LIPSCHITZ SLICES VERSUS LINEAR SLICES IN BANACH SPACES

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ABSTRACT. The aim of this note is to study the topology generated by Lipschitz slices in the unit sphere of a Banach space. We prove that the above topology agrees with the weak topology in the unit sphere and, as a consequence, we obtain easy Lipschitz characterizations of classical linear topics in Banach spaces as the Daugavet property, Radon-Nikodym property, convex point of continuity property and strong regularity, which shows that the above classical linear properties depend only on the natural uniformity in the Banach space given by the metric and the linear structure.

1. INTRODUCTION

Recently considered has been (see [6]) the study of the Daugavet equation for Lipschitz operators, which has resulted in the introduction of the so-called “Lip-slices”, getting a characterization of the Daugavet property in terms of such subsets of the unit sphere. Motivated by these facts, we will define, in the first section, the Lip-topology on the sphere of a Banach space as the topology generated by the Lip-slices of the unit sphere and, even though it seems natural to think that the weak topology is strictly coarser than the Lip-topology, we will actually prove in Theorem 2.4 that both topologies agree, which will allow us to improve the characterization of the Daugavet property given in [6, Theorem 2.6] and to extend the given ones in [9] to a Lipschitz setting. Finally, we get in the second section, as a consequence in Theorem 3.1, that a Banach space X satisfies the Radon-Nikodym property if, and only if, the unit sphere of every equivalent norm in X contains Lipschitz slices with arbitrarily small diameter, which shows that the well known characterization of the Radon-Nikodym property for Banach spaces in terms of slices is uniform; that is, it depends only on the natural uniformity in X given by the metric and the linear structure of the Banach space. Similar characterizations are also obtained for the convex point of continuity property and the strong regularity in Banach spaces. We refer to [4] for background on these classical properties.

We shall introduce some notation. We consider real Banach spaces, B_X (resp. S_X) denotes the closed unit ball (resp. sphere) of the Banach space X , and X^*

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stands for the dual space of X . A slice of a bounded subset C of X is a set of the form

$$S(C, f, \alpha) := \{x \in C : f(x) > M - \alpha\},$$

where $f \in X^*$, $f \neq 0$, $M = \sup_{x \in C} f(x)$ and $\alpha > 0$. w denotes the weak topology of a Banach space, and it is well known that the family of slices is a subbasis of the weak topology for bounded subsets in X .

Let X be a Banach space and $f : X \rightarrow \mathbb{R}$ a Lipschitz function. Recall that the seminorm of f is defined by

$$\|f\| = \sup \left\{ \frac{f(x) - f(y)}{\|x - y\|} : x, y \in X, x \neq y \right\}.$$

According to [2], the generalized derivative of a point $x \in X$ in the direction $v \in X$ is defined by

$$f^\circ(x, v) := \limsup_{y \rightarrow x, t \searrow 0} \frac{f(y + tv) - f(y)}{t}.$$

Such a limit always exists from the Lipschitz condition. Moreover, it is a sublinear and positively homogeneous function in the variable v [2, Proposition 2.1.1].

In addition, the generalized gradient of f at x is defined as

$$\partial f(x) := \{x^* \in X^* : f^\circ(x, v) \geq x^*(v) \ \forall v \in X\}.$$

Given $v \in X$ it follows that [2, Proposition 2.1.2]

$$f^\circ(x, v) = \max_{x^* \in \partial f(x)} x^*(v) \quad \forall x \in X.$$

2. THE WEAK AND LIP-TOPOLOGY

Definition 2.1. Let X be a Banach space. According to [6], a Lip-slice of S_X is a non-empty set of the form

$$S(S_X, f, \varepsilon) := \left\{ \frac{x_1 - x_2}{\|x_1 - x_2\|} : \frac{f(x_1) - f(x_2)}{\|x_1 - x_2\|} > \|f\| - \varepsilon \right\},$$

where f is a Lipschitz real valued function on X .

In [6] it is proved that if a Banach space has the Daugavet property then, for each $x \in S_X$, S a Lip-slice of S_X and $\varepsilon > 0$, there exists $y \in S$ such that

$$\|x + y\| > 2 - \varepsilon.$$

On the other hand, note that the class of Lip-slices defines a subbasis of a topology on S_X . Indeed, define $\mathcal{B}$ to be the class of Lip-slices of S_X . Then

$$\bigcup_{S \in \mathcal{B}} S = S_X$$

as clearly $S(S_X, \|\cdot\|, \alpha) = S_X$ for each $\alpha \in \mathbb{R}^+$ (indeed, given $x \in S_X$ one has that $\frac{2x-x}{\|2x-x\|} \in S$).

Now we can give the following.

Definition 2.2. Let X be a Banach space. We will define the Lip-topology on S_X as the topology generated by the subbasis $\mathcal{B}$.

At this point it seems natural to wonder whether the Lip- and the weak topology on S_X are equal. It is obvious, as slices of S_X are Lip-slices of S_X , that the weak topology is contained in the Lip-topology. Our aim is to prove that both topologies actually agree. The key to the proof will be the following.

Lemma 2.3. *Let X be a Banach space and S a Lip-slice of S_X . Then for each $x \in S$ there exists T , a slice of S_X , such that*

$$x \in T \subseteq S.$$

Proof. Assume that $S := S(S_X, f, \varepsilon)$. Consider $x, y \in X, x \neq y$, such that $\frac{y-x}{\|y-x\|} \in S$, i.e.

$$f(y) - f(x) > (1 - \varepsilon)\|y - x\|.$$

Define $\phi : [0, 1] \rightarrow [x, y]$ by $\phi(t) := ty + (1-t)x, t \in [0, 1]$, and $F := f \circ \phi : [0, 1] \rightarrow \mathbb{R}$. As F is a Lipschitz function we have that F is derivable almost everywhere. Now

$$(1 - \varepsilon)\|y - x\| < f(y) - f(x) = F(1) - F(0) = \int_0^1 F'(t) dt.$$

From here we can find t such that $F'(t)$ exists and is bigger than $(1 - \varepsilon)\|y - x\|$. Thus given $z = \phi(t)$ it follows that

$$\limsup_{h \rightarrow 0} \frac{f(z + h(y-x)) - f(z)}{h} > (1 - \varepsilon)\|y - x\|.$$

Indeed, given $h > 0$ small enough one has

$$\phi(t+h) = (t+h)y + (1-(t+h))x = ty + (1-t)x + h(y-x).$$

As $ty + (1-t)x = \phi(t) = z$, we conclude that

$$\frac{F(t+h) - F(t)}{h} = \frac{f(z + h(y-x)) - f(z)}{h}.$$

As there exists $\lim_{h \rightarrow 0} \frac{F(t+h) - F(t)}{h} = \lim_{h \rightarrow 0} \frac{f(z + h(y-x)) - f(z)}{h}$ we conclude that

$$\lim_{h \rightarrow 0} \frac{f(z + h(y-x)) - f(z)}{h} = \limsup_{h \rightarrow 0} \frac{f(z + h(y-x)) - f(z)}{h} \leq f^\circ(z, y-x).$$

As the generalized derivative is positively homogeneous we conclude that $f^\circ(z, \frac{y-x}{\|y-x\|}) > 1 - \varepsilon$. Hence, from [2, Proposition 2.1.2] there exists $\varphi \in \partial f(z)$ such that $\varphi(\frac{y-x}{\|y-x\|}) > 1 - \varepsilon$. Consequently

$$\frac{y-x}{\|y-x\|} \in T := \{v \in S_X / \varphi(v) > 1 - \varepsilon\}.$$

Now we shall prove that $T \subseteq S$. To this aim pick $v \in T$ so $\varphi(v) > 1 - \varepsilon$. As

$$\varphi(v) \leq f^\circ(z, v) = \limsup_{y \rightarrow z, t \searrow 0} \frac{f(y+tv) - f(y)}{t}$$

we can find $t > 0$ and y close enough to z such that $1 - \varepsilon < \frac{f(y+tv) - f(y)}{t}$. From the definition of S one has

$$S \ni \frac{y+tv-y}{\|y+tv-y\|} = \frac{tv}{\|tv\|} = v,$$

so $\frac{y-x}{\|y-x\|} \in T \subseteq S$. □

As a consequence we get the desired result.

Theorem 2.4. *Let X be a Banach space. Then the weak and the Lip-topology agree on S_X .*

Proof. As we have pointed out it is clear that the weak topology on S_X is contained in the Lip-topology on S_X .

For the reverse inclusion, pick W to be a basic open set of the Lip-topology of S_X . Then $W := \bigcap_{i=1}^n S(S_X, f_i, \varepsilon_i)$ is a finite and non-empty intersection of Lip-slices. Pick $x \in W$. So, for each $i \in \{1, \dots, n\}$, we can find by the above lemma T_i , a slice of S_X , such that

$$x \in T_i \subseteq S(S_X, f_i, \varepsilon_i).$$

So, if we define $U := \bigcap_{i=1}^n T_i$, one has

$$x \in U \subseteq W,$$

so we are done. $\square$

The above result can be applied to improve and clarify the results in [6] and get consequences on Daugavet property which extend the given ones in [9] to a Lipschitz setting in an easy way. Indeed, the following result is the key to doing that, as was done in [9] for linear slices and weak open subsets.

Corollary 2.5. *Let X be a Banach space. Then X has the Daugavet property if, and only if, for every $x \in S_X$, non-empty open subset for the Lip-topology U of S_X and $\varepsilon > 0$ there exists $y \in U$ such that*

$$\|x + y\| > 2 - \varepsilon.$$

Proof. It is a straightforward application of [9, Lemma 3] and the fact that every finite and non-empty intersection of Lip-slices contains a non-empty relatively weak open subset of S_X proved in Theorem 2.4. $\square$

According to [7, Definition 4.10], a locally convex topology τ on a Banach space X is said to be a *Daugavet topology* if given $x, y \in S_X$, U a τ neighborhood of y and $\varepsilon \in \mathbb{R}^+$ there exists an element $z \in U \cap S_X$ such that $\|x + z\| > 2 - \varepsilon$.

The above corollary says that Lip-topology is a Daugavet topology in a more general setting.

3. CHARACTERIZATION OF GEOMETRICAL PROPERTIES USING THE LIP-TOPOLOGY

Now we shall exhibit a characterization of the Radon-Nikodym property (RNP), the convex point of continuity property (CPCP) and strong regularity (SR) in terms of Lip-slices. (We refer to [4] for background on these properties.)

Theorem 3.1. *Let X be a Banach space.*

- (1) *X has the RNP if, and only if, for every equivalent norm on X it follows that its new unit sphere contains Lip-slices of arbitrarily small diameter.*
- (2) *X has the CPCP if, and only if, for every equivalent norm on X it follows that its new unit sphere contains non-empty relatively open subsets, for the Lip-topology, of arbitrarily small diameter.*

- (3) *X is SR if, and only if, for every equivalent norm on X it follows that its new unit ball contains convex combinations of Lip-slices of arbitrarily small diameter.*

The above theorem shows that RNP, CPCP and SR are properties which depend on the natural uniformity of the Banach space, given by the metric and the linear structure. In fact, roughly speaking, we can say that every property in Banach spaces characterized in terms of the diameter of relevant subsets for the weak topology is determined by the natural uniformity in the space and so the property also can be characterized in terms of relevant subsets for the Lip-topology. This is also the case, as we have seen, for the Daugavet property. Also, this idea allows us to get easily Lipschitz characterizations for the diameter two properties, changing linear slices by Lipschitz slices. (Recall that a Banach space satisfies the slice diameter two property, for example, if every slice in the unit ball of the space has diameter two.)

In order to prove Theorem 3.1 we shall need to prove a characterization of previous properties in terms of the diameter of slices (respectively non-empty relatively weak open subsets, convex combination of slices) of the unit ball in each equivalent renorming of the space. It is well known that such characterization holds for the RNP (see [3]). Now we shall prove similar statements for the CPCP and SR. To this aim we shall need the following two preliminary results.

Lemma 3.2. *Let X be a Banach space. Then:*

- (1) *If X fails to be strongly regular, then, for each $\varepsilon > 0$, there exists $K \subseteq X$ a closed, bounded, symmetric and convex subset such that $\text{diam}(K) = 2$ and each convex combination of slices of K has diameter at least $1 - \varepsilon$.*
- (2) *If X fails the CPCP, then, for each $\varepsilon > 0$, there exists $K \subseteq X$ a closed, convex and bounded subset such that $\text{diam}(K) = 2$ and every non-empty relatively weak open subset of K has diameter at least $1 - \varepsilon$.*

Proof. (1) Pick $\varepsilon > 0$. By [8, Proposition 4.10 (b)], consider $C \subseteq X$ a closed and convex subset such that $\text{diam}(C) = 1$ and such that each convex combination of slices has diameter at least $1 - \varepsilon$. Now $K := \overline{\text{co}}(C \cup -C)$ is the desired subset by the proof of [1, Lemma 2.12].

(2) Pick $\varepsilon > 0$. By [8, Proposition 4.10, (a)] consider $C \subseteq X$ a closed, convex subset such that $\text{diam}(C) = 1$ and such that every non-empty relatively weak open subset of C has diameter at least $1 - \varepsilon$. Define $K := \overline{C - C}$. K is obviously closed. Moreover, K is convex because $C - C$ is convex. Moreover, as $C - C$ is symmetric, so is K .

In order to finish the proof pick U a weak open subset of X verifying $U \cap (C - C) \neq \emptyset$, and let us prove that the last set has diameter at least $1 - \varepsilon$. Pick $x_0 \in U \cap (C - C)$, so $x_0 = a - b$ for suitable $a, b \in C$. Now

$$x_0 \in U \Rightarrow a = x_0 + b \in b + U.$$

Consequently $a \in (b + U) \cap C$, so $(b + U) \cap C \neq \emptyset$. By the assumptions on C we conclude that

$$\text{diam}((b + U) \cap C) \geq 1 - \varepsilon.$$

We will show that

$$(3.1) \quad ((b + U) \cap C) - b \subseteq U \cap (C - C).$$

To this aim pick $x \in ((b+U) \cap C) - b$. So there exists $u \in U$ such that $x = b + u - b$. So $x = u \in U$. Moreover, $b + u \in C$ implies that $x = b + u - b \in C - C$. So (3.1) holds. Hence

$$1 - \varepsilon \leq \text{diam}((b+U) \cap C) = \text{diam}(((b+U) \cap C) - b) \leq \text{diam}(U \cap (C - C)).$$

The last inequality proves that each non-empty relatively weak open subset of $C - C$ has diameter at least $1 - \varepsilon$. As each non-empty relatively weak open subset of K necessarily intersects $C - C$, we conclude that each non-empty relatively weak open subset of K has diameter at least $1 - \varepsilon$.

It is also clear that $\text{diam}(K) = \text{diam}(C - C) = 2$, so we are done. $\square$

Now we shall prove a renorming fact for Banach spaces failing CPCP or SR.

Theorem 3.3. *Let $(X, \|\cdot\|)$ be a Banach space and consider $\alpha \in \mathbb{R}^+$.*

- (1) *Assume that $C \subseteq B_X$ is a closed, symmetric and convex subset verifying that for every non-empty relatively weak open subset U of C ,*

$$\text{diam}_{\|\cdot\|}(U) \geq \alpha.$$

Then for each $\varepsilon \in \mathbb{R}^+$, there exists $|\cdot|$, an equivalent norm on X satisfying:

- (a) $|x| \geq \|x\|$ for all $x \in X$.
 (b) *The diameter of every non-empty relatively weak open subset of $B_{|\cdot|}$ is at least $\frac{\alpha}{1+\varepsilon}$.*
 (2) *Assume that $C \subseteq B_X$ is a closed, symmetric and convex subset verifying that for all $T \subseteq C$, a convex combination of slices of C ,*

$$\text{diam}_{\|\cdot\|}(T) \geq \alpha.$$

Then for each $\varepsilon \in \mathbb{R}^+$, there exists $|\cdot|$, an equivalent norm on X satisfying:

- (a) $|x| \geq \|x\|$ for all $x \in X$.
 (b) *The diameter of each convex combination of slices of $B_{|\cdot|}$ is at least $\frac{\alpha}{1+\varepsilon}$.*

Proof. (1) Pick an arbitrary $\varepsilon \in \mathbb{R}^+$, $C \subseteq B_X$ satisfying the hypothesis of the theorem.

Define on X the norm whose unit ball is

$$\frac{1}{1+\varepsilon} \overline{C + \varepsilon B_X}.$$

This new norm on X is equivalent to $\|\cdot\|$. Moreover, as $B_{|\cdot|} \subseteq B_X$, one has

$$\|x\| \leq |x| \quad \forall x \in X.$$

Let us prove (b). To this aim pick $U \subseteq B_{|\cdot|}$, a non-empty relatively weak open set. We can assume that there exist $x_0 \in B_{|\cdot|}$, $\beta \in \mathbb{R}^+$ and $x_1^*, \dots, x_n^* \in X^*$ such that

$$U = (x_0 + V_{\beta, x_1^*, \dots, x_n^*}) \cap \frac{1}{1+\varepsilon} \overline{C + \varepsilon B_X},$$

where $V_{\beta, x_1^*, \dots, x_n^*} := \{x \in X : |x_i^*(x)| < \beta \ \forall i \in \{1, \dots, n\}\}$. From linearity of the weak topology we can ensure the existence of $\gamma \in \mathbb{R}^+$, $y_1^*, \dots, y_m^* \in X^*$ such that

$$(3.2) \quad V_{\gamma, y_1^*, \dots, y_m^*} + V_{\gamma, y_1^*, \dots, y_m^*} \subseteq V_{\beta, x_1^*, \dots, x_n^*}.$$

Consider $v \in (x_0 + V_{\gamma, y_1^*, \dots, y_m^*}) \cap \left(\frac{1}{1+\varepsilon} (C + \varepsilon B_X)\right)$. Then, on the one hand,

$$(3.3) \quad v = \frac{1}{1+\varepsilon} (c + \varepsilon b), \quad c \in C, b \in B_X.$$

On the other hand,

$$(3.4) \quad v = x_0 + x, \quad x \in V_{\gamma, y_1^*, \dots, y_m^*}.$$

Now we can prove the following.

Claim 3.4. The following inclusion holds:

$$(3.5) \quad A := \frac{1}{1+\varepsilon} \left(\left(c + V_{\frac{\gamma(1+\varepsilon)}{2}, y_1^*, \dots, y_m^*} \cap C \right) + \varepsilon \left(b + V_{\frac{\gamma(1+\varepsilon)}{2\varepsilon}, y_1^*, \dots, y_m^*} \cap B_X \right) \right) \\ \subseteq (v + V_{\gamma, y_1^*, \dots, y_m^*}) \cap \left(\frac{1}{1+\varepsilon} (C + \varepsilon B_X) \right).$$

Proof. Pick $z \in A$. Then

$$z = \frac{1}{1+\varepsilon} (c + x + \varepsilon(b + y)),$$

where $x \in V_{\frac{\gamma(1+\varepsilon)}{2}, y_1^*, \dots, y_m^*}$, $y \in V_{\frac{\gamma(1+\varepsilon)}{2\varepsilon}, y_1^*, \dots, y_m^*}$, $c + x \in C$ and $b + y \in B_X$.

As $c + x \in C$ and $b + y \in B_X$, then

$$z = \frac{1}{1+\varepsilon} (c + x + \varepsilon(b + y)) \in \frac{1}{1+\varepsilon} (C + \varepsilon B_X).$$

On the other hand, bearing in mind that $v = \frac{1}{1+\varepsilon} (c + \varepsilon b)$, we conclude that

$$z = v + \frac{x + \varepsilon y}{1+\varepsilon}.$$

So, in order to finish the proof of the claim, it is enough to prove that $\frac{x + \varepsilon y}{1+\varepsilon} \in V_{\gamma, y_1^*, \dots, y_m^*}$. To this aim pick $i \in \{1, \dots, m\}$. Then

$$\left| y_i^* \left(\frac{x + \varepsilon y}{1+\varepsilon} \right) \right| \leq \frac{|y_i^*(x)| + \varepsilon |y_i^*(y)|}{1+\varepsilon} < \frac{\frac{\gamma(1+\varepsilon)}{2} + \varepsilon \frac{\gamma(1+\varepsilon)}{2\varepsilon}}{1+\varepsilon} = \gamma,$$

so we are done. $\square$

Furthermore (3.4) implies that

$$(v + V_{\gamma, y_1^*, \dots, y_m^*}) \cap \left(\frac{1}{1+\varepsilon} (C + \varepsilon B_X) \right) \\ = (x_0 + x + V_{\gamma, y_1^*, \dots, y_m^*}) \cap \left(\frac{1}{1+\varepsilon} (C + \varepsilon B_X) \right) \\ \subseteq (x_0 + V_{\gamma, y_1^*, \dots, y_m^*} + V_{\gamma, y_1^*, \dots, y_m^*}) \cap \left(\frac{1}{1+\varepsilon} (C + \varepsilon B_X) \right) \\ \subseteq (x_0 + V_{\beta, x_1^*, \dots, x_n^*}) \cap \left(\frac{1}{1+\varepsilon} (C + \varepsilon B_X) \right) \subseteq U.$$

By assumption, we have that

$$\text{diam} \left(\frac{1}{1+\varepsilon} \left(\left(c + V_{\frac{\gamma(1+\varepsilon)}{2}, y_1^*, \dots, y_m^*} \cap C \right) + \varepsilon \left(b + V_{\frac{\gamma(1+\varepsilon)}{2\varepsilon}, y_1^*, \dots, y_m^*} \cap B_X \right) \right) \right) \\ \geq \frac{\alpha}{1+\varepsilon},$$

hence

$$\text{diam}_{|\cdot|}(U) \geq \text{diam}_{\|\cdot\|}(U) \geq \frac{\alpha}{1+\varepsilon},$$

which proves (a).

The proof of (2) follows from Lemma 3.2 and [1, Lemma 2.13]. $\square$

From the above results and from [8, Corollary 3.2] we get the following.

Corollary 3.5. *Let X be a Banach space.*

- (1) *X has the RNP if, and only if, each equivalent norm on X verifies that its unit ball contains slices of arbitrarily small diameter.*
- (2) *X has the CPCP if, and only if, each equivalent norm on X verifies that its unit ball contains non-empty relatively weak open subsets of arbitrarily small diameter.*
- (3) *X is SR if, and only if, each equivalent norm on X verifies that its unit ball contains convex combinations of slices of arbitrarily small diameter.*

In order to characterize the CPCP and SR in terms of Lip-slices, we need the following lemma.

Lemma 3.6. *Let X be a Banach space. The following assertions are equivalent:*

- (1) *B_X contains non-empty relatively weak open subsets of arbitrarily small diameter.*
- (2) *S_X contains non-empty relatively weak open subsets of arbitrarily small diameter.*
- (3) *S_X contains non-empty relatively open subsets of the Lip-topology of arbitrarily small diameter.*

Proof. (2) $\Leftrightarrow$ (3) is known by Theorem 2.4.

(1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (1). Consider $\delta > 0$ and $U := \bigcap_{i=1}^n S(S_X, x_i^*, \varepsilon_i)$ a basic non-empty relatively weak open subset of S_X whose diameter is smaller than $\frac{\delta}{2}$. Define

$$V := \bigcap_{i=1}^n S(B_X, x_i^*, \varepsilon_i),$$

which is a non-empty relatively weak open subset of B_X and pick $x \in V \cap S_X$. Pick $0 < \alpha < \min\{\frac{\delta}{2}, 1\}$ and consider by [5, Lemma 2.1], for each $i \in \{1, \dots, n\}$, a functional $y_i^* \in S_{X^*}$ such that

$$x \in \bigcap_{i=1}^n S(B_X, y_i^*, \alpha) \subseteq V.$$

Consequently $x \in \bigcap_{i=1}^n S(S_X, y_i^*, \alpha) \subseteq U$, so $\text{diam} \left(\bigcap_{i=1}^n S(S_X, y_i^*, \alpha) \right) < \frac{\delta}{2}$. Now we will prove

$$\text{diam} \left(\bigcap_{i=1}^n S(B_X, y_i^*, \alpha) \right) < \delta.$$

To this aim pick $x, y \in \bigcap_{i=1}^n S(B_X, y_i^*, \alpha)$. Now $\frac{x}{\|x\|}, \frac{y}{\|y\|} \in \bigcap_{i=1}^n S(S_X, y_i^*, \alpha)$, so

$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| < \frac{\delta}{2}$. Hence

$$\begin{aligned} \|x - y\| &\leq \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| + \left\| x - \frac{x}{\|x\|} \right\| + \left\| y - \frac{y}{\|y\|} \right\| \\ &< \frac{\delta}{2} + 1 - y_1^*(x) + 1 - y_1^*(y) < \frac{\delta}{2} + 2\alpha < \delta. \end{aligned}$$

□

A similar result can be stated replacing “weakly open sets” with “convex combination of slices” (and, consequently, with “slices”).

Lemma 3.7. *Let X be a Banach space. The following assertions are equivalent:*

- (1) B_X contains convex combinations of slices of arbitrarily small diameter.
- (2) B_X contains convex combinations of slices of S_X of arbitrarily small diameter.
- (3) B_X contains convex combinations of Lip-slices of arbitrarily small diameter.

Proof. (2) $\Leftrightarrow$ (3) is known by Theorem 2.4.

(1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1) is similar to (2) $\Rightarrow$ (1) in Lemma 3.6, but we shall provide a proof for the sake of completeness.

Consider an arbitrary $\delta > 0$ and $C := \sum_{i=1}^n \lambda_i S(S_X, x_i^*, \alpha)$ a convex combination of slices whose diameter is less than $\frac{\delta}{2}$. We can assume, considering smaller numbers, that $0 < \alpha < \min\{\frac{\delta}{4}, 1\}$. We shall prove that $\text{diam}(\sum_{i=1}^n \lambda_i S(B_X, x_i^*, \alpha)) \leq \delta$. To this aim pick $\sum_{i=1}^n \lambda_i x_i, \sum_{i=1}^n \lambda_i y_i \in \sum_{i=1}^n \lambda_i S(B_X, x_i^*, \alpha)$. Now it is clear that

$$\sum_{i=1}^n \lambda_i \frac{x_i}{\|x_i\|}, \sum_{i=1}^n \lambda_i \frac{y_i}{\|y_i\|} \in \sum_{i=1}^n \lambda_i S(S_X, x_i^*, \alpha) \Rightarrow \left\| \sum_{i=1}^n \lambda_i \frac{x_i}{\|x_i\|} - \sum_{i=1}^n \lambda_i \frac{y_i}{\|y_i\|} \right\| < \frac{\delta}{2}.$$

Now

$$\begin{aligned} & \left\| \sum_{i=1}^n \lambda_i (x_i - y_i) \right\| \\ & \leq \left\| \sum_{i=1}^n \lambda_i \frac{x_i}{\|x_i\|} - \sum_{i=1}^n \lambda_i \frac{y_i}{\|y_i\|} \right\| + \sum_{i=1}^n \lambda_i \left\| x_i - \frac{x_i}{\|x_i\|} \right\| + \sum_{i=1}^n \lambda_i \left\| y_i - \frac{y_i}{\|y_i\|} \right\| \\ & < \frac{\delta}{2} + \sum_{i=1}^n \lambda_i \alpha + \sum_{i=1}^n \lambda_i \alpha = \frac{\delta}{2} + 2\alpha < \delta. \end{aligned}$$

Consequently, $\text{diam}(\sum_{i=1}^n \lambda_i S(B_X, x_i^*, \alpha)) \leq \delta$, as desired. $\square$

Now using the above lemmata and Theorem 3.3 we get the desired characterization of the RNP, CPCP and SR in Theorem 3.1.

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