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ X & \mathbf{0} \end{bmatrix} : X \in \mathrm{M}(n \times n, \mathbb{R}) \right\}.$$

It is easy to see that $Z_H(A)$ normalizes $U^+(A)$ and $U^-(A)$. In fact, for

$$z = \begin{bmatrix} B & \\ & C \end{bmatrix} \in Z_H(A),$$

we have that

$$\mathrm{Ad}_z(\mathfrak{n}^+(X)) = \mathfrak{n}^+(BXC^{-1}),$$

and

$$\mathrm{Ad}_z(\mathfrak{n}^-(X)) = \mathfrak{n}^-(CXB^{-1}).$$

The above two equations give two different actions of $Z_H(A)$ on $M(n \times n, \mathbb{R})$. Let us denote them by Ad^+ and Ad^- , respectively.

Define

$$\mathfrak{a} := \begin{bmatrix} \mathrm{I}_n & \mathbf{0} \\ \mathbf{0} & -\mathrm{I}_n \end{bmatrix}.$$

For any $X \in \mathrm{GL}(n, \mathbb{R})$, $\{\mathfrak{n}^+(X), \mathfrak{n}^-(X^{-1}), \mathfrak{a}\}$ makes an $\mathfrak{sl}(2, \mathbb{R})$ triple. Therefore, there is an embedding of $\mathrm{SL}(2, \mathbb{R})$ into H that sends $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ to $\exp(\mathfrak{n}^+(X)) = u(X)$, $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ to $\exp(\mathfrak{n}^-(X^{-1})) = u^-(X^{-1})$, and $\begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}$ to $\exp(t\mathfrak{a}) = a(t)$. We call the image of this $\mathrm{SL}(2, \mathbb{R})$ embedding $\mathrm{SL}(2, X) \subset H$. Let us denote

$$\sigma(X) := \begin{bmatrix} & -X \\ X^{-1} & \end{bmatrix} \in \mathrm{SL}(2, X).$$

In this $\mathrm{SL}(2, \mathbb{R})$ copy, $\sigma(X)$ corresponds to $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in \mathrm{SL}(2, \mathbb{R})$.

2.2. Unipotent invariance of limit measures. Recall that for $t > 0$, μ_t denotes the normalized Lebesgue measure on the curve $a(t)u(\varphi(I))x$, and μ_G denotes the unique G invariant probability measure on G/Γ . Our aim is to prove that $\mu_t \rightarrow \mu_G$ as $t \rightarrow +\infty$. We first modify the measures μ_t to another measure λ_t and show that if $\lambda_t \rightarrow \mu_G$, then $\mu_t \rightarrow \mu_G$. Then we could study $\{\lambda_t : t > 0\}$ instead. The motivation for this modification is that any accumulation point of $\{\lambda_t : t > 0\}$ is invariant under a unipotent subgroup.

The measure λ_t is defined as follows:

Definition 2.1. For $t > 0$ and a closed subinterval $J \subset I$, suppose that the derivative $\varphi^{(1)}(s) \in \mathrm{GL}(n, \mathbb{R})$ for all $s \in J$. We define an analytic curve $z : J \rightarrow Z_H(A)$ such that $\mathrm{Ad}_{z(s)}^+(\varphi^{(1)}(s)) = \mathrm{I}_n$ for all $s \in J$. Then we define λ_t^J to be the normalized Lebesgue measure on $\{z(s)a(t)u(\varphi(s))x : s \in J\}$, i.e., for $f \in C_c(G/\Gamma)$,

$$\int f d\lambda_t^J := \frac{1}{|J|} \int_{s \in J} f(z(s)a(t)u(\varphi(s))x) ds.$$

Remark 2.2. Definition 2.1 is taken from [Sha09c] and [Sha09b].

For a subinterval $J \subset I$, we similarly define μ_t^J to be the normalized Lebesgue measure on $a(t)u(\varphi(J))x$.

Proposition 2.3. Suppose for any closed subinterval $J \subset I$ such that λ_t^J is defined, $\lambda_t^J \rightarrow \mu_G$ as $t \rightarrow +\infty$. Then $\mu_t \rightarrow \mu_G$ as $t \rightarrow +\infty$.

Proof. Let $s_1, s_2, \dots, s_l \in I$ be all the points where $\varphi^{(1)}(s)$ is not invertible. For any fixed $f \in C_c(G/\Gamma)$ and $\epsilon > 0$, we want to show that for $t > 0$ large enough,

$$\int f d\mu_t \approx_{4\epsilon} \int_{G/\Gamma} f d\mu_G.$$

For each $i \in \{1, 2, \dots, l\}$, one could choose an open subinterval $B_i \subset I$ containing s_i , such that

$$(2.1) \quad \left| \left(\sum_{i=1}^l |B_i| \right) \int_{G/\Gamma} f d\mu_G \right| \leq \epsilon |I|,$$

and for any $t > 0$,

$$(2.2) \quad \left| \int_{\bigcup_{i=1}^l B_i} f(a(t)u(\varphi(s))x) ds \right| \leq \epsilon |I|.$$

Since f is uniformly continuous, there exists a constant $\delta > 0$, such that if $x_1 \stackrel{\delta}{\approx} x_2$, then $f(x_1) \stackrel{\epsilon}{\approx} f(x_2)$.

We cut $I \setminus \bigcup_{i=1}^l B_i$ into several small closed subintervals $J_1, J_2, \dots, J_p$, such that, for every J_r , $z^{-1}(s_1)z(s_2) \stackrel{\delta}{\approx} e$ for any $s_1, s_2 \in J_r$.

Now for a fixed $J_r \subset I \setminus \bigcup_{i=1}^l B_i$, we choose $s_0 \in J_r$ and define $f_0(x) = f(z^{-1}(s_0)x)$. Then for any $s \in J_r$, because $z^{-1}(s_0)z(s)a(t)u(\varphi(s))x \stackrel{\delta}{\approx} a(t)u(\varphi(s))x$, we have

$$f_0(z(s)a(t)u(\varphi(s))x) = f(z^{-1}(s_0)z(s)a(t)u(\varphi(s))x) \stackrel{\epsilon}{\approx} f(a(t)u(\varphi(s))x).$$

Therefore

$$\int f_0 d\lambda_t^{J_r} \stackrel{\epsilon}{\approx} \int f d\mu_t^{J_r}.$$

Because $\int f_0 d\lambda_t^{J_r} \rightarrow \int_{G/\Gamma} f_0(x) d\mu_G(x)$ as $t \rightarrow +\infty$, and $\int_{G/\Gamma} f_0(x) d\mu_G(x) = \int_{G/\Gamma} f(z^{-1}(s_0)x) d\mu_G(x) = \int_{G/\Gamma} f(x) d\mu_G$ (because μ_G is G -invariant), we have that there exists a constant $T_r > 0$, such that for $t > T_r$,

$$\int f_0 d\lambda_t^{J_r} \stackrel{\epsilon}{\approx} \int_{G/\Gamma} f d\mu_G.$$

Therefore, for $t > T_r$,

$$\int f d\mu_t^{J_r} \stackrel{2\epsilon}{\approx} \int_{G/\Gamma} f d\mu_G,$$

i.e.,

$$\int_{J_r} f(a(t)u(\varphi(s))x) ds \stackrel{2\epsilon|J_r|}{\approx} |J_r| \int_{G/\Gamma} f d\mu_G.$$

Then for $t > \max_{1 \leq r \leq p} T_r$, we could sum up the above approximations for $r = 1, 2, \dots, p$ and get

$$\int_{I \setminus \bigcup_{i=1}^l B_i} f(a(t)u(\varphi(s))x) ds \stackrel{2\epsilon|I|}{\approx} (|I| - \sum_{i=1}^l |B_i|) \int_{G/\Gamma} f d\mu_G.$$

Combined with (2.1) and (2.2), the above approximation implies that

$$\int_I f(a(t)u(\varphi(s))x) ds \stackrel{4\epsilon|I|}{\approx} |I| \int_{G/\Gamma} f d\mu_G,$$

which is equivalent to

$$\int f d\mu_t \stackrel{4\epsilon}{\approx} \int_{G/\Gamma} f d\mu_G.$$

Because $\epsilon > 0$ can be arbitrarily small, the proof is completed. $\square$

By Proposition 2.3, to prove $\mu_t \rightarrow \mu_G$ as $t \rightarrow +\infty$, it suffices to show that for any closed subinterval $J \subset I$ where λ_t^J is defined, $\lambda_t^J \rightarrow \mu_G$ as $t \rightarrow +\infty$. In particular, if we could prove the equidistribution of $\{\lambda_t := \lambda_t^I : t > 0\}$ as $t \rightarrow +\infty$ assuming $\varphi^{(1)}(s)$ is invertible for all $s \in I$, then the equidistribution of $\{\mu_t : t > 0\}$ as $t \rightarrow +\infty$ will follow. Therefore, later in this paper, we will assume that $\varphi^{(1)}(s)$ is invertible for all $s \in I$ and define λ_t to be the normalized Lebesgue measure on the curve $\{z(s)a(t)u(\varphi(s))x : s \in I\}$. Our goal is to show that $\lambda_t \rightarrow \mu_G$ as $t \rightarrow +\infty$.

We will show that any limit measure of $\{\lambda_t : t > 0\}$ is invariant under the unipotent subgroup $W = \{u(tI_n) : t \in \mathbb{R}\}$.

Proposition 2.4 (see [Sha09c]). *Let $t_i \rightarrow +\infty$ be a sequence such that $\lambda_{t_i} \rightarrow \mu_\infty$ in weak-* topology; then μ_∞ is invariant under the W -action.*

Proof. Given any $f \in C_c(G/\Gamma)$, and $r \in \mathbb{R}$, we want to show that

$$\int f(u(rI_n)x) d\mu_\infty = \int f(x) d\mu_\infty.$$

Since $z(s)$ and $\varphi(s)$ are analytic and defined on closed interval $I = [a, b]$, there exists a constant $T_1 > 0$ such that for $t \geq T_1$, $z(s)$ and $\varphi(s)$ can be extended to analytic curves defined on $[a - |r|e^{-2t}, b + |r|e^{-2t}]$. Throughout the proof, we always assume that $t_i \geq T_0$. Then $z(s + re^{-2t_i})$ and $\varphi(s + re^{-2t_i})$ are both well defined for $s \in I$.

From the definition of μ_∞ we have

$$\int f(u(rI_n)x) d\mu_\infty = \lim_{t_i \rightarrow +\infty} \frac{1}{|I|} \int_{s \in I} f(u(rI_n)z(s)a(t_i)u(\varphi(s))x) ds.$$

We want to argue that

$$u(rI_n)z(s)a(t_i)u(\varphi(s)) \stackrel{O(e^{-2t_i})}{\approx} z(s + re^{-2t_i})a(t_i)u(\varphi(s + re^{-2t_i})).$$

Since $z(s + re^{-2t_i}) \stackrel{O(e^{-2t_i})}{\approx} z(s)$ for t_i large enough, it suffices to show that

$$u(rI_n)z(s)a(t_i)u(\varphi(s)) \stackrel{O(e^{-2t_i})}{\approx} z(s)a(t_i)u(\varphi(s + re^{-2t_i})).$$

In fact,

$$\begin{aligned} & z(s)a(t_i)u(\varphi(s + re^{-2t_i})) \\ &= z(s)a(t_i)u(\varphi(s) + re^{-2t_i}\varphi'(s) + O(e^{-4t_i})) \\ &= z(s)u(r\varphi'(s))u(O(e^{-2t_i}))a(t_i)u(\varphi(s)). \end{aligned}$$

By the definition of $z(s)$, we have the above is equal to

$$u(O(e^{-2t_i}))u(rI_n)z(s)a(t_i)u(\varphi(s)).$$

This shows that

$$u(rI_n)z(s)a(t_i)u(\varphi(s)) \stackrel{O(e^{-2t_i})}{\approx} z(s + re^{-2t_i})a(t_i)u(\varphi(s + re^{-2t_i})).$$

Therefore, for any $\delta > 0$, there is some constant $T \geq T_1$, such that for $t_i \geq T$,

$$u(rI_n)z(s)a(t_i)u(\varphi(s)) \stackrel{\delta}{\approx} z(s + re^{-2t_i})a(t_i)u(\varphi(s + re^{-2t_i})).$$

Given $\epsilon > 0$, we choose $\delta > 0$ such that whenever $x_1 \stackrel{\delta}{\approx} x_2$, we have $f(x_1) \stackrel{\epsilon}{\approx} f(x_2)$. Let $T > 0$ be the constant as above. Then from the above argument, for $t_i > T$, we have

$$f(u(rI_n)z(s)a(t_i)u(\varphi(s))x) \stackrel{\epsilon}{\approx} f(z(s + re^{-2t_i})a(t_i)u(\varphi(s + re^{-2t_i}))x),$$

therefore,

$$\begin{aligned} & \frac{1}{|I|} \int_{s \in I} f(u(rI_n)z(s)a(t_i)u(\varphi(s))x) ds \\ & \stackrel{\epsilon}{\approx} \frac{1}{|I|} \int_{s \in I} f(z(s + re^{-2t_i})a(t_i)u(\varphi(s + re^{-2t_i}))x) ds \\ & = \frac{1}{|I|} \int_{a+re^{-2t_i}}^{b+re^{-2t_i}} f(z(s)a(t_i)u(\varphi(s))x) ds. \end{aligned}$$

It is easy to see that when t_i is large enough,

$$\frac{1}{|I|} \int_{a+re^{-2t_i}}^{b+re^{-2t_i}} f(z(s)a(t_i)u(\varphi(s))x) ds \stackrel{\epsilon}{\approx} \frac{1}{|I|} \int_a^b f(z(s)a(t_i)u(\varphi(s))x) ds.$$

Therefore, for t_i large enough,

$$\int f(u(rI_n)x) d\lambda_{t_i} \stackrel{2\epsilon}{\approx} \int f(x) d\lambda_{t_i}.$$

Letting $t_i \rightarrow +\infty$, we have

$$\int f(u(rI_n)x) d\mu_\infty \stackrel{2\epsilon}{\approx} \int f(x) d\mu_\infty.$$

Since the above approximation is true for arbitrary $\epsilon > 0$, we have that μ_∞ is W -invariant. $\square$

2.3. Nondivergence of limit measures. We will prove that any limit measure μ_∞ of $\{\lambda_t : t > 0\}$ is still a probability measure of G/Γ .

To show the nondivergence of limit measures, it suffices to prove the following proposition:

Proposition 2.5 (see [Sha09c]). *For any $\epsilon > 0$, there exists a compact subset $\mathcal{K}_\epsilon \subset G/\Gamma$ such that $\lambda_t(\mathcal{K}_\epsilon) \geq 1 - \epsilon$ for all $t > 0$.*

The proof of the proposition is due to Nimish Shah [Sha09c]. Here we just modify the proof to fit our needs.

Definition 2.6. Let $\mathfrak{g}$ denote the Lie algebra of G , and let $d = \dim G$. We define

$$\mathcal{V} := \bigoplus_{i=1}^d \bigwedge^i \mathfrak{g},$$

and fix a norm $\|\cdot\|$ on $\mathcal{V}$. Let G act on $\mathcal{V}$ via $\bigoplus_{i=1}^d \bigwedge^i \text{Ad}(\cdot)$. This defines a linear representation of G :

$$G \rightarrow \text{GL}(\mathcal{V}).$$

The following theorem due to Kleinbock and Margulis [KM98] is the basic tool to prove the proposition:

Theorem 2.7 (see [Dan84] and [KM98]). *Fix a norm $\|\cdot\|$ on $\mathcal{V}$. There exist finitely many vectors $v_1, v_2, \dots, v_r \in \mathcal{V}$ such that for each $i = 1, 2, \dots, r$, the orbit Γv_i is discrete, and the following holds: for any $\epsilon > 0$ and $R > 0$, there exists a*

compact set $K \subset G/\Gamma$ such that for any $t > 0$ and any subinterval $J \subset I$, one of the following holds:

S.1 There exist $\gamma \in \Gamma$ and $j \in \{1, \dots, r\}$ such that

$$\sup_{s \in J} \|a(t)u(\varphi(s))g\gamma v_j\| < R.$$

S.2

$$|\{s \in J : a(t)u(\varphi(s))x \in K\}| \geq (1 - \epsilon)|J|.$$

Remark 2.8. The proof for polynomial curves is due to Dani [Dan84], and the proof for analytic curves is due to Kleinbock and Margulis [KM98]. The crucial part to prove the above theorem is to find constants $C > 0$ and $\alpha > 0$ such that in this particular representation, all the coordinate functions of $a(t)u(\varphi(\cdot))$ are (C, α) -good. Here a function $f : I \rightarrow \mathbb{R}$ is called (C, α) -good if for any subinterval $J \subset I$ and any $\epsilon > 0$, the following holds:

$$|\{s \in J : |f(s)| < \epsilon\}| \leq C \left(\frac{\epsilon}{\sup_{s \in J} |f(s)|} \right)^\alpha |J|.$$

The following basic lemma on representations of $\mathrm{SL}(2, \mathbb{R})$ due to Nimish Shah is crucial in the proof:

Lemma 2.9 (see [Sha09c]). *Let V be a representation of $\mathrm{SL}(2, \mathbb{R})$, fix a norm $\|\cdot\|$ on V . We define*

$$A = \left\{ a(t) = \begin{bmatrix} e^t & \\ & e^{-t} \end{bmatrix} : t \in \mathbb{R} \right\},$$

and

$$U^+(A) = \left\{ u(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} : t \in \mathbb{R} \right\}.$$

Then for any $t > 0$, there exists a constant $\kappa = \kappa(t) > 0$ such that for any $v \in V$,

$$\max\{\|v^+(A)\|, \|(u(t)v)^{+0}(A)\|\} \geq \kappa\|v\|.$$

In $H = \mathrm{SL}(2n, \mathbb{R})$, for any $X \in \mathrm{GL}(n, \mathbb{R})$, $u(X) \in \mathrm{SL}(2, X) \subset H$ corresponds to $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ in $\mathrm{SL}(2, \mathbb{R})$, where $\mathrm{SL}(2, X) \cong \mathrm{SL}(2, \mathbb{R})$ is defined in §2.1. Lemma 2.9 easily implies the following:

Corollary 2.10 (see [Sha09c]). *Let V be a linear representation of $H = \mathrm{SL}(2n, \mathbb{R})$. Fix a norm $\|\cdot\|$ on V . Let $A = \{a(t) : t \in \mathbb{R}\} \subset H$ be the one-parameter diagonal subgroup defined in (1.1). Then for any $X \in \mathrm{GL}(n, \mathbb{R})$, there exists a constant $\kappa > 0$ such that for any $v \in V$,*

$$\max\{\|v^+(A)\|, \|(u(X)v)^{+0}(A)\|\} \geq \kappa\|v\|.$$

In particular, for any $t > 0$, and any $v \in V$,

$$\max\{\|a(t)v\|, \|a(t)u(X)v\|\} \geq \kappa\|v\|.$$

Proof of Proposition 2.5. Fix $s_0 \in I$ and a closed subinterval J_{s_0} such that $\varphi(s) - \varphi(s_0) \subset \mathrm{GL}(n, \mathbb{R})$ for $s \in J_{s_0}$. Let us fix $s_1 \in J_{s_0}$. Let $\kappa > 0$ be the constant given in Corollary 2.10 with X replaced by $\varphi(s_1) - \varphi(s_0)$.

For any $\epsilon > 0$ and $R > 0$, by Theorem 2.7, there exists a compact subset $K \subset G/\Gamma$, such that for any $t > 0$, one of the following holds:

S.1 There exist $\gamma \in \Gamma$ and $j \in \{1, \dots, r\}$ such that

$$\sup_{s \in I} \|a(t)u(\varphi(s))g\gamma v_j\| < R.$$

S.2

$$|\{s \in I : a(t)u(\varphi(s))x \in K\}| \geq (1 - \epsilon)|I|.$$

Then because Γv_i is discrete in $\mathcal{V}$, there exists a uniform constant $r > 0$ such that

$$\|u(\varphi(s_0))g\gamma v_i\| \geq r,$$

for any v_i and $\gamma \in \Gamma$. Applying Corollary 2.10 with v replaced by $u(\varphi(s_0))g\gamma v_i$ and X replaced by $\varphi(s_1) - \varphi(s_0)$, we get that for any $v_i, \gamma \in \Gamma$ and $t > 0$,

$$\sup_{s \in I} \|a(t)u(\varphi(s))g\gamma v_i\| \geq \max\{a(t)u(\varphi(s_0))g\gamma v_i, a(t)u(\varphi(s_1))g\gamma v_i\} \geq \kappa r.$$

If we choose $R < \kappa r$, then case **S.1** above cannot hold. This shows that

$$|\{s \in I : a(t)u(\varphi(s))x \in K\}| \geq (1 - \epsilon)|I|.$$

Let M be a compact subset of $Z_H(A)$ containing $z(I)$. Define $\mathcal{K}_\epsilon := MK$. Since $z(s) \in M$, we have that

$$|\{s \in I : z(s)a(t)u(\varphi(s))x \in \mathcal{K}_\epsilon\}| \geq (1 - \epsilon)|I|,$$

i.e., $\lambda_t(\mathcal{K}_\epsilon) \geq 1 - \epsilon$ for all $t > 0$.

This completes the proof. $\square$

3. RATNER'S THEOREM AND THE LINEARIZATION TECHNIQUE

Take any convergent subsequence $\lambda_{t_i} \rightarrow \mu_\infty$. By Proposition 2.4 and Proposition 2.5, μ_∞ is a W -invariant probability measure on G/Γ .

In order to apply Ratner's Theorem and the linearization technique, we first introduce some notation.

Definition 3.1. Let $\mathcal{L}$ be the collection of analytic subgroups $L < G$ such that $L \cap \Gamma$ is a lattice of L . One could prove that $\mathcal{L}$ is a countable set (see [Rat91a]).

For $L \in \mathcal{L}$, let us define

$$N(L, W) := \{g \in G : g^{-1}Wg \subset L\},$$

and

$$S(L, W) := \bigcup_{L' \in \mathcal{L}, L' \subsetneq L} N(L', W).$$

We recall Ratner's Measure Classification Theorem as follows:

Theorem 3.2 (see [Rat91a]). *Recall that π denotes the projection from G to $X = G/\Gamma$. Given the W -invariant probability measure μ on G/Γ , there exists $L \in \mathcal{L}$ such that*

$$(3.1) \quad \mu(\pi(N(L, W))) > 0 \quad \text{and} \quad \mu(\pi(S(L, W))) = 0.$$

Moreover, almost every W -ergodic component of μ on $\pi(N(L, W))$ is a measure of the form $g\mu_L$ where $g \in N(L, W) \setminus S(L, W)$, μ_L is a finite L -invariant measure on $\pi(L)$, and $g\mu_L(E) = \mu_L(g^{-1}E)$ for all Borel sets $E \subset G/\Gamma$. In particular, if $L \triangleleft G$, then the restriction of μ to $\pi(N(L, W))$ is L -invariant.

If $\mu_\infty = \mu_G$, then there is nothing to prove. So we assume $\mu_\infty \neq \mu_G$. By Ratner's Theorem, there exists $L \in \mathcal{L}$, $L \subsetneq G$, such that $\mu_\infty(\pi(N(L, W))) > 0$ and $\mu_\infty(\pi(S(L, W))) = 0$. Now we apply the linearization technique.

Definition 3.3. Let $\mathcal{V}$ be the finite dimensional representation of G defined as in Definition 2.6. A subset $\mathcal{C} \subset \mathcal{V}$ is called symmetric if $\mathcal{C} = -\mathcal{C}$. For $L \in \mathcal{L}$, we choose a basis $\mathfrak{e}_1, \mathfrak{e}_2, \dots, \mathfrak{e}_l$ of the Lie algebra $\mathfrak{l}$ of L , and define

$$p_L = \bigwedge_{i=1}^l \mathfrak{e}_i \in \mathcal{V}.$$

Let $N_G(L)$ denote the normalizer of L in G . It is easy to see that

$$N_G(L) = \{g \in G : gp_L \in \mathbb{R} \cdot p_L\}.$$

Define

$$N_G^1(L) := \{g \in G : gp_L = p_L\},$$

and

$$\Gamma_L := \{\gamma \in \Gamma : \gamma p_L = \pm p_L\}.$$

The action of G on p_L defines a map:

$$\begin{aligned} \eta : G &\rightarrow \mathcal{V}, \\ g &\mapsto gp_L. \end{aligned}$$

We define $\mathcal{A}$ to be the Zariski closure of $\eta(N(L, W))$. For any compact subset $\mathcal{D} \subset \mathcal{A}$, we define

$$S(\mathcal{D}) := \{g \in N(L, W) : \eta(g\gamma) \in \mathcal{D} \text{ for some } \gamma \in \Gamma \setminus \Gamma_L\}.$$

We need the following propositions:

Proposition 3.4 (see [Sha09c], Proposition 4.5). *$S(\mathcal{D}) \subset S(L, W)$ and $\pi(S(\mathcal{D}))$ is closed in G/Γ . Moreover, for any compact set $\mathcal{K} \in G/\Gamma \setminus \pi(S(\mathcal{D}))$, there exists a neighborhood Φ of $\mathcal{D}$ in $\mathcal{V}$ such that, for any $g \in G$ and $\gamma_1, \gamma_2 \in \Gamma$, if $\pi(g) \in \mathcal{K}$ and $\eta(g\gamma_i) \in \Phi$, $i = 1, 2$, then $\eta(\gamma_1) = \pm\eta(\gamma_2)$.*

Proposition 3.5 (see [Sha09c], Proposition 4.6). *Given a symmetric compact subset $\mathcal{C} \subset \mathcal{A}$ and $\epsilon > 0$, there exists a symmetric compact set $\mathcal{D} \subset \mathcal{A}$ containing $\mathcal{C}$ such that, given a symmetric neighborhood Φ of $\mathcal{D}$ in $\mathcal{V}$, there exists a symmetric neighborhood Ψ of $\mathcal{C}$ in $\mathcal{V}$ contained in Φ such that for any $t > 0$, for any $v \in \mathcal{V}$, and for any interval $J \subset I$, one of the following holds:*

SS.1 $a(t)u(\varphi(s))v \in \Phi$ for all $s \in J$.

SS.2 $|\{s \in J : a(t)u(\varphi(s))v \in \Psi\}| \leq \epsilon |\{s \in J : a(t)u(\varphi(s))v \in \Phi\}|$.

Remark 3.6. The proof is similar to Theorem 2.7, and also follows from the fact that all coordinate functions of $a(t)u(\varphi(\cdot))v$ are (C, α) -good for some constants $C > 0$ and $\alpha > 0$.

The following proposition is the aim of this section.

Proposition 3.7. *There exists a $\gamma \in \Gamma$ such that*

$$u(\varphi(s))g\gamma p_L \in \mathcal{V}^{-0}(A),$$

for all $s \in I$.

Proof. Take a compact subset $C \subset N(L, W) \setminus S(L, W)$ such that $\mu_\infty(\pi(C)) > c_0 > 0$ for some constant c_0 . Define $\mathcal{C} := \eta(C) \cup (-\eta(C))$; then $\mathcal{C} \subset \mathcal{A}$ is a symmetric compact subset. Choose a compact subset $\mathcal{K} \subset G/\Gamma \setminus \pi(S(L, W))$ containing $\pi(C)$ in its interior. Applying Proposition 3.5, we can find a symmetric compact subset $\mathcal{D} \subset \mathcal{A}$ containing $\mathcal{C}$ such that the conclusion of Proposition 3.5 holds for $\mathcal{C}$, $\mathcal{D}$ and some small $0 < \epsilon < \frac{c_0}{2}$. Applying Proposition 3.4 to $\mathcal{D}$ and $\mathcal{K}$, we have that there exists an open neighborhood Φ of $\mathcal{D}$ such that the conclusion of Proposition 3.4 holds. Choose a neighborhood Ψ of $\mathcal{C}$ according to Proposition 3.5.

We claim that for any $i \in \mathbb{N}$ large enough, there exists $\gamma_i \in \Gamma$ such that

$$a(t_i)u(\varphi(I))g\gamma_i p_L \subset \Phi.$$

For contradiction, we assume that it is not the case, i.e., there exists an infinite subsequence of $\mathbb{N}$ such that for any i in this subsequence, and any $\gamma \in \Gamma$, $a(t_i)u(\varphi(I))g\gamma p_L$ is not contained in Φ . This implies that case **SS.1** in Proposition 3.5 does not hold for $v = g\gamma p_L$ and $J = I$. We define

$$J_i := \{s \in I : a(t_i)u(\varphi(s))x \in \mathcal{K} : a(t_i)u(\varphi(s))g\gamma p_L \cap \Psi \neq \emptyset\}.$$

Since $\mu_\infty(\pi(C)) > c_0$, we have that for i large enough, $|J_i| > c_0|I|$.

By Proposition 3.4, for any $s \in J_i$, up to $\pm$ sign, there exists unique $\gamma(s)p_L$ such that $a(t_i)u(\varphi(s))g\gamma(s)p_L \in \Psi$. Let $I_{\gamma(s)}$ be the maximal interval I containing s such that

$$a(t_i)u(\varphi(I_{\gamma(s)}))g\gamma(s)p_L \subset \Phi.$$

By Proposition 3.4, there is no other $\gamma'p_L$ other than $\pm\gamma(s)p_L$ and $s \in I_{\gamma(s)} \cap J_i$ such that

$$a(t_i)u(\varphi(s))g\gamma'p_L \in \Psi.$$

Therefore J_i is covered by at most countably many intervals $I_{\gamma(s)}$'s which covers the whole interval I at most twice, namely, every point belongs to at most two different intervals (this is because for any $s_1 < s_2 \in J_i$, $I_{\gamma(s_1)} \cap I_{\gamma(s_2)} \subset (s_1, s_2)$). Because case **SS.1** in Proposition 3.5 does not hold, we have that **SS.2** must hold, i.e.,

$$|J_i \cap I_{\gamma(s)}| < \epsilon|I_{\gamma(s)}|.$$

This shows that

$$|J_i| < 2\epsilon|I|.$$

This contradicts the fact that $|J_i| > c_0|I|$. This shows the claim.

Since Γp_L is discrete in $\mathcal{V}$, one of the following will happen:

- (1) $\|\gamma_i p_L\| \rightarrow +\infty$ as $i \rightarrow \infty$.
- (2) $\gamma_i p_L$ remains the same for all large i .

If the first case happens, we define a unit vector $v_i = \frac{\gamma_i p_L}{\|\gamma_i p_L\|}$ for each i . Because

$$a(t_i)u(\varphi(I))\gamma_i p_L \subset \Phi,$$

there is a constant R such that

$$\sup_{s \in I} \|a(t_i)u(\varphi(s))v_i\| \leq \frac{R}{\|\gamma_i p_L\|} \rightarrow 0.$$

Suppose $v_i \rightarrow v_\infty$ passing to some subsequence; then we have that

$$\sup_{s \in I} \|a(t_i)u(\varphi(s))v_\infty\| \rightarrow 0,$$

as $i \rightarrow \infty$. This is impossible by Corollary 2.10. Therefore $\gamma_i p_L = \gamma p_L$ remains the same for all large i . This means that for all i large enough,

$$\sup_{s \in I} \|a(t_i)u(\varphi(s))g\gamma p_L\| \leq R.$$

This implies that for $v = g\gamma p_L$,

$$u(\varphi(s))v \in \mathcal{V}^{-0}(A).$$

This completes the proof. $\square$

4. CONCLUSION

In this section we will prove Theorem 1.4. By the correspondence between homogeneous dynamics and Diophantine approximation discussed in §1, Theorem 1.2 follows from Theorem 1.4.

We need the following basic lemma on $\mathrm{SL}(2, \mathbb{R})$ representations.

Lemma 4.1 (see [Yan13], Lemma 5.1). *Let V be a finite dimensional linear representation of $\mathrm{SL}(2, \mathbb{R})$. Denote*

$$A := \left\{ a(t) := \begin{bmatrix} e^t & \\ & e^{-t} \end{bmatrix} : t \in \mathbb{R} \right\},$$

and

$$U := \left\{ u(s) := \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \right\}.$$

Suppose there is a nonzero vector $v \in V^{-0}(A)$ satisfying

$$u(r)v \in V^{-0}(A),$$

for some $r \in \mathbb{R}$; then $(u(r)v)^0(A) = \sigma v^0(A)$, where σ denotes the matrix

$$\sigma = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Proof of Theorem 1.4. We start with the linear algebraic condition we get from Proposition 3.7:

$$u(\varphi(s))v \in \mathcal{V}^{-0}(A), \quad \text{for all } s \in I.$$

We first prove the following claim:

Claim 4.2. $(u(\varphi(s))v)^0(A)$ is invariant under the action of the unipotent subgroup $\{u(r\varphi^{(1)}(s)) : r \in \mathbb{R}\}$.

Proof of the claim. On the one hand, since $u(\varphi(s))v \in \mathcal{V}^{-0}(A)$ for all $s \in I$, for any fixed $r \in \mathbb{R}$,

$$\lim_{t \rightarrow +\infty} a(t)u(\varphi(s + re^{-2t}))v = \lim_{t \rightarrow +\infty} (u(\varphi(s + re^{-2t}))v)^0(A) = (u(\varphi(s))v)^0(A).$$

On the other hand,

$$\begin{aligned} & \lim_{t \rightarrow +\infty} a(t)u(\varphi(s + re^{-2t}))v \\ &= \lim_{t \rightarrow +\infty} a(t)u(\varphi(s) + r\varphi^{(1)}(s)e^{-2t} + O(e^{-4t}))v \\ &= \lim_{t \rightarrow +\infty} a(t)u(r\varphi^{(1)}(s)e^{-2t} + O(e^{-4t}))a(-t)a(t)u(\varphi(s))v \\ &= \lim_{t \rightarrow +\infty} u(r\varphi^{(1)}(s) + O(e^{-2t}))a(t)u(\varphi(s))v \\ &= u(r\varphi^{(1)}(s))(u(\varphi(s))v)^0(A). \end{aligned}$$

This implies $(u(\varphi(s))v)^0(A) = u(r\varphi^{(1)}(s))(u(\varphi(s))v)^0(A)$ for all $r \in \mathbb{R}$.

This proves the claim. $\square$

Fix $s_0 \in I$ and a closed subinterval $J_{s_0} \subset I$ such that $\varphi^{(1)}(s_0)$ is invertible, and $\varphi(s) - \varphi(s_0)$ is invertible for all $s \in J_{s_0}$. For $s \in J_{s_0}$, let us denote $X(s) = \varphi(s) - \varphi(s_0)$, and consider the subgroup $\mathrm{SL}(2, X(s)) \cong \mathrm{SL}(2, \mathbb{R})$. Note that in $\mathrm{SL}(2, X(s))$,

$$\sigma(X(s)) = \begin{bmatrix} & -X(s) \\ X^{-1}(s) & \end{bmatrix}$$

corresponds to

$$\sigma = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in \mathrm{SL}(2, \mathbb{R}).$$

Applying Lemma 4.1 with $\mathrm{SL}(2, \mathbb{R})$ replaced by $\mathrm{SL}(2, X(s))$, v replaced by $u(\varphi(s_0))v$ and $u(r)$ replaced by $u(X(s))$, we conclude that

$$(4.1) \quad (u(\varphi(s))v)^0(A) = \sigma(X(s))(u(\varphi(s_0))v)^0(A).$$

Let us denote $w = (u(\varphi(s_0))v)^0(A)$ and define

$$\mathcal{S} := \{X \in \mathrm{M}(n \times n, \mathbb{R}) : u^-(X)w = w\}.$$

It is easy to see that $\mathcal{S}$ is a subspace of $\mathrm{M}(n \times n, \mathbb{R})$.

Claim 4.3. $\mathcal{S}$ is a proper subspace of $\mathrm{M}(n \times n, \mathbb{R})$.

Proof of the claim. Suppose not, then w is fixed by the whole horospherical subgroup $U^-(A)$. w is also fixed by A since it is in $\mathcal{V}^0(A)$. This implies that w is fixed by $H = \mathrm{SL}(2n, \mathbb{R})$. Let us denote $u(\varphi(s_0))v = w + w^-$, where $w^- = u(\varphi(s_0))v - w \in \mathcal{V}^-(A)$. We claim that $w^- = \mathbf{0}$. In fact, if $w^- \neq \mathbf{0}$, then for $s \in J$,

$$u(X(s))w^- = u(\varphi(s))v - u(X(s))w = u(\varphi(s))v - w \in \mathcal{V}^{-0}(A).$$

This contradicts Lemma 2.9 with $\mathrm{SL}(2, \mathbb{R})$ replaced by $\mathrm{SL}(2, X(s))$, $u(r)$ replaced by $u(X(s))$, and v replaced by w^- . This shows that $w^- = \mathbf{0}$. Thus, $u(\varphi(s_0))v = w$ is fixed by the whole group $H = \mathrm{SL}(2n, \mathbb{R})$. Then $v = g\gamma p_L$ is fixed by H . Hence p_L is fixed by the action of $\gamma^{-1}g^{-1}Hg\gamma$. Thus

$$\begin{aligned} \Gamma p_L &= \overline{\Gamma p_L} \text{ (since } \Gamma p_L \text{ is discrete)} \\ &= \overline{\Gamma \gamma^{-1}g^{-1}Hg\gamma p_L} \\ &= \overline{\Gamma g^{-1}Hg\gamma p_L} \\ &= Gg\gamma p_L \text{ (since } \overline{Hg\Gamma} = G) \\ &= Gp_L. \end{aligned}$$

This implies that $G_0 p_L = p_L$ where G_0 denotes the connected component of e in G . In particular, $\gamma^{-1}g^{-1}Hg\gamma \subset G_0$ and $G_0 \subset N_G^1(L)$. By [Sha09a, Theorem 2.3], there exists a closed subgroup $F_1 \subset N_G^1(L)$ containing all Ad-unipotent one-parameter subgroups of G contained in $N_G^1(L)$ such that $F_1 \cap \Gamma$ is a lattice in F_1 and $\pi(F_1)$ is closed. If we put $F = g\gamma F_1 \gamma^{-1}g^{-1}$, then $H \subset F$ since H is generated by its unipotent one-parameter subgroups. Moreover, $Fx = g\gamma\pi(F_1)$ is closed and admits a finite F -invariant measure. Then since $\overline{Hx} = G/\Gamma$, we have $F = G$. This implies $F_1 = G$ and thus $L \triangleleft G$. Therefore $N(L, W) = G$. In particular, $W \subset L$, and thus $L \cap H$ is a normal subgroup of H containing W . Since H is a simple group, we have that $H \subset L$. Since L is a normal subgroup of G and $\pi(L)$ is a closed orbit with finite L -invariant measure, every orbit of L on G/Γ is also closed and admits a finite L -invariant measure, in particular, Lx is closed. But since Hx is dense in G/Γ , Lx is also dense. This shows that $L = G$, which contradicts our hypothesis that $\mu_\infty \neq \mu_G$. This proves the claim. $\square$

Let $\langle \cdot, \cdot \rangle$ denote the standard inner product on $M(n \times n, \mathbb{R}) \cong \mathbb{R}^{n^2}$. Then there exists a nonzero vector $\mathbf{Y} \in M(n \times n, \mathbb{R})$ such that any $X \in \mathcal{S}$ satisfies $\langle X, \mathbf{Y} \rangle = 0$.

We have proved that $(u(\varphi(s))v)^0(A)$ is fixed by $\{u(r\varphi^{(1)}(s)) : r \in \mathbb{R}\}$. By (4.1), we have that $w = (u(\varphi(s_0))v)^0(A)$ is fixed by

$$(\sigma(X(s)))^{-1}u(\varphi^{(1)}(s_0))\sigma(X(s)) = \begin{bmatrix} \mathbf{I}_n & \\ H(s) & \mathbf{I}_n \end{bmatrix},$$

where $H(s) := -X(s)^{-1}\varphi^{(1)}(s)X(s)^{-1}$. This means $\langle -X(s)^{-1}\varphi^{(1)}(s)X(s)^{-1}, \mathbf{Y} \rangle = 0$. Note that

$$((X(s))^{-1})^{(1)} = -X(s)^{-1}\varphi^{(1)}(s)X(s)^{-1}.$$

This implies that $\langle (X(s))^{-1}, \mathbf{Y} \rangle$ is a constant, i.e., $\{X(s)^{-1} = (\varphi(s) - \varphi(s_0))^{-1} : s \in J\}$ is contained in a proper affine subspace of $M(n \times n, \mathbb{R})$ orthogonal to $\mathbf{Y}$. Because this holds for arbitrary $s_0 \in I$, we get a contradiction to condition **A.2** in Theorem 1.2.

This completes the proof of Theorem 1.4. $\square$

Remark 4.4. We call a proper subspace $\mathcal{S}$ of $M(n \times n, \mathbb{R})$ **observable** if there exists a nonzero vector $v \in \mathcal{V}$ fixed by A such that $\mathcal{S} = \{X \in M(n \times n, \mathbb{R}) : u^-(X)v = v\}$. A proper affine subspace of $M(n \times n, \mathbb{R})$ is called **observable** if it is parallel to an **observable** subspace. From the proof we see that if the equidistribution does not hold, then $\{(\varphi(s) - \varphi(s_0))^{-1} : s \in J_{s_0}\}$ must be contained in an **observable** affine subspace. Therefore, to ensure the equidistribution, we only need $\{(\varphi(s) - \varphi(s_0))^{-1} : s \in J_{s_0}\}$ to avoid **observable** affine subspaces of $M(n \times n, \mathbb{R})$.

APPENDIX A. RELATION BETWEEN THE CONDITION GIVEN IN [ABRdS14] AND CONDITION **A.1** IN THEOREM 1.2

We will discuss the condition given in [ABRdS14] and its relation to condition **A.1** in Theorem 1.2. Because in this paper we only consider the case $m = n$, we only discuss this special case.

We denote $M(s) = [\mathbf{I}_n, \varphi(s)] \in M(n \times 2n, \mathbb{R})$. Given a subspace $W \subset \mathbb{R}^{2n}$ and $0 < r < \frac{\dim W}{2}$, we define the pencil $\mathcal{P}_{W,r}$ as follows:

$$\mathcal{P}_{W,r} := \{M \in M(n \times 2n, \mathbb{R}) : \dim MW = r\}.$$

In [ABRdS14], the following theorem is announced: if a submanifold is not contained in a pencil defined as above, then the submanifold is extremal. In our setup, it says that if the curve $\{M(s) : s \in I\}$ is not contained in any pencil $\mathcal{P}_{W,r}$, then the curve is extremal. It is easy to see that if W is a rational subspace, then $\mathcal{P}_{W,r}$ is not extremal. So this condition is almost optimal.

Proposition A.1. *If the curve $\varphi(I)$ satisfies that for some $s_0 \in I$, $\varphi(s) - \varphi(s_0)$ is invertible for s in a subinterval J of I , then the curve $\{M(s) : s \in I\}$ is not contained in any pencil $\mathcal{P}_{W,r}$.*

Proof. Suppose not; then the curve $\{[\mathbf{I}_n, \varphi(s)] : s \in I\}$ is contained in some pencil $\mathcal{P}_{W,r}$ where $r < \frac{\dim W}{2}$. This means that if we denote $M(s) = [\mathbf{I}_n, \varphi(s)]$, the intersection of W and the kernel of $M(s)$ has dimension greater than $\frac{\dim W}{2}$, i.e.,

$$\dim \text{Ker}(M(s)) \cap W > \frac{\dim W}{2}.$$

Let us denote $W(s) := \text{Ker}(M(s)) \cap W$. Then for any $s_1 \neq s_2 \in I$, $W(s_1) \cap W(s_2) \neq \{\mathbf{0}\}$ because the sum of their dimensions is greater than $\dim W$. This means that the intersection $\text{Ker}(M(s_1)) \cap \text{Ker}(M(s_2)) \neq \{\mathbf{0}\}$. It is easy to see that the kernel of $M(s)$ is $\{(-\varphi(s)w, w) : w \in \mathbb{R}^n\}$, so there exist $w_1, w_2 \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ such that

$$(-\varphi(s_1)w_1, w_1) = (-\varphi(s_2)w_2, w_2).$$

This implies that $w_1 = w_2$, and $-\varphi(s_1)w_1 = -\varphi(s_2)w_2$. This implies that $(\varphi(s_1) - \varphi(s_2))w_1 = \mathbf{0}$ for nonzero vector w_1 . This shows that $\varphi(s_1) - \varphi(s_2)$ is not invertible, for any $s_1 \neq s_2 \in I$. This gives a contradiction. $\square$

This shows that condition **A.1** implies the condition given in [ABRdS14].

ACKNOWLEDGEMENT

The author would like to express deep gratitude to his advisor, Professor Nimish Shah, for suggesting this problem, and for his continuous advice and support during the process of the work. The author also would like to thank Professor Kleinbock for reading an earlier version of this paper and providing many comments and suggestions. For example, he drew [KMW10] and [ABRdS14] to the author's attention and suggested the problem of figuring out the relation between the conditions given in this paper and that in [ABRdS14].

Thanks are due to the referees for valuable suggestions.

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