NOTES ON ENDPOINT ESTIMATES FOR MULTILINEAR FRACTIONAL INTEGRAL OPERATORS

YASUO KOMORI-FURUYA

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ABSTRACT. We prove the strong type estimates for the multilinear fractional integral operators of Kenig and Stein type at critical indices. Our results are optimal.

1. INTRODUCTION

Since Lacey and Thiele [5] proved the L^p boundedness of the bilinear Hilbert transform, and Grafakos and Torres [2] proved the L^p boundedness of multilinear singular integrals, considerable attention has been paid to the study of multilinear operators. Kenig and Stein [4] proved the boundedness of multilinear fractional integral operators on L^p spaces and weak type estimates on L^1 . Tang [6] showed the BMO estimates at critical indices for multilinear fractional integral operators. In this paper we consider strong type estimates on L^1 and L^∞ estimates at critical indices. We also show that our results are optimal by giving counterexamples.

Ordinary fractional integral operators are defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy \quad \text{where } 0 < \alpha < n.$$

We know the following (see, for example, [1]):

$$\begin{aligned} I_\alpha : L^p \rightarrow L^q & \quad \text{when } \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n} > 0 \quad \text{and } p > 1, \\ I_\alpha : L^1 \rightarrow L^{q,\infty} & \quad \text{when } \frac{1}{q} = 1 - \frac{\alpha}{n}, \\ I_\alpha : L^{n/\alpha} & \not\rightarrow L^\infty. \end{aligned}$$

We define the multilinear fractional integral operators.

Definition 1.

$$I_{m,\alpha}(f_1, \dots, f_m)(x) := \int_{(\mathbb{R}^n)^m} \frac{f_1(y_1) \cdots f_m(y_m)}{(|x-y_1| + \cdots + |x-y_m|)^{mn-\alpha}} dy_1 \cdots dy_m,$$

where $m \geq 2$ and $0 < \alpha < mn$.

Remark. Kenig and Stein [4] define $I_{m,\alpha}$ by

$$\int f_1(y_1) \cdots f_m(y_m) / (|x-y_1|^2 + \cdots + |x-y_m|^2)^{-(mn-\alpha)/2} dy_1 \cdots dy_m,$$

but we use our definition for simplicity of notation.

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We know the following two results.

Theorem (Kenig and Stein [4]). *Let*

$$\frac{1}{q} = \sum_{i=1}^m \frac{1}{p_i} - \frac{\alpha}{n} > 0.$$

If each $p_i > 1$, then $I_{m,\alpha}$ is bounded from $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. If $p_i = 1$ for some i , then $I_{m,\alpha}$ is bounded from $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n)$ to $L^{q,\infty}(\mathbb{R}^n)$.

Remark. A simple example shows that $I_{m,\alpha}$ is not bounded from $L^1 \times \cdots \times L^1$ to L^q where $1/q = m - \alpha/n$ (see the counterexample (19)).

Theorem (Tang [6]). *If $\sum_{i=1}^m 1/p_i = \alpha/n$, then $I_{m,\alpha}$ is bounded from $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n)$ to $BMO(\mathbb{R}^n)$.*

Note that $L^\infty \subsetneq BMO$.

In this paper we shall prove that $I_{m,\alpha}$ is bounded from $\prod_{i=1}^m L^{p_i}$ to L^q even if some p_i are equal to one under additional conditions on α . Furthermore we shall consider L^∞ estimates for $I_{m,\alpha}$.

2. THEOREMS

We state our results. In the following we always assume that $1 \leq p_1 \leq p_2 \leq \cdots \leq p_m \leq \infty$.

Theorem 1. *Let $p_1 = \cdots = p_k = 1$ and $1 < p_{k+1}, \dots, p_m < \infty$ for some $1 \leq k \leq m-1$, and*

$$\frac{1}{q} = k + \sum_{i=k+1}^m \frac{1}{p_i} - \frac{\alpha}{n} > 0.$$

Assume that

$$(1) \quad kn \leq \alpha < mn.$$

Then

$$\|I_{m,\alpha}(f_1, \dots, f_m)\|_q \leq C \prod_{i=1}^k \|f_i\|_1 \prod_{i=k+1}^m \|f_i\|_{p_i}.$$

Throughout this paper, C is a positive constant which is independent of essential parameters and not necessarily the same at each occurrence.

Remark. As we shall see in the proof, the main interest of the theorem is the case when $\alpha = kn$.

The following corollary is easily obtained from Theorem 1.

Corollary 1. *Let $p_1 = \cdots = p_k = 1$, $1 < p_{k+1}, \dots, p_{m-l} < \infty$ and $p_{m-l+1} = \cdots = p_m = \infty$ for some $0 \leq k < m-l$, and*

$$\frac{1}{q} = k + \sum_{i=k+1}^{m-l} \frac{1}{p_i} - \frac{\alpha}{n} > 0.$$

Assume that

$$(2) \quad kn \leq \alpha < (m-l)n.$$

Then

$$\|I_{m,\alpha}(f_1, \dots, f_m)\|_q \leq C \prod_{i=1}^k \|f_i\|_1 \prod_{i=k+1}^{m-l} \|f_i\|_{p_i} \prod_{i=m-l+1}^m \|f_i\|_\infty.$$

If $k = 0$, then we assume that $1 < p_i$ for all i . If $l = 0$, then we assume that $p_i < \infty$ for all i .

Remark. $I_{m,\alpha}$ is not bounded from $L^1 \times \dots \times L^1 \times L^\infty \times \dots \times L^\infty$ to L^q (see the counterexamples (20) and (21)).

When $q = \infty$ we obtain the following result.

Theorem 2. Let

$$\sum_{i=1}^m \frac{1}{p_i} = \frac{\alpha}{n} \quad \text{and} \quad 1 < p_i < \infty.$$

Assume that

$$(3) \quad n \leq \alpha < mn.$$

Then

$$\|I_{m,\alpha}(f_1, \dots, f_m)\|_\infty \leq C \prod_{i=1}^m \|f_i\|_{p_i}.$$

The following corollary is easily obtained from Theorem 2.

Corollary 2. Let $p_1 = \dots = p_k = 1$, $1 < p_{k+1}, \dots, p_{m-l} < \infty$ and $p_{m-l+1} = \dots = p_m = \infty$ for some $0 \leq k < m-l-1$, and

$$k + \sum_{i=k+1}^{m-l} \frac{1}{p_i} = \frac{\alpha}{n}.$$

Assume that

$$(4) \quad (k+1)n \leq \alpha < (m-l)n.$$

Then

$$\|I_{m,\alpha}(f_1, \dots, f_m)\|_\infty \leq C \prod_{i=1}^k \|f_i\|_1 \prod_{i=k+1}^{m-l} \|f_i\|_{p_i} \prod_{i=m-l+1}^m \|f_i\|_\infty.$$

If $k = 0$, then we assume that $1 < p_i$ for all i . If $l = 0$, then we assume that $p_i < \infty$ for all i .

Remark. $I_{m,\alpha}$ is not bounded from $L^1 \times \dots \times L^1 \times L^{p_i} \times L^\infty \times \dots \times L^\infty$ to L^∞ (see the counterexample (27)).

We shall show that the conditions (1) $\sim$ (4) are optimal by giving counterexamples in Section 4.

Proof of Corollary 1. We may assume that $f_i \geq 0$ for all i . Since

$$\begin{aligned} & \int_{(\mathbb{R}^n)^l} \frac{1}{(|x - y_{m-l+1}| + |x - y_{m-l+2}| + \dots + |x - y_m| + A)^{mn-\alpha}} dy_{m-l+1} \dots dy_m \\ & \leq \frac{C}{A^{mn-\alpha-ln}} \quad \text{where} \quad A > 0, \end{aligned}$$

we have

$$\begin{aligned} & I_{m,\alpha}(f_1, \dots, f_m)(x) \\ & \leq \prod_{i=m-l+1}^m \|f_i\|_\infty \int_{(\mathbb{R}^n)^{m-l}} \frac{f_1(y_1) \cdots f_{m-l}(y_{m-l})}{(|x - y_1| + \cdots + |x - y_{m-l}|)^{mn-\alpha-ln}} dy_1 \cdots dy_{m-l} \\ & \leq \prod_{i=m-l+1}^m \|f_i\|_\infty I_{m-l,\alpha}(f_1, \dots, f_{m-l})(x). \end{aligned}$$

Note that $kn \leq \alpha < (m-l)n$ and $1/q = k + \sum_{i=k+1}^{m-l} 1/p_i - \alpha/n$. By Theorem 1 or Theorem (Kenig and Stein) if $k = 0$, we have

$$\|I_{m-l,\alpha}(f_1, \dots, f_{m-l})\|_q \leq C \prod_{i=1}^k \|f_i\|_1 \prod_{i=k+1}^{m-l} \|f_i\|_{p_i}.$$

□

Proof of Corollary 2. We may assume that $f_i \geq 0$ for all i . Then

$$\begin{aligned} & I_{m,\alpha}(f_1, \dots, f_m)(x) \\ & \leq C \prod_{i=1}^k \|f_i\|_1 \prod_{i=m-l+1}^m \|f_i\|_\infty \\ & \quad \times \int_{(\mathbb{R}^n)^{m-k-l}} \frac{f_{k+1}(y_{k+1}) \cdots f_{m-l}(y_{m-l})}{(|x - y_{k+1}| + \cdots + |x - y_{m-l}|)^{mn-\alpha-ln}} dy_{k+1} \cdots dy_{m-l} \\ & \leq C \prod_{i=1}^k \|f_i\|_1 \prod_{i=m-l+1}^m \|f_i\|_\infty I_{m-k-l,\alpha-kn}(f_{k+1}, \dots, f_{m-l})(x). \end{aligned}$$

Since $n \leq \alpha - kn < (m-k-l)n$ and $\sum_{i=k+1}^{m-l} 1/p_i = (\alpha - kn)/n$, we have by Theorem 2

$$\|I_{m-k-l,\alpha-kn}(f_{k+1}, \dots, f_{m-l})\|_\infty \leq C \prod_{i=k+1}^{m-l} \|f_i\|_{p_i}.$$

□

3. PROOFS OF THEOREMS

This section is organized as follows. First we prove Theorem 1 where $kn < \alpha < mn$. This is easy. For Theorem 2 and Theorem 1 where $kn = \alpha$, we prove them only for $m = 2$ and $m = 3$, because when $m \geq 4$ we can prove them by induction on m . Finally we show outlines of proofs when $m \geq 4$.

In the following we assume that $f_i \geq 0$ for all i .

Proof of Theorem 1 where $\alpha > kn$. When $k < m - 1$, we have

$$\begin{aligned} & I_{m,\alpha}(f_1, \dots, f_m)(x) \\ & \leq \prod_{i=1}^k \|f_i\|_1 \int_{(\mathbb{R}^n)^{m-k}} \frac{f_{k+1}(y_{k+1}) \cdots f_m(y_m)}{(|x - y_{k+1}| + \cdots + |x - y_m|)^{mn-\alpha}} dy_{k+1} \cdots dy_m \\ & \leq \prod_{i=1}^k \|f_i\|_1 I_{m-k,\alpha-kn}(f_{k+1}, \dots, f_m)(x). \end{aligned}$$

Since $0 < \alpha - kn < (m - k)n$ and $1/q = \sum_{i=k+1}^m 1/p_i - (\alpha - kn)/n$, we have by Theorem (Kenig and Stein)

$$\|I_{m-k, \alpha-kn}(f_{k+1}, \dots, f_m)(x)\|_q \leq C \prod_{i=k+1}^m \|f_i\|_{p_i}.$$

When $k = m - 1$, we have $I_{m, \alpha}(f_1, \dots, f_m)(x) \leq \prod_{i=1}^{m-1} \|f_i\|_1 I_{\alpha-kn}(f_m)(x)$, where $I_{\alpha-kn}$ is the ordinary fractional integral operator. Since $0 < \alpha - kn < n$ and $1/q = 1/p_m - (\alpha - kn)/n$, we have by the boundedness of $I_{\alpha-kn}$,

$$\|I_{\alpha-kn}(f_m)\|_q \leq C \|f_m\|_{p_m}.$$

□

Now we consider the cases when $m = 2$ and $m = 3$. First we prove Theorem 2 and then we prove Theorem 1 where $kn = \alpha$ by using Theorem 2.

Theorem 2 ($m = 2$). *Let $1 < p_1, p_2 < \infty$ and $1/p_1 + 1/p_2 = \alpha/n$ where $n \leq \alpha < 2n$. Then the bilinear fractional integral operator $I_{2, \alpha}$ is bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$.*

Proof. This is essentially proved in [3, Theorem 340, p. 253] when $n = 1$. We give a proof for the sake of completeness. Let

$$(5) \quad J_{2, \alpha}(f_1, f_2) := \int_{(\mathbb{R}^n)^2} \frac{f_1(y_1) f_2(y_2)}{(|y_1| + |y_2|)^{2n - \alpha}} dy_1 dy_2.$$

It suffices to show that

$$J_{2, \alpha}(f_1, f_2) \leq C \|f_1\|_{p_1} \|f_2\|_{p_2}.$$

We write

$$J_{2, \alpha}(f_1, f_2) = \int_{|y_2| \leq |y_1|} + \int_{|y_2| \geq |y_1|} =: J^1 + J^2.$$

We estimate J^1 . J^2 can be estimated similarly. Since

$$(-2n + \alpha)p'_1 + np'_2 + n(p'_1 - p_2)/p'_2 = 0$$

and $p'_1 \geq p_2$, we have

$$\begin{aligned} J^1 &\leq \int_{\mathbb{R}^n} \frac{f_1(y_1)}{|y_1|^{2n - \alpha}} \left(\int_{|y_2| \leq |y_1|} f_2(y_2) dy_2 \right) dy_1 \\ &\leq \|f_1\|_{p_1} \left[\int_{\mathbb{R}^n} \frac{1}{|y_1|^{(2n - \alpha)p'_1}} \left(\int_{|y_2| \leq |y_1|} f_2(y_2) dy_2 \right)^{p'_1} dy_1 \right]^{1/p'_1} \\ &\leq \|f_1\|_{p_1} \left[\int_{\mathbb{R}^n} |y_1|^{(-2n + \alpha)p'_1 + np'_2} \left(\frac{1}{|y_1|^n} \int_{|y_2| \leq |y_1|} f_2(y_2) dy_2 \right)^{p_2} \right. \\ &\quad \left. \times \left(\int_{|y_2| \leq |y_1|} f_2(y_2) dy_2 \right)^{p'_1 - p_2} dy_1 \right]^{1/p'_1} \\ &\leq C \|f_1\|_{p_1} \|f_2\|_{p_2}^{(p'_1 - p_2)/p'_1} \left[\int_{\mathbb{R}^n} \left(\frac{1}{|y_1|^n} \int_{|y_2| \leq |y_1|} f_2(y_2) dy_2 \right)^{p_2} dy_1 \right]^{1/p'_1} \\ &\leq C \|f_1\|_{p_1} \|f_2\|_{p_2}^{(p'_1 - p_2)/p'_1} \left(\int_{\mathbb{R}^n} M f_2(y_1)^{p_2} dy_1 \right)^{1/p'_1} \leq C \|f_1\|_{p_1} \|f_2\|_{p_2}, \end{aligned}$$

where M is the Hardy–Littlewood maximal operator.

□

Theorem 2 ($m = 3$). *Let $1 < p_1 \leq p_2 \leq p_3 < \infty$, $1/p_1 + 1/p_2 + 1/p_3 = \alpha/n$ and $n \leq \alpha < 3n$. Then the trilinear fractional integral operator $I_{3,\alpha}$ is bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n) \times L^{p_3}(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$.*

Proof. Let

$$(6) \quad J_{3,\alpha}(f_1, f_2, f_3) := \int_{(\mathbb{R}^n)^3} \frac{f_1(y_1)f_2(y_2)f_3(y_3)}{(|y_1| + |y_2| + |y_3|)^{3n-\alpha}} dy_1 dy_2 dy_3.$$

Note that

$$\begin{cases} 1/p_1 + 1/p_2 = \alpha/n - 1/p_3, \\ 1/p_1 + 1/p_2 \geq 2/p_3, \end{cases} \quad \text{and} \quad \begin{cases} 1/p_2 + 1/p_3 = \alpha/n - 1/p_1, \\ 1/p_2 + 1/p_3 \leq 2/p_1. \end{cases}$$

By a simple calculation we have the following:

$$(7) \quad \text{If } \frac{3n}{2} \leq \alpha < 3n, \quad \text{then } \frac{1}{p_1} + \frac{1}{p_2} \geq 1.$$

$$(8) \quad \text{If } n \leq \alpha < \frac{3n}{2}, \quad \text{then } \frac{1}{p_2} + \frac{1}{p_3} < 1.$$

The case (7): $3n/2 \leq \alpha < 3n$.

By Hölder's inequality we have

$$\begin{aligned} J_{3,\alpha}(f_1, f_2, f_3) &\leq C \|f_3\|_{p_3} \int \frac{f_1(y_1)f_2(y_2)}{(|y_1| + |y_2|)^{3n-\alpha-n/p_3}} dy_1 dy_2 \\ &\leq C \|f_3\|_{p_3} J_{2,\alpha-n/p_3}(f_1, f_2); \end{aligned}$$

see the definition (5). Since $n \leq \alpha - n/p_3 < 2n$ and $1/p_1 + 1/p_2 = (\alpha - n/p_3)/n$, we can apply Theorem 2 ($m = 2$) and we obtain the desired result.

The case (8): $n \leq \alpha < 3n/2$.

We write

$$J_{3,\alpha}(f_1, f_2, f_3) = \int_{|y_3| \leq |y_2|} dy_1 dy_2 dy_3 + \int_{|y_3| \geq |y_2|} dy_1 dy_2 dy_3 =: J^1 + J^2.$$

It suffices to estimate J^1 . J^2 can be estimated similarly. We have

$$\begin{aligned} J^1 &\leq \int \frac{f_1(y_1)f_2(y_2)}{(|y_1| + |y_2|)^{2n-\alpha}} \left(\frac{1}{|y_2|^n} \int_{|y_3| \leq |y_2|} f_3(y_3) dy_3 \right) dy_1 dy_2 \\ &\leq C \int \frac{f_1(y_1)f_2(y_2)Mf_3(y_2)}{(|y_1| + |y_2|)^{2n-\alpha}} dy_1 dy_2 = CJ_{2,\alpha}(f_1, f_2 \cdot Mf_3). \end{aligned}$$

If $1/r = 1/p_2 + 1/p_3$, then $r > 1$ and $1/p_1 + 1/r = \alpha/n$. Therefore we can apply Theorem 2 ($m = 2$) and we obtain

$$J_{2,\alpha}(f_1, f_2 \cdot Mf_3) \leq C \|f_1\|_{p_1} \|f_2 \cdot Mf_3\|_r \leq C \prod_{i=1}^3 \|f_i\|_{p_i}.$$

□

We prove Theorem 1 for $\alpha = kn$ where $m = 2$ and $m = 3$.

Theorem 1 ($m = 2, k = 1$). *Let $1 < p_2 < \infty$. Then the bilinear fractional integral operator $I_{2,n}$ is bounded from $L^1(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^{p_2}(\mathbb{R}^n)$.*

Proof. Let $f_1 \in L^1$ and $f_2 \in L^{p_2}$. For any nonnegative function $g \in L^{p'_2}$,

$$\int g(x) I_{2,n}(f_1, f_2)(x) dx = \int f_1(y_1) dy_1 \int \frac{f_2(y_2) g(x)}{(|x - y_1| + |x - y_2|)^n} dy_2 dx.$$

Therefore it suffices to show that

$$\int \frac{f_2(y_2) g(x)}{(|x - y_1| + |x - y_2|)^n} dy_2 dx \leq C \|f_2\|_{p_2} \|g\|_{p'_2} \quad \text{for all } y_1 \in \mathbb{R}^n.$$

Note that

$$\int \frac{f_2(y_2) g(x)}{(|x - y_1| + |x - y_2|)^n} dy_2 dx = \int \frac{f_2(y_1 + y_2) g(x + y_1)}{(|x| + |x - y_2|)^n} dy_2 dx.$$

Therefore we need to show that

$$(9) \quad \int \frac{f_2(y_2) g(x)}{(|x| + |x - y_2|)^n} dy_2 dx \leq C \|f_2\|_{p_2} \|g\|_{p'_2}.$$

We have

$$\int \frac{f_2(y_2) g(x)}{(|x| + |x - y_2|)^n} dy_2 dx \leq C \int \frac{f_2(y_2) g(x)}{(|x| + |y_2|)^n} dy_2 dx = C J_{2,n}(f_2, g);$$

see (5) for the definition of $J_{2,n}$. Since $1/p_2 + 1/p'_2 = n/n$, we can apply Theorem 2 ($m = 2$) and we obtain

$$J_{2,n}(f_2, g) \leq C \|f_2\|_{p_2} \|g\|_{p'_2}.$$

□

Theorem 1 ($m = 3, k = 1$). *Let $1 < p_2, p_3 < \infty$ and $1/q = 1/p_2 + 1/p_3$. Then the trilinear fractional integral operator $I_{3,n}$ is bounded from $L^1(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n) \times L^{p_3}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.*

Proof. Note that q may be less than one. Therefore we cannot use the same argument as above. We write

$$\begin{aligned} I_{3,n}(f_1, f_2, f_3)(x) &= \int_{|y_2| \leq |y_3|} \frac{f_1(x - y_1) f_2(x - y_2) f_3(x - y_3)}{(|y_1| + |y_2| + |y_3|)^{2n}} dy_1 dy_2 dy_3 \\ &\quad + \int_{|y_2| \geq |y_3|} \frac{f_1(x - y_1) f_2(x - y_2) f_3(x - y_3)}{(|y_1| + |y_2| + |y_3|)^{2n}} dy_1 dy_2 dy_3 \\ &=: I^1(x) + I^2(x). \end{aligned}$$

It suffices to estimate I^1 ,

$$\begin{aligned} I^1(x) &\leq \int \left(\frac{1}{|y_3|^n} \int_{|y_2| \leq |y_3|} f_2(x - y_2) dy_2 \right) \frac{f_1(x - y_1) f_3(x - y_3)}{(|y_1| + |y_3|)^n} dy_1 dy_3 \\ &\leq C M f_2(x) \cdot I_{2,n}(f_1, f_3)(x). \end{aligned}$$

We have by Theorem 1 ($m = 2, k = 1$)

$$\|I_{2,n}(f_1, f_3)\|_{p_3} \leq C \|f_1\|_1 \|f_3\|_{p_3}.$$

Therefore we obtain

$$\|I^1\|_q \leq \|M f_2\|_{p_2} \|I_{2,n}(f_1, f_3)\|_{p_3} \leq C \|f_1\|_1 \|f_2\|_{p_2} \|f_3\|_{p_3}.$$

□

Theorem 1 ($m = 3, k = 2$). *Let $1 < p_3 < \infty$. Then the trilinear fractional integral operator $I_{3,2n}$ is bounded from $L^1(\mathbb{R}^n) \times L^1(\mathbb{R}^n) \times L^{p_3}(\mathbb{R}^n)$ to $L^{p_3}(\mathbb{R}^n)$.*

Proof. We use the same argument as in the proof of Theorem 1 ($m = 2, k = 1$). We will show that for any nonnegative function $g \in L^{p'_3}$,

$$\int \frac{f_2(y_2)f_3(y_3)g(x)}{(|x-y_1|+|x-y_2|+|x-y_3|)^n} dy_2 dy_3 dx \leq C \|f_2\|_1 \|f_3\|_{p_3} \|g\|_{p'_3} \quad \text{for all } y_1 \in \mathbb{R}^n.$$

It suffices to show the following:

$$\int \frac{f_2(y_2)f_3(y_3)g(x)}{(|x|+|x-y_2|+|x-y_3|)^n} dy_2 dy_3 dx \leq C \|f_2\|_1 \|f_3\|_{p_3} \|g\|_{p'_3}.$$

Therefore it is sufficient to show that

$$\int \frac{f_3(y_3)g(x)}{(|x|+|x-y_2|+|x-y_3|)^n} dy_3 dx \leq C \|f_3\|_{p_3} \|g\|_{p'_3} \quad \text{for all } y_2 \in \mathbb{R}^n.$$

We have

$$(10) \quad \int \frac{f_3(y_3)g(x)}{(|x|+|x-y_2|+|x-y_3|)^n} dy_3 dx \leq C \int \frac{f_3(y_3)g(x)}{(|y_2-y_3|+|y_2-x|)^n} dy_3 dx = C I_{2,n}(f_3, g)(y_2).$$

Since $1/p_3 + 1/p'_3 = n/n$, we can apply Theorem 2 ($m = 2$) and we obtain the desired result. $\square$

When $m \geq 4$, we repeat the same argument as above, therefore we show only outlines of proofs.

An outline of the proof of Theorem 2 for $m \geq 4$. Let

$$(11) \quad J_{m,\alpha}(f_1, \dots, f_m) := \int_{(\mathbb{R}^n)^m} \frac{f_1(y_1) \cdots f_m(y_m)}{(|y_1| + \cdots + |y_m|)^{mn-\alpha}} dy_1 \cdots dy_m.$$

It suffices to show that

$$(12) \quad J_{m,\alpha}(f_1, \dots, f_m) \leq C \prod_{i=1}^m \|f_i\|_{p_i} \quad \text{where } n \leq \alpha < mn.$$

We use induction on m . We assume that (12) holds for some $m \geq 3$. Let

$$\sum_{i=1}^{m+1} \frac{1}{p_i} = \frac{\alpha}{n} \quad \text{and} \quad n \leq \alpha < (m+1)n.$$

We will estimate $J_{m+1,\alpha}(f_1, \dots, f_{m+1})$. We have the following:

$$(13) \quad \text{If } \frac{(m+1)n}{m} \leq \alpha < (m+1)n, \quad \text{then } \frac{1}{p_1} + \cdots + \frac{1}{p_m} \geq 1.$$

$$(14) \quad \text{If } n \leq \alpha < \frac{(m+1)n}{m}, \quad \text{then } \frac{1}{p_m} + \frac{1}{p_{m+1}} < 1.$$

The case (13): $(m+1)n/m \leq \alpha < (m+1)n$.

By Hölder's inequality we have

$$J_{m+1,\alpha}(f_1, \dots, f_{m+1}) \leq C \|f_{m+1}\|_{p_{m+1}} J_{m,\alpha-n/p_{m+1}}(f_1, \dots, f_m).$$

Since $n \leq \alpha - n/p_{m+1} < mn$ and $1/p_1 + \cdots + 1/p_m = (\alpha - n/p_{m+1})/n$, we obtain the desired result by the hypothesis of induction.

The case (14): $n \leq \alpha < (m+1)n/m$.

We write

$$J_{m+1,\alpha}(f_1, \dots, f_{m+1}) = \int_{|y_{m+1}| \leq |y_m|} + \int_{|y_{m+1}| \geq |y_m|} =: J^1 + J^2.$$

It suffices to estimate J^1 . We have

$$J^1 \leq C J_{m,\alpha}(f_1, \dots, f_{m-1}, f_m \cdot M f_{m+1}).$$

If $1/r = 1/p_m + 1/p_{m+1}$, then $r > 1$. By the hypothesis of induction we obtain

$$J_{m,\alpha}(f_1, \dots, f_{m-1}, f_m \cdot M f_{m+1}) \leq C \prod_{i=1}^{m-1} \|f_i\|_{p_i} \|f_m \cdot M f_{m+1}\|_r \leq C \prod_{i=1}^{m+1} \|f_i\|_{p_i}.$$

□

An outline of the proof of Theorem 1 for $m \geq 4$ and $\alpha = kn$. First we prove the following:

$$(15) \quad \|I_{m,(m-1)n}(f_1, \dots, f_m)\|_q \leq C \prod_{i=1}^{m-1} \|f_i\|_1 \|f_m\|_{p_m} \quad \text{where } 1 < p_m < \infty.$$

Note that $q = p_m$, therefore $q > 1$. For any nonnegative function $g \in L^{q'}$, we estimate

$$\int I_{m,(m-1)n}(f_1, \dots, f_m)(x) g(x) dx.$$

The proof is reduced to the L^∞ estimate of $I_{m-1,(m-2)n}(f_3, \dots, f_m, g)(y_2)$ (see (10)). Applying Corollary 2 for the indices $\underbrace{1, \dots, 1}_{m-3}, p_m, q'$, we obtain the desired

result.

In general cases we use induction on m . Let $m \geq 3$ be fixed. We assume that for any $1 \leq k \leq m-1$,

$$\|I_{m,kn}(f_1, \dots, f_m)\|_q \leq C \prod_{i=1}^k \|f_i\|_1 \prod_{i=k+1}^m \|f_i\|_{p_i}$$

where $1/q = \sum_{i=k+1}^m 1/p_i$.

We will estimate $I_{m+1,kn}(f_1, \dots, f_{m+1})$ where $1/q = \sum_{i=k+1}^{m+1} 1/p_i$ and $1 \leq k \leq m$. By (15) it suffices to consider the cases when $1 \leq k \leq m-1$. Then $p_m, p_{m+1} > 1$. We write

$$\begin{aligned} I_{m+1,kn}(f_1, \dots, f_{m+1})(x) &= \int_{|y_m| \leq |y_{m+1}|} dy_1 \cdots dy_{m+1} \\ &+ \int_{|y_m| \geq |y_{m+1}|} dy_1 \cdots dy_{m+1} =: I^1(x) + I^2(x). \end{aligned}$$

We have

$$\begin{aligned} I^1(x) &\leq C M f_m(x) \cdot I_{m,kn}(f_1, \dots, f_{m-1}, f_{m+1})(x), \\ I^2(x) &\leq C M f_{m+1}(x) \cdot I_{m,kn}(f_1, \dots, f_m)(x). \end{aligned}$$

We obtain the desired result by Hölder's inequality and the hypothesis of induction.

□

4. COUNTEREXAMPLES

We show that the conditions (1) $\sim$ (4) are optimal by giving counterexamples. We consider the case when $n = 1$ and $m = 3$. Let

$$I_{3,\alpha}(f_1, f_2, f_3)(x) := \int_{(\mathbb{R}^1)^3} \frac{f_1(y_1)f_2(y_2)f_3(y_3)}{(|x - y_1| + |x - y_2| + |x - y_3|)^{3-\alpha}} dy_1 dy_2 dy_3,$$

where $0 < \alpha < 3$ and $1 < p_1 \leq p_2 \leq p_3 < \infty$.

Counterexamples for Theorem 1 and Corollary 1.

(16) If $0 < \alpha < 1$ and $1/q = 1 + 1/p_2 + 1/p_3 - \alpha$, then

$$I_{3,\alpha} : L^1 \times L^{p_2} \times L^{p_3} \not\rightarrow L^q.$$

(17) If $0 < \alpha < 2$, and $1/q = 2 + 1/p_3 - \alpha$, then

$$I_{3,\alpha} : L^1 \times L^1 \times L^{p_3} \not\rightarrow L^q.$$

(18) If $0 < \alpha < 1$ and $1/q = 1 + 1/p_2 - \alpha$, then

$$I_{3,\alpha} : L^1 \times L^{p_2} \times L^\infty \not\rightarrow L^q.$$

(19) If $1/q = 3 - \alpha$, then $I_{3,\alpha} : L^1 \times L^1 \times L^1 \not\rightarrow L^q$.

(20) If $1/q = 2 - \alpha$, then $I_{3,\alpha} : L^1 \times L^1 \times L^\infty \not\rightarrow L^q$.

(21) If $1/q = 1 - \alpha$, then $I_{3,\alpha} : L^1 \times L^\infty \times L^\infty \not\rightarrow L^q$.

Counterexamples for Theorem 2 and Corollary 2.

(22) If $0 < \alpha < 1$ and $1/p_1 + 1/p_2 + 1/p_3 = \alpha$, then

$$I_{3,\alpha} : L^{p_1} \times L^{p_2} \times L^{p_3} \not\rightarrow L^\infty.$$

(23) If $0 < \alpha < 2$ and $1 + 1/p_2 + 1/p_3 = \alpha$, then

$$I_{3,\alpha} : L^1 \times L^{p_2} \times L^{p_3} \not\rightarrow L^\infty.$$

(24) If $0 < \alpha < 1$ and $1/p_1 + 1/p_2 = \alpha$, then

$$I_{3,\alpha} : L^{p_1} \times L^{p_2} \times L^\infty \not\rightarrow L^\infty.$$

(25) If $1/p_1 = \alpha$, then $I_{3,\alpha} : L^{p_1} \times L^\infty \times L^\infty \not\rightarrow L^\infty$.

(26) If $2 + 1/p_3 = \alpha$, then $I_{3,\alpha} : L^1 \times L^1 \times L^{p_3} \not\rightarrow L^\infty$.

(27) If $1 + 1/p_2 = \alpha$, then $I_{3,\alpha} : L^1 \times L^{p_2} \times L^\infty \not\rightarrow L^\infty$.

Proof. (16). Let $\varepsilon = (q(1/p_2 + 1/p_3))^{-1}$. Since $\alpha < 1$ we have $\varepsilon > 1$. We define

$$f_1(x) = \chi_{(0,1)}(x) \text{ and } f_i(x) = x^{-1/p_i}(\log x)^{-\varepsilon/p_i} \chi_{\{x \geq 10\}}(x) \text{ for } i = 2, 3,$$

where χ_E is the characteristic function of a set E . Then $f_1 \in L^1, f_2 \in L^{p_2}$ and $f_3 \in L^{p_3}$. If $x \geq 10$, we have

$$\begin{aligned} I_{3,\alpha}(f_1, f_2, f_3)(x) &\geq C \int_{(\mathbb{R}^1)^2} \frac{f_2(y_2)f_3(y_3)dy_2dy_3}{(|x| + |x - y_2| + |x - y_3|)^{3-\alpha}} \\ &\geq C \int_{2x}^{3x} dy_2 \int_{2x}^{3x} \frac{f_2(y_2)f_3(y_3)}{(|x| + |x - y_2| + |x - y_3|)^{3-\alpha}} dy_3 \\ &\geq C x^{-1/q} (\log x)^{-1/q} \notin L^q. \end{aligned}$$

(17). Let

$$f_i(x) = \chi_{(0,1)}(x) \text{ for } i = 1, 2 \text{ and } f_3(x) = x^{-1/p_3}(\log x)^{-1/q} \chi_{\{x \geq 10\}}(x).$$

Since $p_3 > q$ we have $f_3 \in L^{p_3}$. If $x \geq 10$, we have

$$I_{3,\alpha}(f_1, f_2, f_3)(x) \geq C \int_{2x}^{3x} \frac{f_3(y_3) dy_3}{(|x| + |x - y_3|)^{3-\alpha}} \geq C x^{-1/q} (\log x)^{-1/q} \notin L^q.$$

(18). Let

$$f_1(x) = \chi_{(0,1)}(x), f_2(x) = x^{-1/p_2} (\log x)^{-1/q} \chi_{\{x \geq 10\}}(x) \text{ and } f_3 \equiv 1.$$

(19). Let $f_i(x) = \chi_{(0,1)}(x)$ for $i = 1, 2, 3$.

(20). Let $f_i(x) = \chi_{(0,1)}(x)$ for $i = 1, 2$ and $f_3 \equiv 1$.

(21). Let $f_1(x) = \chi_{(0,1)}(x)$ and $f_i \equiv 1$ for $i = 2, 3$.

(22). Let

$$f_i(x) = x^{-1/p_i} (\log x)^{-/\alpha p_i} \chi_{\{x \geq 10\}}(x) \text{ for } i = 1, 2, 3,$$

$$J_{3,\alpha}(f_1, f_2, f_3) \geq \int_{10}^{\infty} dy_1 \int_{y_1}^{2y_1} dy_2 \int_{y_1}^{2y_1} dy_3 \geq C \int_{10}^{\infty} \frac{dy_1}{y_1 \log y_1} = \infty.$$

For the definition of $J_{3,\alpha}$, see (6).

(23). Note that $1 < \alpha < 2$. Let $\varepsilon = (1/p_2 + 1/p_3)^{-1}$. Then $\varepsilon > 1$. Let

$$f_1(x) = \chi_{(0,1)}(x) \text{ and } f_i(x) = x^{-1/p_i} (\log x)^{-\varepsilon/p_i} \chi_{\{x \geq 10\}}(x) \text{ for } i = 2, 3,$$

$$J_{3,\alpha}(f_1, f_2, f_3) \geq C \int_{10}^{\infty} \frac{1}{y_3 \log y_3} dy_3 = \infty.$$

(24). Let

$$f_i(x) = x^{-1/p_i} (\log x)^{-/\alpha p_i} \chi_{\{x \geq 10\}}(x) \text{ for } i = 1, 2 \text{ and } f_3 \equiv 1.$$

(25). Let

$$f_1(x) = x^{-1/p_1} (\log x)^{-1} \chi_{\{x \geq 10\}} \text{ and } f_i \equiv 1 \text{ for } i = 2, 3.$$

(26). Let

$$f_i(x) = \chi_{(0,1)}(x) \text{ for } i = 1, 2 \text{ and } f_3(x) = x^{-1/p_3} (\log x)^{-1} \chi_{\{x \geq 10\}}(x),$$

$$J_{3,\alpha}(f_1, f_2, f_3) \geq C \int_{10}^{\infty} \frac{f_3(y_3) dy_3}{y_3^{3-\alpha}} = \infty.$$

(27). Note that $\alpha < 2$. Let

$$f_1(x) = \chi_{(0,1)}(x), f_2(x) = x^{-1/p_2} (\log x)^{-1} \chi_{\{x \geq 10\}}(x) \text{ and } f_3 \equiv 1,$$

$$J_{3,\alpha}(f_1, f_2, f_3) \geq C \int_{10}^{\infty} \frac{f_2(y_2) dy_2}{y_2^{2-\alpha}} = \infty.$$

□

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DEPARTMENT OF MATHEMATICS, SCHOOL OF SCIENCE, TOKAI UNIVERSITY, HIRATSUKA, KANAGAWA 259-1299 JAPAN

E-mail address: komori@tokai-u.jp