STRICT SINGULARITY OF A VOLTERRA-TYPE INTEGRAL OPERATOR ON H^p

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ABSTRACT. We prove that the Volterra-type integral operator

$$T_g f(z) = \int_0^z f(\zeta) g'(\zeta) d\zeta, \quad z \in \mathbb{D},$$

defined on the Hardy spaces H^p fixes an isomorphic copy of ℓ^p if it is not compact. In particular, the strict singularity of T_g coincides with its compactness on spaces H^p . As a consequence, we obtain a new proof for the equivalence of the compactness and the weak compactness of T_g on H^1 . Moreover, a non-compact T_g acting on the space $BMOA$ fixes an isomorphic copy of c_0 .

1. INTRODUCTION

Let g be a fixed analytic function in the open unit disc $\mathbb{D}$ of the complex plane $\mathbb{C}$. We consider a linear integral operator T_g defined for analytic functions f in $\mathbb{D}$ by

$$T_g f(z) = \int_0^z f(\zeta) g'(\zeta) d\zeta, \quad z \in \mathbb{D}.$$

Ch. Pommerenke was the first author to consider the boundedness of the operator T_g on the Hardy space H^2 and he characterized it in connection to exponentials of $BMOA$ functions [14]. A systematic study of T_g was initiated by A. Aleman and A. G. Siskakis in [3], who showed that T_g is bounded (compact) on the Hardy spaces H^p , $1 \leq p < \infty$, if and only if $g \in BMOA$ ($g \in VMOA$). The same boundedness characterization of the operator T_g on H^p , $0 < p < 1$, spaces was obtained by Aleman and J. Cima in [2]. Here $BMOA$ and $VMOA$ denote the spaces of analytic functions in $\mathbb{D}$ with boundary values of bounded mean oscillation and vanishing mean oscillation respectively. See also surveys [1] and [15] for more background and results on T_g .

A bounded operator $S: X \rightarrow Y$ between Banach spaces is strictly singular if its restriction to any infinite-dimensional closed subspace is not an isomorphism onto its range. This notion generalizes the concept of compact operators and it was introduced by T. Kato in [9]. Canonical examples of non-compact strictly singular operators are the inclusion mappings $i_{p,q}: \ell^p \hookrightarrow \ell^q$, where $1 \leq p < q < \infty$.

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There also exist non-compact strictly singular operators on H^p spaces for $1 \leq p < \infty$, $p \neq 2$. To construct such an operator, one may consider cases $1 \leq p < 2$ and $2 < p < \infty$ separately and use the fact that H^p , $1 \leq p < \infty$, contains complemented copies of ℓ^2 and ℓ^p ; see e.g. [18] for $p = 1$. In the first case, one considers a bounded projection from H^p onto its closed subspace M , which is isomorphic to ℓ^p . Then one utilizes the inclusion mapping $i_{p,2}$ and the fact that ℓ^2 can be embedded in H^p . In the second case, one interchanges the roles of ℓ^p and ℓ^2 and repeats the reasoning above.

The aim of this note is to show that every non-compact operator T_g acting on a Hardy space H^p , $1 \leq p < \infty$, fixes an isomorphic copy of ℓ^p . In particular, this implies that T_g is strictly singular on H^p if and only if it is compact. This article was partly motivated by [11], where the same question was studied in connection to composition operators. For the case $p = \infty$; see also [5].

Our main result is the following theorem.

Theorem 1.1. *Let $g \in BMOA \setminus VMOA$ and $1 \leq p < \infty$. Then the operator $T_g: H^p \rightarrow H^p$ fixes an isomorphic copy of ℓ^p inside H^p , that is, there exists a subspace $M \subset H^p$ which is isomorphic to ℓ^p and such that the restriction of T_g to M is an isomorphism onto its range. In particular, T_g is not strictly singular.*

We should point out that there is a striking extrapolation result concerning strict singularity by Hernández, Semenov, and Tradacete in [7, Theorem 3.3]. It states that if an operator S is bounded on $L^p_{\mathbb{R}}(E)$ and $L^q_{\mathbb{R}}(E)$ for some $1 < p < q < \infty$ and strictly singular on $L^r_{\mathbb{R}}(E)$ for some $p < r < q$, then it is compact on $L^s_{\mathbb{R}}(E)$ for all $p < s < q$. Here $L^p_{\mathbb{R}}(E)$ stands for the L^p space of real-valued functions on a finite measure space E . Taking the complex-valued counterpart of this result for granted, we may deduce the equivalence of strict singularity and compactness of T_g on H^p for $1 < p < \infty$ by using the Riesz projection in the following way: Recall that strictly singular operators form a two-sided (closed) ideal in the space $\mathcal{L}(L^p)$ of bounded operators on $L^p = L^p_{\mathbb{C}}(\mathbb{T})$, where $\mathbb{T} = \partial\mathbb{D}$. Therefore the strict singularity of $T_g: H^p \rightarrow H^p$ implies that $T_g R: L^p \rightarrow L^p$ is strictly singular, where $R: L^p \rightarrow H^p$ is the Riesz projection and we have identified $T_g: H^p \rightarrow H^p$ with $T_g: H^p \rightarrow L^p$. Since the condition $g \in BMOA$ characterizes the boundedness of T_g on every H^q space and the Riesz projection is bounded on the scale $1 < q < \infty$, we get that $T_g R$ is bounded on every L^q , $1 < q < \infty$, space. Now assuming that the complex version of the interpolation result is valid, it follows that $T_g R$ is compact on L^p and consequently the restriction $T_g R|_{H^p} = T_g$ is compact on H^p .

Theorem 1.1, however, makes a stronger statement: a non-compact operator T_g on H^p fixes an isomorphic copy of ℓ^p . This holds for $p = 1$ as well, in which case we obtain a new proof for the equivalence of the compactness and the weak compactness of T_g on H^1 : If $g \in BMOA \setminus VMOA$, i.e., the operator T_g is not compact, then by Theorem 1.1 T_g fixes an isomorphic copy of ℓ^1 and consequently it is not weakly compact.

After giving some preliminaries in Section 2, we give the proof of Theorem 1.1 in Section 3. In Section 4, we consider the case of T_g acting on the space $BMOA$ and we make some remarks on strict singularity of T_g on other spaces.

Finally, we point out that the notions of strict singularity and compactness of an operator can also be defined in a more general setting, for example in quasi-Banach spaces; see [8, Section 7.1]. Examples of such spaces are the Hardy spaces H^p when $0 < p < 1$. The same compactness and boundedness characterizations of T_g as in

the case $1 \leq p < \infty$ also hold when $0 < p < 1$; see [2]. On the scale $0 < p < 1$, the triangle inequality is replaced by an inequality of the type $\|f+g\|_p \leq C(\|f\|_p + \|g\|_p)$ for some constant $C > 1$ for all $f, g \in H^p$. This brings only some (multiplicative) constants to the proofs of results and lemmas in this article. So a non-compact $T_g: H^p \rightarrow H^p$ fixes an isomorphic copy of ℓ^p also in the case $0 < p < 1$.

2. PRELIMINARIES

In this section, we briefly recall some spaces of analytic functions that appear later.

Let $H(\mathbb{D})$ be the algebra of analytic functions in $\mathbb{D}$. We define the Hardy spaces

$$H^p = \left\{ f \in H(\mathbb{D}) : \|f\|_p = \left(\sup_{0 \leq r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p dt \right)^{1/p} < \infty \right\}.$$

The space $BMOA$ consists of functions $f \in H(\mathbb{D})$ with

$$\|f\|_* = \sup_{a \in \mathbb{D}} \|f \circ \sigma_a - f(a)\|_2 < \infty,$$

where $\sigma_a(z) = (a-z)/(1-\bar{a}z)$ is the Möbius automorphism of $\mathbb{D}$ that interchanges the origin and the point $a \in \mathbb{D}$. Its closed subspace $VMOA$ consists of those $f \in BMOA$ with

$$\limsup_{|a| \rightarrow 1} \|f \circ \sigma_a - f(a)\|_2 = 0.$$

Every $BMOA$ function f satisfies “a reverse Hölder’s inequality”, which implies that for each $0 < p < \infty$ we have that

$$(1) \quad \|f\|_* \simeq \sup_{a \in \mathbb{D}} \|f \circ \sigma_a - f(a)\|_p < \infty,$$

where each side is bounded above by a constant multiple of the other. See e.g. [6] for more information on the spaces $BMOA$ and $VMOA$.

3. PROOF OF THE MAIN RESULT

Our goal is to show that a non-compact operator $T_g: H^p \rightarrow H^p$, $1 \leq p < \infty$, $g \in BMOA \setminus VMOA$, fixes an isomorphic copy of ℓ^p and hence the compactness and strict singularity are equivalent for T_g on H^p . This is done by constructing bounded operators $V: \ell^p \rightarrow H^p$ and $U: \ell^p \rightarrow H^p$ such that the diagram in Figure 1 commutes ($U = T_g V$), where $V(\ell^p) = M$ is the closed linear span of suitably chosen test functions $f_{a_k} \in H^p$ and the operator U is an isomorphism onto its range $U(\ell^p) = T_g(M)$.

$$\begin{array}{ccc} \ell^p & & \\ V \downarrow & \searrow U & \\ H^p & \xrightarrow{T_g} & H^p \end{array}$$

FIGURE 1. Operators U, V and T_g

The strategy for choosing the suitable test functions in Propositions 3.2 and 3.5 below is similar to the one used by Laitila et al. in [11], where they utilized these test functions to show that a non-compact composition operator $C_\varphi: H^p \rightarrow H^p$, where $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ is analytic, fixes an isomorphic copy of ℓ^p . However, due to the distinct nature of operators T_g and C_φ , a different kind of analysis is needed in our case.

Before proving our main result (Theorem 1.1), we provide some preparatory material. We first state a localization lemma for the standard test functions in H^p , $1 \leq p < \infty$, defined by

$$f_a(z) = \left[\frac{1 - |a|^2}{(1 - \bar{a}z)^2} \right]^{1/p}, \quad z \in \mathbb{D},$$

for each $a \in \mathbb{D}$. Observe that $\|f_a\|_p = 1$ for all $a \in \mathbb{D}$. The proof of the lemma is straightforward and therefore omitted.

Lemma 3.1. *Let $1 \leq p < \infty$ and m be the normalized Lebesgue measure on $\mathbb{T}$. Define*

$$A_\varepsilon = \{e^{i\theta} : |e^{i\theta} - 1| < \varepsilon\}$$

for $\varepsilon > 0$. Then

$$(i) \quad \lim_{a \rightarrow 1} \int_{\mathbb{T} \setminus A_\varepsilon} |f_a|^p dm = 0 \text{ for each } \varepsilon > 0.$$

$$(ii) \quad \lim_{\varepsilon \rightarrow 0} \int_{A_\varepsilon} |f_a|^p dm = 0 \text{ for each } a \in \mathbb{D}.$$

Next, utilizing test functions $f_{a_k}, a_k \in \mathbb{D}$, for which $|a_k| \rightarrow 1$ sufficiently fast, we construct a certain type of bounded operator $V: \ell^p \rightarrow H^p$.

Proposition 3.2. *Let $1 \leq p < \infty$ and $(a_n) \subset \mathbb{D}$ be a sequence such that $a_n \rightarrow \omega \in \mathbb{T}$. Then there exists a subsequence (b_n) of (a_n) so that the mapping*

$$V: \ell^p \rightarrow H^p, \quad V(\alpha) = \sum_{n=1}^{\infty} \alpha_n f_{b_n},$$

where $\alpha = (\alpha_n) \in \ell^p$, is bounded.

Proof. We may assume that $\omega = 1$. For each $\varepsilon > 0$, we consider the set A_ε defined in Lemma 3.1. Using the fact that $\|f_a\|_p = 1$ for all $a \in \mathbb{D}$ and Lemma 3.1, we choose positive numbers ε_n with $\varepsilon_1 = 2\pi > \varepsilon_2 > \dots > 0$ and a subsequence (b_n) of (a_n) such that the following conditions hold:

$$(i) \quad \left(\int_{A_n} |f_{b_j}|^p dm \right)^{1/p} < 4^{-n}, \quad j = 1, \dots, n-1;$$

$$(ii) \quad \left(\int_{\mathbb{T} \setminus A_n} |f_{b_n}|^p dm \right)^{1/p} < 4^{-n},$$

for every $n \geq 1$, where $A_n = A_{\varepsilon_n}$.

Using conditions (i)-(ii) and the fact that $\left(\int_{A_n} |f_{b_n}|^p dm\right)^{1/p} \leq 1$ for all n , we show the upper bound $\|V\alpha\|_p \leq C\|\alpha\|_{\ell^p}$ for all $\alpha = (\alpha_j) \in \ell^p$, where $C > 0$ may depend on p :

$$\begin{aligned} \|V\alpha\|_p^p &= \int_{\mathbb{T}} \left| \sum_{j=1}^{\infty} \alpha_j f_{b_j} \right|^p dm = \sum_{n=1}^{\infty} \int_{A_n \setminus A_{n+1}} \left| \sum_{j=1}^{\infty} \alpha_j f_{b_j} \right|^p dm \\ &\leq \sum_{n=1}^{\infty} \left(\sum_{j=1}^{\infty} |\alpha_j| \left(\int_{A_n \setminus A_{n+1}} |f_{b_j}|^p dm \right)^{1/p} \right)^p \\ &\leq \sum_{n=1}^{\infty} \left(|\alpha_n| \left(\int_{A_n \setminus A_{n+1}} |f_{b_n}|^p dm \right)^{1/p} \right. \\ &\quad \left. + \sum_{j \neq n} |\alpha_j| \left(\int_{A_n \setminus A_{n+1}} |f_{b_j}|^p dm \right)^{1/p} \right)^p, \end{aligned}$$

where

$$(2) \quad \left(\int_{A_n \setminus A_{n+1}} |f_{b_j}|^p dm \right)^{1/p} \leq \left(\int_{A_n} |f_{b_j}|^p dm \right)^{1/p} < 4^{-n}$$

for $j < n$ by condition (i) and

$$(3) \quad \left(\int_{A_n \setminus A_{n+1}} |f_{b_j}|^p dm \right)^{1/p} \leq \left(\int_{\mathbb{T} \setminus A_j} |f_{b_j}|^p dm \right)^{1/p} < 4^{-j}$$

for $j > n$ by condition (ii). Thus by estimates (2) and (3), we have that

$$(4) \quad \left(\int_{A_n \setminus A_{n+1}} |f_{b_j}|^p dm \right)^{1/p} < 2^{-n-j}$$

for $j \neq n$. By using estimate (4) we get

$$\begin{aligned} \|V(\alpha)\|_p^p &\leq \sum_{n=1}^{\infty} \left(|\alpha_n| \left(\int_{A_n \setminus A_{n+1}} |f_{b_n}|^p dm \right)^{1/p} + \sum_{j \neq n} |\alpha_j| 2^{-n-j} \right)^p \\ &\leq \sum_{n=1}^{\infty} (|\alpha_n| + \|\alpha\|_{\ell^p} 2^{-n})^p \\ &\leq 2^p \left(\sum_{n=1}^{\infty} |\alpha_n|^p + \|\alpha\|_{\ell^p}^p \sum_{n=1}^{\infty} 2^{-np} \right) \leq 2^{p+1} \|\alpha\|_{\ell^p}^p. \quad \square \end{aligned}$$

The next result is an observation by Aleman and Cima [2] and it states that for a non-compact operator T_g on H^p we can find a sequence (f_{a_k}) of test functions so that the sequence $(\|T_g f_{a_k}\|_p)$ converges to a positive constant. Its proof is based on the fact that for all $0 < p < \infty$ and $t \in (0, p/2)$ there exists a constant $C = C(p, t) > 0$ such that

$$\|T_g f_a\|_p \geq C \|g \circ \sigma_a - g(a)\|_t$$

for all $a \in \mathbb{D}$, where $\sigma_a(z) = (a - z)/(1 - \bar{a}z)$; see [2, Theorem 3].

Proposition 3.3. *Let $g \in BMOA \setminus VMOA$ and $1 \leq p < \infty$. Then*

$$c := \limsup_{|a| \rightarrow 1} \|T_g f_a\|_p > 0.$$

In particular, there exists a sequence $(a_k) \subset \mathbb{D}$, $a_k \rightarrow \omega \in \mathbb{T}$ so that

$$\lim_{k \rightarrow \infty} \|T_g f_{a_k}\|_p = c.$$

Next, we prove a localization result for the images $T_g f_a$ of the test functions f_a (cf. Lemma 3.1).

Lemma 3.4. *Let $g \in BMOA$, $1 \leq p < \infty$, and $(a_k) \subset \mathbb{D}$ be a sequence such that $a_k \rightarrow \omega \in \mathbb{T}$. Define*

$$A_\varepsilon = \{e^{i\theta} : |e^{i\theta} - \omega| < \varepsilon\}$$

for each $\varepsilon > 0$. Then

$$\begin{aligned} (i) \quad & \lim_{k \rightarrow \infty} \int_{\mathbb{T} \setminus A_\varepsilon} |T_g f_{a_k}|^p dm = 0 \text{ for every } \varepsilon > 0. \\ (ii) \quad & \lim_{\varepsilon \rightarrow 0} \int_{A_\varepsilon} |T_g f_{a_k}|^p dm = 0 \text{ for each } k. \end{aligned}$$

Proof. (i) Let $\varepsilon > 0$. For the simplicity of notation, we may assume that $\omega = 1$. Also, we assume that $g(0) = 0$. We have that

$$|1 - \bar{a}_k s e^{i\theta}| \geq \delta$$

for all k , $0 \leq s \leq 1$, and $\varepsilon \leq |\theta| \leq \pi$, where $\delta > 0$. Thus for those s and θ we have

$$|f_{a_k}(s e^{i\theta})|^p = \frac{1 - |a_k|^2}{|1 - \bar{a}_k s e^{i\theta}|^2} \leq \frac{1 - |a_k|^2}{\delta^2}$$

and

$$|f'_{a_k}(s e^{i\theta})|^p \leq C \frac{1 - |a_k|^2}{|1 - \bar{a}_k s e^{i\theta}|^{2+p}} \leq C \frac{1 - |a_k|^2}{\delta^{2+p}}$$

for all k , where $C > 0$ may depend on p . For $\zeta \in \mathbb{T} \setminus A_\varepsilon$, we obtain

$$\begin{aligned} |T_g f_{a_k}(\zeta)|^p &= \left| \int_0^1 f_{a_k}(s\zeta) g'(s\zeta) \zeta ds \right|^p \\ &\leq C \left(|f_{a_k}(\zeta) g(\zeta)|^p + \left(\int_0^1 |f'_{a_k}(s\zeta) g(s\zeta)| ds \right)^p \right) \\ &\leq C \left(\frac{1 - |a_k|}{\delta^2} |g(\zeta)|^p + \frac{1 - |a_k|}{\delta^{2+p}} \left(\int_0^1 |g(s\zeta)| ds \right)^p \right), \end{aligned}$$

where constants $C > 0$ may depend on p and change from one instance to another. Since $g \in BMOA$, we have that $|g(z)| \leq C \log \left(\frac{1}{1-|z|} \right) \|g\|_*$ for some $C > 0$, and consequently $\int_0^1 |g(s\zeta)| ds \leq C \|g\|_*$. Therefore

$$\int_{\mathbb{T} \setminus A_\varepsilon} |T_g f_{a_k}|^p dm \leq C \left(\frac{1 - |a_k|}{\delta^2} \|g\|_p^p + \frac{1 - |a_k|}{\delta^{2+p}} \|g\|_*^p \right) \rightarrow 0$$

as $k \rightarrow \infty$, where $\|g\|_p \leq \sup_{b \in \mathbb{D}} \|g \circ \sigma_b - g(b)\|_p \simeq \|g\|_* < \infty$; see (1).

(ii) If k is fixed, then it follows from the absolute continuity of the measure $B \mapsto \int_B |T_g f_{a_k}|^p dm$ that $\int_{A_\varepsilon} |T_g f_{a_k}|^p dm \rightarrow 0$ as $\varepsilon \rightarrow 0$. $\square$

As a final step towards the proof of Theorem 1.1, we construct an isomorphism from ℓ^p into H^p using a non-compact T_g and test functions.

Proposition 3.5. *Let $g \in BMOA \setminus VMOA$, $1 \leq p < \infty$, and $(a_n) \subset \mathbb{D}$ be the sequence from Proposition 3.3. Then there exists a subsequence (b_n) of (a_n) such that the mapping*

$$U: \ell^p \rightarrow H^p, U(\alpha) = \sum_{n=1}^{\infty} \alpha_n T_g f_{b_n},$$

where $\alpha = (\alpha_n) \in \ell^p$, is an isomorphism onto its range.

Proof. By Proposition 3.2 there exists a subsequence (c_n) of (a_n) inducing a bounded operator

$$S: \ell^p \rightarrow H^p, S(\alpha) = \sum_{n=1}^{\infty} \alpha_n f_{c_n},$$

and for any subsequence (b_n) of (c_n) the operator $V: \ell^p \rightarrow H^p$, $V(\alpha) = \sum_{n=1}^{\infty} \alpha_n f_{b_n}$ is bounded. After finding the suitable sequence (b_n) , the operator $U = T_g V$ will be bounded as a composition of two bounded operators.

Before proving that U is bounded from below, we provide some preparatory material. Since (c_n) is a subsequence of (a_n) , we have that $c_n \rightarrow \omega \in \mathbb{T}$ and there exists a number $c > 0$ such that

$$\lim_{n \rightarrow \infty} \|T_g f_{c_n}\|_p = c$$

by Proposition 3.3. Using Proposition 3.3 and Lemma 3.4, we choose positive numbers ε_n with $\varepsilon_1 > \varepsilon_2 > \dots > 0$ and a subsequence (b_n) of (c_n) such that the following conditions hold:

- (i) $\left(\int_{A_n} |T_g f_{b_j}|^p dm \right)^{1/p} < 4^{-n} \delta c, \quad j = 1, \dots, n-1,$
- (ii) $\left(\int_{\mathbb{T} \setminus A_n} |T_g f_{b_n}|^p dm \right)^{1/p} < 4^{-n} \delta c,$
- (iii) $\frac{c}{2} \leq \left(\int_{A_n} |T_g f_{b_n}|^p dm \right)^{1/p} \leq 2c,$

for every $n \geq 1$, where

$$A_n = A_{\varepsilon_n} = \{e^{i\theta} : |e^{i\theta} - \omega| < \varepsilon_n\}$$

and $\delta > 0$ is a constant whose value is determined later.

Now we are ready to prove that U is bounded from below. Using conditions (ii) and (iii), we get

$$\begin{aligned}
\|U\alpha\|_p^p &= \int_{\mathbb{T}} \left| \sum_{j=1}^{\infty} \alpha_j T_g f_{b_j} \right|^p dm \geq \sum_{n=1}^{\infty} \int_{A_n \setminus A_{n+1}} \left| \sum_{j=1}^{\infty} \alpha_j T_g f_{b_j} \right|^p dm \\
&\geq \sum_{n=1}^{\infty} \left| |\alpha_n| \left(\int_{A_n \setminus A_{n+1}} |T_g f_{b_n}|^p dm \right)^{1/p} \right. \\
&\quad \left. - \left(\int_{A_n \setminus A_{n+1}} \left| \sum_{j \neq n} \alpha_j T_g f_{b_j} \right|^p dm \right)^{1/p} \right|^p \\
&\geq \sum_{n=1}^{\infty} \left(2^{-p} |\alpha_n|^p \left(\frac{c}{2} - 4^{-n-1} \delta c \right)^p - \int_{A_n \setminus A_{n+1}} \left| \sum_{j \neq n} \alpha_j T_g f_{b_j} \right|^p dm \right) \\
&\geq \sum_{n=1}^{\infty} \left(2^{-p} |\alpha_n|^p \left(\frac{c}{2} - 4^{-n-1} \delta c \right)^p \right. \\
&\quad \left. - \left(\sum_{j \neq n} |\alpha_j| \left(\int_{A_n \setminus A_{n+1}} |T_g f_{b_j}|^p dm \right)^{1/p} \right)^p \right),
\end{aligned}$$

where

$$\left(\int_{A_n \setminus A_{n+1}} |T_g f_{b_j}|^p dm \right)^{1/p} \leq \left(\int_{A_n} |T_g f_{b_j}|^p dm \right)^{1/p} < 4^{-n} \delta c$$

for $j < n$ by condition (i) and

$$\left(\int_{A_n \setminus A_{n+1}} |T_g f_{b_j}|^p dm \right)^{1/p} \leq \left(\int_{\mathbb{T} \setminus A_j} |T_g f_{b_j}|^p dm \right)^{1/p} < 4^{-j} \delta c$$

for $j > n$ by condition (ii). Thus we have that

$$\left(\int_{A_n \setminus A_{n+1}} |T_g f_{b_j}|^p dm \right)^{1/p} < 2^{-n-j} \delta c$$

for $j \neq n$. Consequently, we can estimate

$$\begin{aligned}
\|U\alpha\|_p^p &\geq \sum_{n=1}^{\infty} \left(2^{-p} |\alpha_n|^p \left(\frac{c}{2} - 4^{-n-1} \delta c \right)^p - \left(\sum_{j=1}^{\infty} |\alpha_j| 2^{-n-j} \delta c \right)^p \right) \\
&\geq \sum_{n=1}^{\infty} \left(2^{-p} |\alpha_n|^p \left(\frac{c}{2} - 4^{-n-1} \delta c \right)^p - 2^{-n} \delta^p c^p \|\alpha\|_{\ell^p}^p \right) \\
&\geq 2^{-p} \sum_{n=1}^{\infty} |\alpha_n|^p \left(\frac{c}{2} - \frac{1}{16} \delta c \right)^p - \delta^p c^p \|\alpha\|_{\ell^p}^p \\
&\geq 2^{-p} \left(\frac{7}{16} \right)^p c^p \|\alpha\|_{\ell^p}^p - \delta^p c^p \|\alpha\|_{\ell^p}^p \\
&= \left(\left(\frac{7}{32} \right)^p - \delta^p \right) c^p \|\alpha\|_{\ell^p}^p = \left(\frac{1}{8} \right)^p c^p \|\alpha\|_{\ell^p}^p,
\end{aligned}$$

when we choose $0 < \delta < 1$ such that $\delta^p = \left(\frac{7}{32} \right)^p - \left(\frac{1}{8} \right)^p$, i.e., $\delta = \frac{(7^p - 4^p)^{1/p}}{32}$. Thus the bounded operator U is also bounded from below and consequently it is an isomorphism onto its range. $\square$

Now our main result follows.

Proof of Theorem 1.1. By Propositions 3.2 and 3.5, we can choose a sequence $(b_n) \subset \mathbb{D}$ that induces a bounded operator

$$V: \ell^p \rightarrow H^p, \quad V(\alpha) = \sum_{n=1}^{\infty} \alpha_n f_{b_n},$$

where $\alpha = (\alpha_n) \in \ell^p$, and an isomorphism $U: \ell^p \rightarrow H^p$, $U = T_g V$ onto its range.

Define $M = \overline{\text{span}\{f_{b_n}\}}$, where the closure is taken in H^p . Since U is bounded from below, we have that the restriction $T_g|_M$ is bounded from below. Thus $T_g|_M: M \rightarrow T_g(M)$ is an isomorphism and consequently M is isomorphic to ℓ^p . In particular, the operator T_g is not strictly singular. $\square$

We conclude this section by posing two questions concerning the ℓ^2 -singularity of T_g . To be specific, define $S(H^p)$ to be the class of strictly singular operators on H^p and $S_r(H^p)$ to be the class of ℓ^r -singular operators on H^p , i.e., those bounded operators which do not fix a copy of the space ℓ^r . Then $S(H^p) = S_2(H^p) \cap S_p(H^p)$ for $1 < p < \infty$; see [17]. For a non-compact operator T_g on H^p it follows from Theorem 1.1 that $T_g \notin S_p(H^p)$. However, we did not pursue the following questions.

Questions. Does a non-compact $T_g: H^p \rightarrow H^p$ satisfy $T_g \notin S_2(H^p)$? Can we characterize those g such that $T_g \in S_2(H^p)$, where $1 \leq p < \infty$, $p \neq 2$?

4. STRICT SINGULARITY ON $BMOA$ AND OTHER SPACES

In this section, we consider the strict singularity of T_g on $BMOA$, the Bergman spaces A^p , $1 \leq p < \infty$, and the Bloch space $\mathcal{B}$. The case of $BMOA$ essentially follows from the reasoning done in [10], where we utilize an idea of Leĭbov [12] that there exist isomorphic copies of the space c_0 of null sequences inside $VMOA$. This fact will imply that the strict singularity of T_g on $BMOA$ or on $VMOA$ is equivalent to the compactness of T_g on the same space. The sketch of the proof is the following.

We recall that the boundedness of T_g on $BMOA$ is characterized by the condition $g \in LMOA$, where $LMOA$ is the “logarithmic $BMOA$ ” space; see [16].

By the proof of Theorem 2 and Lemma 6 in [10] we can find a sequence (h_n) in $VMOA$ which is equivalent to the standard basis of c_0 . If T_g is non-compact on $VMOA$, it follows from the estimate (4.6) in [10] that $\|T_g h_n\|_* > c > 0$ for some constant c for all n . Since $g \in LMOA \subset BMOA$, the operator T_g is bounded on H^2 and consequently $\|T_g h_n\|_2 \rightarrow 0$, as $n \rightarrow \infty$. Now we apply Lemma 6 in [10] again to obtain (by passing to a subsequence, if needed) that $(T_g h_n)$ is equivalent to the standard basis of c_0 . Hence $T_g|_M$, where $M = \overline{\text{span}\{h_n\}}$, is an isomorphism onto its range and T_g is not strictly singular on $VMOA$ (or on $BMOA$).

In the Bergman spaces, boundedness and compactness of T_g was characterized in [4]. It is known that A^p , $1 \leq p < \infty$, are isomorphic to ℓ^p ; see e.g. [19, Chapter 3.A, Theorem 11]. Hence the strict singularity of the operator T_g coincides with the compactness, since all strictly singular operators on ℓ^p are compact; see [13, Proposition 2.c.3] and a comment thereafter. The boundedness and compactness of T_g acting on Bloch spaces was investigated in [20]. In this case, we can deduce as follows: If T_g acting on $\mathcal{B}$ is strictly singular, then its restriction to the little Bloch space $\mathcal{B}_0$ is also strictly singular. Since $\mathcal{B}_0$ is isomorphic to c_0 and strictly singular operators on c_0 are compact, the restriction $T_g|_{\mathcal{B}_0}$ is compact. Also, the bidual $(\mathcal{B}_0)^{**}$ can be identified with $\mathcal{B}$, so the operator T_g can be identified with the biadjoint operator $(T_g|_{\mathcal{B}_0})^{**}$. Therefore T_g is compact.

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