ON COMPLETE METRIZABILITY OF THE HAUSDORFF METRIC TOPOLOGY

LÁSZLÓ ZSILINSZKY

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ABSTRACT. It is shown that there exists a nonseparable completely metrizable bounded metric space (X, d) such that the hyperspace $CL(X)$ of the nonempty closed subsets of X endowed with the Hausdorff metric H_d is not completely metrizable.

1. INTRODUCTION

The main purpose of the paper is to demonstrate and clarify the differences between the hyperspace of a nonseparable versus a separable metric space regarding complete metrizability of the Hausdorff metric hypertopology. The *Hausdorff metric topology* τ_{H_d} on the hyperspace $CL(X)$ of nonempty closed subsets of a given metric space (X, d) is one of the oldest and best-studied hypertopologies due to its applicability to various areas of mathematics [1, 2, 4, 20]. Various completeness-type properties of the Hausdorff metric topology are stock theorems in topology; e.g. $(CL(X), \tau_{H_d})$ is compact (resp. totally bounded) iff X is. More recently, local compactness [12] and cofinal completeness [6] have been characterized for $(CL(X), \tau_{H_d})$, but a characterization of complete metrizability of $(CL(X), \tau_{H_d})$ is unknown. There are partial results, however; indeed, the main reason for the interest in the Hausdorff metric is the following useful fact: *if (X, d) is a bounded complete metric space, then $(CL(X), H_d)$ is a complete metric space*, where

$$(1.1) \quad H_d(A_0, A_1) = \sup\{|d(x, A_0) - d(x, A_1)| : x \in X\}, \text{ for } A_0, A_1 \in CL(X),$$

and $d(x, A) = \inf\{d(x, a) : a \in A\}$ is the distance from $x \in X$ to $A \in CL(X)$. If d is not bounded, H_d is only an infinite-valued distance, which generates the topology τ_{H_d} on $CL(X)$. Moreover, $d' = \min\{1, d\}$ is an equivalent to d , bounded metric on X and $\tau_{H_{d'}} = \tau_{H_d}$, so we get

Theorem 1.1. *If (X, d) is complete, then $(CL(X), \tau_{H_d})$ is completely metrizable.*

It is the purpose of this note to show that the previous theorem does not extend to a completely metrizable (X, d) , contrary to some claims in the literature [3]. In the main results, proved in Section 3, we use topological games—the so-called strong Choquet game and the Banach-Mazur game, reviewed in Section 2—to show that a nonseparable bounded completely metrizable metric space (X, d) exists for which the corresponding Hausdorff metric topology is not completely metrizable. More

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generally, $(CL(X), \tau_{H_d})$ lacks any closed-hereditary completeness property, since it contains a closed copy of the rationals; however, the hyperspace still contains a dense completely metrizable subspace, and thus is a Baire space.

To put this result in perspective, recall that it was Effros [16, Lemma] who showed that for $(CL(X), \tau_{H_d})$ to be Polish (i.e. completely metrizable and separable), it is sufficient that (X, d) is completely metrizable and totally bounded, which is in turn also necessary, since separability of $(CL(X), \tau_{H_d})$ is equivalent to total boundedness of X [4, Theorem 3.2.3.], and X sits in $(CL(X), \tau_{H_d})$ as a closed subspace. It is possible to improve on this result using the work of Costantini [10] about another related hyperspace topology, the so-called *Wijsman topology* τ_{W_d} : to explain, it is useful to view $CL(X)$ as sitting in the space $C(X)$ of real-valued continuous functions defined on X via the identification $A \leftrightarrow d(\cdot, A)$, since, by (1.1), $(CL(X), \tau_{H_d})$ is then a subspace of $C(X)$ with the uniform topology, while $(CL(X), \tau_{W_d})$ is a subspace of $C(X)$ with the topology of pointwise convergence. This immediately implies that $\tau_{W_d} \subseteq \tau_{H_d}$, which yields

Theorem 1.2. *If (X, d) is Polish, then $(CL(X), \tau_{H_d})$ is completely metrizable.*

Proof. If $(\tilde{X}, \tilde{d})$ denotes the completion of (X, d) , it follows from [10] that the set $\{\text{cl}_{\tilde{X}}(A) : A \in CL(X)\}$ is G_δ in $(CL(\tilde{X}), \tau_{W_{\tilde{d}}})$; thus, it is G_δ in $(CL(\tilde{X}), \tau_{H_{\tilde{d}}})$ as well. Therefore, by Theorem 1.1, $(CL(X), \tau_{H_d})$ is completely metrizable, since mapping $A \in CL(X)$ onto the $\tilde{X}$ -closure of A is an isometric embedding of $(CL(X), H_d)$ into $(CL(\tilde{X}), H_{\tilde{d}})$ [16]. $\square$

The Hausdorff distance is sensitive to its generating metric, because $\tau_{H_d} = \tau_{H_{d'}}$ iff d, d' are uniformly equivalent metrics on X [4, Theorem 3.3.2.]. Thus, it is not automatic to argue that *complete metrizability* of (X, d) is sufficient for complete metrizability of $(CL(X), \tau_{H_d})$ even though it is clearly necessary. Knowing that $\tau_{W_d} = \tau_{H_d}$ on $CL(X)$ iff (X, d) is totally bounded [4], it is not surprising that the Hausdorff metric and Wijsman topologies interact in studying complete metrizability of the hyperspaces. However, when a totally bounded metric on X is not available, i.e. when X is not separable, the two topologies have no effect on each other, and so the wealth of completeness results on the Wijsman topology [5, 8, 11, 13, 27] is not applicable.

2. PRELIMINARIES

Given a metric space (X, d) , $A \in CL(X)$ and $\varepsilon > 0$, denote by

$$B_d(A, \varepsilon) = \{x \in X : d(x, A) < \varepsilon\}$$

the open ε -hull of A , and use $B_d(x, \varepsilon)$ instead of $B_d(\{x\}, \varepsilon)$ for the open ε -ball about x . In addition to (1.1), there is an equivalent definition for the Hausdorff distance:

$$H_d(A_0, A_1) = \inf\{\varepsilon > 0 : A_0 \subseteq B_d(A_1, \varepsilon) \text{ and } A_1 \subseteq B_d(A_0, \varepsilon)\}$$

whenever $A_0, A_1 \in CL(X)$ [4, 17].

In the *strong Choquet game* $Ch(Z)$ [9, 19] players α and β take turns in choosing objects in the topological space Z with an open base $\mathcal{B}$: β starts by picking (z_0, V_0) from $\mathcal{E} = \{(z, V) \in Z \times \mathcal{B} : z \in V\}$, and α responds by $U_0 \in \mathcal{B}$ with $z_0 \in U_0 \subseteq V_0$. The next choice of β is $(z_1, V_1) \in \mathcal{E}$ with $V_1 \subseteq U_0$, and again α picks U_1 with $z_1 \in U_1 \subseteq V_1$, etc. Player α wins the run $(z_0, V_0), U_0, \dots, (z_n, V_n), U_n, \dots$ provided

$\bigcap_n U_n = \bigcap_n V_n \neq \emptyset$; otherwise, β wins. A *strategy* in $Ch(Z)$ for α (resp. β) is a function $\sigma : \mathcal{E}^{<\omega} \rightarrow \mathcal{B}$ (resp. $\sigma : \mathcal{B}^{<\omega} \rightarrow \mathcal{E}$) such that

$$\begin{aligned} z_n \in \sigma((z_0, V_0), \dots, (z_n, V_n)) &\subseteq V_n \text{ for all } ((z_0, V_0), \dots, (z_n, V_n)) \in \mathcal{E}^{<\omega} \\ (\text{resp. } \sigma(\emptyset) = (z_0, V_0) \text{ and } V_n &\subseteq U_{n-1}, \text{ where } \sigma(U_0, \dots, U_{n-1}) = (z_n, V_n) \\ &\text{for all } (U_0, \dots, U_{n-1}) \in \mathcal{B}^n, n \geq 1). \end{aligned}$$

A strategy σ for α (resp. β) is a *winning strategy* if α (resp. β) wins every run of $Ch(Z)$ compatible with σ , i.e. such that $\sigma(z_0, V_0), \dots, (z_n, V_n) = U_n$ for all $n < \omega$ (resp. $\sigma(\emptyset) = (z_0, V_0)$ and $\sigma(U_0, \dots, U_{n-1}) = (z_n, V_n)$ for all $n \geq 1$). The strong Choquet game $Ch(Z)$ is α -, β -favorable, respectively, provided α , resp. β , has a winning strategy in $Ch(Z)$. This game has been studied in general topological spaces [7, 14, 15, 22, 28]; however, the two fundamental results about it concern metrizable ones:

- **Choquet** [9, 19]. A metrizable space X is completely metrizable if and only if $Ch(X)$ is α -favorable.
- **Debs-Porada-Telgársky** [14, 24, 25]. A metrizable space X contains a closed copy of the rationals if and only if $Ch(X)$ is β -favorable.

The *Banach-Mazur game* $BM(Z)$ (see [18], also referred to as the Choquet game [19]) is played as the strong Choquet game, except β 's choice is only a nonempty open set contained in the previous choice of α . The notions of α -, β -favorability of $BM(Z)$ are defined analogously to those of $Ch(Z)$. Two key results about the Banach-Mazur game are as follows:

- **Oxtoby** [23, 26]. A metrizable space X contains a dense completely metrizable subspace if and only if $BM(X)$ is α -favorable.
- **Oxtoby-Krom** [18, 19, 23]. A topological space X is a Baire space (i.e. countable intersections of dense open subsets of X are dense) if and only if $BM(X)$ is not β -favorable.

3. MAIN RESULTS

Our main result is as follows:

Theorem 3.1. *There exists a nonseparable bounded metric space (X, d) such that*

- (1) *X is completely metrizable;*
- (2) *$(CL(X), H_d)$ contains a closed copy of the rationals;*
in particular, $(CL(X), H_d)$ is not completely metrizable;
- (3) *$(CL(X), H_d)$ is α -favorable in the Banach-Mazur game;*
in particular, $(CL(X), H_d)$ is a Baire space.

Proof. (1) Consider the product space $\mathbb{R}^\omega$, where $\mathbb{R}$ has the discrete topology. This topology is metrizable by the Baire metric [17, Example 4.2.12(6)]

$$d(x, y) = \frac{1}{\min\{n + 1 : x(n) \neq y(n)\}}$$

for $x, y \in \mathbb{R}^\omega$. Denote

$$F = \{x \in \mathbb{R}^\omega : x(0) \neq 0 \text{ and } x(k) = 0 \text{ for all } k > 0\},$$

and put $X = \mathbb{R}^\omega \setminus F$. It is clear that F is closed in $\mathbb{R}^\omega$, so X is an open subspace of the complete space $(\mathbb{R}^\omega, d)$, and hence (X, d) is completely metrizable.

(2) By the Debs-Porada-Telgársky Theorem, we need to show that $(CL(X), H_d)$ is β -favorable in the strong Choquet game: let $\{I_n^0 \subset \mathbb{R} \setminus \{0\} : n < \omega\}$ be a sequence of pairwise disjoint closed bounded intervals, and denote by I_0 their union. For each $t \in I_0$ define $x_t^0 \in X$ via

$$x_t^0(k) = \begin{cases} t, & \text{if } t \in I_n^0, \ k = 0 \text{ or } k > n + 1, \\ 0, & \text{if } t \in I_n^0, \ 1 \leq k \leq n + 1. \end{cases}$$

Define $A_0 = \{x_t^0 : t \in I_0\} \in CL(X)$, $\mathbf{V}_0 = B_{H_d}(A_0, 1)$, and let $(A_0, \mathbf{V}_0)$ be β 's initial step in $Ch(CL(X), H_d)$. Assume that $\mathbf{U}_0 = B_{H_d}(A_0, \frac{1}{n_0})$ is α 's response, where $1 \leq n_0 < \omega$.

Let $\{I_n^1 \subset I_{n_0+1}^0 : n < \omega\}$ be a sequence of pairwise disjoint closed bounded intervals with union I_1 . For each $t \in I_0 \setminus I_1$ define $x_t^1 = x_t^0$, and for $t \in I_1$ put

$$x_t^1(k) = \begin{cases} x_t^0(k), & \text{if } t \in I_n^1, \ k \leq n_0, \\ 0, & \text{if } t \in I_n^1, \ n_0 < k \leq n_0 + n + 1, \\ t, & \text{if } t \in I_n^1, \ k > n_0 + n + 1. \end{cases}$$

Define $A_1 = \{x_t^1 : t \in I_0\} \in CL(X)$ and $\mathbf{V}_1 = B_{H_d}(A_1, \frac{1}{1+n_0})$, and let $(A_1, \mathbf{V}_1)$ be β 's next step in $Ch(CL(X), H_d)$.

Assume we have defined a partial run $(A_0, \mathbf{V}_0), \mathbf{U}_0, \dots, (A_m, \mathbf{V}_m), \mathbf{U}_m$ of the strong Choquet game in $(CL(X), H_d)$ for some $m < \omega$, where

$$\mathbf{U}_i = B_{H_d}\left(A_i, \frac{1}{\sum_{j \leq i} n_j}\right)$$

for some $1 \leq n_i < \omega$ whenever $i \leq m$. Moreover, for each $1 \leq i \leq m$ a sequence $\{I_n^i \subset I_{n_{i-1}+1}^{i-1} : n < \omega\}$ of pairwise disjoint closed bounded intervals with union I_i is chosen, as well as $x_t^i \in X$ for each $t \in I_0$ so that $x_t^i = x_t^{i-1}$ whenever $t \in I_0 \setminus I_i$, and for $t \in I_i$,

$$x_t^i(k) = \begin{cases} x_t^{i-1}(k), & \text{if } t \in I_n^i, \ k \leq \sum_{j < i} n_j, \\ 0, & \text{if } t \in I_n^i, \ \sum_{j < i} n_j < k \leq 1 + n + \sum_{j < i} n_j, \\ t, & \text{if } t \in I_n^i, \ k > 1 + n + \sum_{j < i} n_j. \end{cases}$$

Then $A_i = \{x_t^i : t \in I_0\}$ and $\mathbf{V}_i = B_{H_d}(A_i, \frac{1}{1+\sum_{j < i} n_j})$.

Choose a sequence of pairwise disjoint closed bounded intervals $\{I_n^{m+1} \subset I_{n_m+1}^m : n < \omega\}$ with union I_{m+1} , define $x_t^{m+1} = x_t^m$ for each $t \in I_0 \setminus I_{m+1}$, and for $t \in I_{m+1}$ put

$$x_t^{m+1}(k) = \begin{cases} x_t^m(k), & \text{if } t \in I_n^{m+1}, \ k \leq \sum_{i \leq m} n_i, \\ 0, & \text{if } t \in I_n^{m+1}, \ \sum_{i \leq m} n_i < k \leq 1 + n + \sum_{i \leq m} n_i, \\ t, & \text{if } t \in I_n^{m+1}, \ k > 1 + n + \sum_{i \leq m} n_i. \end{cases}$$

Define $A_{m+1} = \{x_t^{m+1} : t \in I_0\}$ and $\mathbf{V}_{m+1} = B_{H_d}(A_{m+1}, \frac{1}{1+\sum_{i \leq m} n_i})$.

Claim 3.1.1. $\mathbf{V}_{m+1} \subseteq \mathbf{U}_m$.

Indeed, if $A \in \mathbf{V}_{m+1}$, then $A \subseteq B_d(A_{m+1}, \frac{1}{1 + \sum_{i \leq m} n_i})$, so for all $a \in A$ there is some $x_t^{m+1} \in A_{m+1}$ with $d(a, x_t^{m+1}) < \frac{1}{1 + \sum_{i \leq m} n_i}$, which implies that

$$(3.1) \quad a(k) = x_t^{m+1}(k) \text{ for all } k \leq \sum_{i \leq m} n_i.$$

If $t \in I_0 \setminus I_{m+1}$, then

$$d(a, x_t^m) = d(a, x_t^{m+1}) < \frac{1}{1 + \sum_{i \leq m} n_i} < \frac{1}{\sum_{i \leq m} n_i};$$

if $t \in I_{m+1}$, then $t \in I_n^{m+1}$ for some $n < \omega$. It follows from the definition of x_t^{m+1} and (3.1) that

$$d(a, x_t^m) \leq \frac{1}{1 + \sum_{i \leq m} n_i} < \frac{1}{\sum_{i \leq m} n_i},$$

so we have $A \subseteq B_d(A_m, \frac{1}{\sum_{i \leq m} n_i})$. A similar argument shows that

$$A_{m+1} \subseteq B_d\left(A, \frac{1}{1 + \sum_{i \leq m} n_i}\right) \text{ implies that } A_m \subseteq B_d\left(A, \frac{1}{\sum_{i \leq m} n_i}\right),$$

thus $A \in \mathbf{U}_m$. As a consequence of Claim 3.1.1, we have that putting

$$\sigma_{Ch}(\emptyset) = (A_0, \mathbf{V}_0) \text{ and } \sigma_{Ch}(\mathbf{U}_0, \dots, \mathbf{U}_m) = (A_{m+1}, \mathbf{V}_{m+1}) \text{ whenever } m < \omega$$

defines a strategy for player β in the strong Choquet game on $(CL(X), H_d)$. We will be done if we prove

Claim 3.1.2. σ_{Ch} is a winning strategy for β in $Ch(CL(X), H_d)$.

To show this, consider a run

$$(A_0, \mathbf{V}_0), \mathbf{U}_0, \dots, (A_m, \mathbf{V}_m), \mathbf{U}_m, \dots$$

of $Ch(CL(X), H_d)$ compatible with σ_{Ch} , and assume $A \in \bigcap_{m < \omega} \mathbf{V}_m$. If we choose some $t \in \bigcap_{m < \omega} I_{n_m+1}^m$, note that for every $m < \omega$,

$$(3.2) \quad x_t^m(k) = 0 \text{ for all } 0 < k \leq 1 + \sum_{i \leq m} n_i.$$

Since $A \in \mathbf{V}_0$, there is some $a \in A$ with $d(x_t^0, a) < 1$, thus

$$(3.3) \quad a(0) = x_t^0(0) = t.$$

Since $a \in X$, there exists $0 < k$ so that

$$(3.4) \quad a(k) \neq 0.$$

Choose $m < \omega$ so that $k \leq 1 + \sum_{i \leq m} n_i$. Since $A \in \mathbf{V}_m$, there exists an $x_{t'}^m \in A_m$

with $d(a, x_{t'}^m) < \frac{1}{1 + \sum_{i \leq m} n_i}$, which implies that

$$(3.5) \quad x_{t'}^m(0) = a(0) \text{ and}$$

$$(3.6) \quad x_{t'}^m(k) = a(k).$$

Using (3.3), (3.5) we get

$$t' = x_{t'}^m(0) = a(0) = x_t^0(0) = t,$$

so $t' = t$. This would yield, by (3.6), (3.2), that

$$a(k) = x_{t'}^m(k) = x_t^m(k) = 0,$$

which contradicts (3.4). In conclusion, $\bigcap_{m < \omega} \mathbf{V}_m = \emptyset$, which means that β wins.

(3) Let $\mathbf{V}_0$ be β 's initial step in $BM(CL(X), H_d)$, where $\mathbf{V}_0 = B_{H_d}(A_0, \frac{1}{n_0})$ for some $A_0 \in CL(X)$ and $n_0 \geq 1$. For each $a_0 \in A_0$ define $x_{a_0} \in X$ via

$$x_{a_0}(k) = \begin{cases} a_0(k), & \text{if } k < 2n_0 - 1, \\ n_0 + 1, & \text{if } k \geq 2n_0 - 1, \end{cases}$$

and put $C_0 = \{x_{a_0} : a_0 \in A_0\}$. Then $C_0 \in CL(X)$, and $H_d(A_0, C_0) \leq \frac{1}{2n_0}$. Define $\mathbf{U}_0 = B_{H_d}(C_0, \frac{1}{2n_0})$. Then $\mathbf{U}_0 \subseteq \mathbf{V}_0$ (since if $A \in \mathbf{U}_0$, then $H_d(A, C_0) < \frac{1}{2n_0}$, so $H_d(A_0, A) \leq H_d(A, C_0) + H_d(C_0, A_0) < \frac{1}{n_0}$), so we can take $\mathbf{U}_0$ as α 's first step in $BM(CL(X), H_d)$. Let $\mathbf{V}_1 = B_{H_d}(A_1, \frac{1}{n_1}) \subseteq \mathbf{U}_0$ be β 's next step, where $A_1 \in CL(X)$ and $n_1 \geq 2n_0$. Then $H_d(A_1, C_0) < \frac{1}{2n_0}$, so for all $a_0 \in A_0$ there exists $a_1 \in A_1$ such that $d(a_1, x_{a_0}) < \frac{1}{2n_0}$, thus

$$(3.7) \quad a_1(k) = \begin{cases} a_0(k), & \text{if } k < 2n_0 - 1, \\ n_0 + 1, & \text{if } k = 2n_0 - 1. \end{cases}$$

For each $a_1 \in A_1$ define $x_{a_1} \in X$ via

$$x_{a_1}(k) = \begin{cases} a_1(k), & \text{if } k < 2n_1 - 1, \\ 1 + n_0 + n_1, & \text{if } k \geq 2n_1 - 1, \end{cases}$$

and put $C_0 = \{x_{a_1} : a_1 \in A_1\}$. Then $C_1 \in CL(X)$, and $H_d(A_1, C_1) \leq \frac{1}{2n_1}$, so $\mathbf{U}_1 = B_{H_d}(C_1, \frac{1}{2n_1}) \subseteq \mathbf{V}_1$.

Assume we have defined a partial run $\mathbf{V}_0, \mathbf{U}_0, \dots, \mathbf{V}_m, \mathbf{U}_m$ of the Banach-Mazur game in $(CL(X), H_d)$, where

$$\mathbf{V}_i = B_{H_d}\left(A_i, \frac{1}{n_i}\right) \text{ and } \mathbf{U}_i = B_{H_d}\left(C_i, \frac{1}{2n_i}\right)$$

for some $2n_{i-1} \leq n_i < \omega$ whenever $i \leq m$ (for convenience, define $n_{-1} = \frac{1}{2}$). Moreover, for each $i \leq m$ let $C_i = \{x_{a_i} : a_i \in A_i\}$, where

$$(3.8) \quad x_{a_i}(k) = \begin{cases} a_i(k), & \text{if } k < 2n_i - 1, \\ 1 + \sum_{j \leq i} n_j, & \text{if } k \geq 2n_i - 1. \end{cases}$$

Take $\mathbf{V}_{m+1} = B_{H_d}(A_{m+1}, \frac{1}{n_{m+1}}) \subseteq \mathbf{U}_m$. For any $a_{m+1} \in A_{m+1}$ define

$$y_{a_{m+1}}(k) = \begin{cases} a_{m+1}(k), & \text{if } k < n_{m+1}, \\ 2 + \sum_{i \leq m} n_i, & \text{if } k \geq n_{m+1}. \end{cases}$$

Then $\{y_{a_{m+1}} : a_{m+1} \in A_{m+1}\} \in \mathbf{V}_{m+1} \subseteq \mathbf{U}_m$, so there exists $x_{a_m} \in C_m$ for some $a_m \in A_m$ so that $d(y_{a_{m+1}}, x_{a_m}) < \frac{1}{2n_m}$. If $n_{m+1} < 2n_m$, then

$$y_{a_{m+1}}(2n_m - 1) = 2 + \sum_{i \leq m} n_i \quad \text{and} \\ x_{a_m}(2n_m - 1) = 1 + \sum_{i \leq m} n_i,$$

so $d(y_{a_{m+1}}, x_{a_m}) \geq \frac{1}{2n_m}$, which is impossible, thus $n_{m+1} \geq 2n_m$. It also follows from $\mathbf{V}_{m+1} \subseteq \mathbf{U}_m$ that $H_d(A_{m+1}, C_m) < \frac{1}{2n_m}$. Hence, for each $a_m \in A_m$ there exists $a_{m+1} \in A_{m+1}$ with $d(x_{a_m}, a_{m+1}) < \frac{1}{2n_m}$, so

$$(3.9) \quad a_{m+1}(k) = \begin{cases} a_m(k), & \text{if } k < 2n_m - 1, \\ 1 + \sum_{i \leq m} n_i, & \text{if } k = 2n_m - 1. \end{cases}$$

Define $C_{m+1} = \{x_{a_{m+1}} : a_{m+1} \in A_{m+1}\}$, where

$$(3.10) \quad x_{a_{m+1}}(k) = \begin{cases} a_{m+1}(k), & \text{if } k < 2n_{m+1} - 1, \\ 1 + \sum_{i \leq m+1} n_i, & \text{if } k \geq 2n_{m+1} - 1, \end{cases}$$

and put $\mathbf{U}_{m+1} = B_{H_d}(C_{m+1}, \frac{1}{2n_{m+1}})$. Note that $H_d(A_{m+1}, C_{m+1}) \leq \frac{1}{2n_{m+1}}$, so $\mathbf{U}_{m+1} \subseteq \mathbf{V}_{m+1}$, since if $A \in \mathbf{U}_{m+1}$, then $H_d(C_{m+1}, A) < \frac{1}{2n_{m+1}}$, thus, $H_d(A_{m+1}, A) \leq H_d(A_{m+1}, C_{m+1}) + H_d(C_{m+1}, A) < \frac{1}{n_{m+1}}$. This means that putting

$$\sigma_{BM}(\mathbf{V}_0, \dots, \mathbf{V}_m) = \mathbf{U}_m \text{ for all } m < \omega$$

defines a strategy for α in $BM(CL(X), H_d)$.

Claim 3.1.3. σ_{BM} is a winning strategy for α in $BM(CL(X), H_d)$.

To show this, consider a run $\mathbf{V}_0, \mathbf{U}_0, \dots, \mathbf{V}_m, \mathbf{U}_m, \dots$ of the Banach-Mazur game in $(CL(X), H_d)$ compatible with σ_{BM} . For any $m < \omega$ and $a_m \in A_m$ we get an $a_{m+1} \in A_{m+1}$ satisfying (3.9). For any $a_0 \in A_0$, define

$$A_1[a_0] = \{a_1 \in A_1 : a_1(k) = a_0(k) \text{ for all } k < 2n_0 - 1\},$$

which is nonempty by (3.7). Assume by induction that we have defined $A_m[a_{m-1}] \neq \emptyset$ for some $a_{m-1} \in A_{m-1}$ and $m \geq 1$. For every $a_m \in A_m[a_{m-1}]$ put

$$A_{m+1}[a_m] = \{a_{m+1} \in A_{m+1} : a_{m+1}(k) = a_m(k) \text{ for all } k < 2n_m - 1\},$$

which is nonempty by (3.9); for convenience, also define $A_0[a_{-1}] = A_0$. Denote

$$P = \{(a_m)_{m \geq 0} : a_m \in A_m[a_{m-1}] \text{ for all } m \geq 0\},$$

and for any $p = (a_m)_{m \geq 0} \in P$ define s_p as follows:

$$(3.11) \quad s_p(k) = \begin{cases} a_0(k), & \text{if } k < 2n_0 - 1, \\ a_m(k), & \text{if } 2n_{m-1} - 1 \leq k < 2n_m - 1, \ m \geq 1. \end{cases}$$

Note, by (3.9), that $s_p(2n_m - 1) = 1 + \sum_{i \leq m} n_i$ for every $m < \omega$, so $s_p \in X$ for each $p \in P$. Denote by S the X -closure of the set $\{s_p : p \in P\}$.

Given any $s_p \in S$, we have a sequence $p = (a_m)_{m \geq 0} \in P$ such that $a_i(k) = a_{i-1}(k)$ for all $1 \leq i \leq m$ and $k < 2n_i - 1$, which implies by (3.11) that $a_m(k) = s_p(k)$ for all $k < 2n_m - 1$.

It follows that $d(s_p, A_m) \leq d(s_p, a_m) \leq \frac{1}{2n_m}$, so $d(s, A_m) \leq \frac{1}{2n_m} < \frac{1}{n_m}$ for each $s \in S$, thus

$$(3.12) \quad S \subseteq B_{H_d} \left(A_m, \frac{1}{n_m} \right).$$

Furthermore, for each $1 \leq i \leq m$, $A_i \in \mathbf{V}_i \subseteq \mathbf{U}_{i-1}$, so for any $a_i \in A_i$ there exists $a_{i-1} \in A_{i-1}$ with $d(x_{a_{i-1}}, a_i) < \frac{1}{2n_{i-1}}$, which means that $a_i(k) = x_{a_{i-1}}(k)$ for each $k \leq 2n_{i-1} - 1$, so by (3.8),

$$(3.13) \quad a_i(k) = a_{i-1}(k) \text{ for each } k < 2n_{i-1} - 1;$$

moreover, if $i > m$ we can choose, by (3.9), $a_i \in A_i$ so that (3.13) is satisfied. It follows that $a_i \in A_i[a_{i-1}]$ for all $1 \leq i$; thus $p = (a_i)_{i \geq 0} \in P$ and $s_p(k) = a_m(k)$ for all $k < 2n_m - 1$.

This implies that $d(a_m, S) \leq d(a_m, s_p) \leq \frac{1}{2n_m} < \frac{1}{n_m}$, so

$$(3.14) \quad A_m \subseteq B_{H_d} \left(S, \frac{1}{n_m} \right).$$

In conclusion, by (3.12), (3.14) we have that $H_d(A_m, S) < \frac{1}{n_m}$; thus $S \in \mathbf{V}_m$, which implies that $S \in \bigcap_{m < \omega} \mathbf{V}_m$, and so α wins. $\square$

Corollary 3.2. *There exists a completely metrizable bounded metric space X with compatible metrics d, d' so that $Ch(CL(X), H_d)$ is α -favorable and $Ch(CL(X), H_{d'})$ is β -favorable.*

REFERENCES

- [1] Jean-Pierre Aubin, *Applied abstract analysis*, Exercises by Bernard Cornet and Hervé Moulin, translated from the French by Carole Labrousse, Pure and Applied Mathematics, Wiley-Interscience [John Wiley & Sons], New York-London-Sydney, 1977. MR470034
- [2] C. Castaing and M. Valadier, *Convex analysis and measurable multifunctions*, Lecture Notes in Mathematics, Vol. 580, Springer-Verlag, Berlin-New York, 1977. MR0467310
- [3] T. Banach and R. Voytsitsky, *Characterizing metric spaces whose hyperspaces are homeomorphic to l_2* , Colloq. Math. **113** (2008), no. 2, 223–229, DOI 10.4064/cm113-2-4. MR2425083
- [4] Gerald Beer, *Topologies on closed and closed convex sets*, Mathematics and its Applications, vol. 268, Kluwer Academic Publishers Group, Dordrecht, 1993. MR1269778
- [5] Gerald Beer, *A Polish topology for the closed subsets of a Polish space*, Proc. Amer. Math. Soc. **113** (1991), no. 4, 1123–1133, DOI 10.2307/2048792. MR1065940
- [6] Gerald Beer and Giuseppe Di Maio, *Confinal completeness of the Hausdorff metric topology*, Fund. Math. **208** (2010), no. 1, 75–85, DOI 10.4064/fm208-1-5. MR2609221
- [7] H. R. Bennett, D. J. Lutzer, and G. M. Reed, *Domain representability and the Choquet game in Moore and BCO-spaces*, Topology Appl. **155** (2008), no. 5, 445–458, DOI 10.1016/j.topol.2007.10.005. MR2380929
- [8] J. Chaber and R. Pol, *Note on the Wijsman hyperspaces of completely metrizable spaces* (English, with English and Italian summaries), Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8) **5** (2002), no. 3, 827–832. MR1934383
- [9] Gustave Choquet, *Lectures on analysis. Vol. I: Integration and topological vector spaces*, Edited by J. Marsden, T. Lance and S. Gelbart, W. A. Benjamin, Inc., New York-Amsterdam, 1969. MR0250011
- [10] C. Costantini, *Every Wijsman topology relative to a Polish space is Polish*, Proc. Amer. Math. Soc. **123** (1995), no. 8, 2569–2574, DOI 10.2307/2161290. MR1273484
- [11] Camillo Costantini, *On the hyperspace of a non-separable metric space*, Proc. Amer. Math. Soc. **126** (1998), no. 11, 3393–3396, DOI 10.1090/S0002-9939-98-04956-9. MR1618729

- [12] Camillo Costantini, Sandro Levi, and Jan Pelant, *Compactness and local compactness in hyperspaces*, Topology Appl. **123** (2002), no. 3, 573–608, DOI 10.1016/S0166-8641(01)00222-X. MR1924053
- [13] Jiling Cao and A. H. Tomita, *The Wijsman hyperspace of a metric hereditarily Baire space is Baire*, Topology Appl. **157** (2010), no. 1, 145–151, DOI 10.1016/j.topol.2009.04.039. MR2556089
- [14] G. Debs, *Espaces héréditairement de Baire* (French, with English summary), Fund. Math. **129** (1988), no. 3, 199–206. MR962541
- [15] François G. Dorais and Carl Mummert, *Stationary and convergent strategies in Choquet games*, Fund. Math. **209** (2010), no. 1, 59–79, DOI 10.4064/fm209-1-5. MR2652592
- [16] Edward G. Effros, *Convergence of closed subsets in a topological space*, Proc. Amer. Math. Soc. **16** (1965), 929–931. MR0181983
- [17] Ryszard Engelking, *General topology*, 2nd ed., Sigma Series in Pure Mathematics, vol. 6, Heldermann Verlag, Berlin, 1989. Translated from the Polish by the author. MR1039321
- [18] R. C. Haworth and R. A. McCoy, *Baire spaces*, Dissertationes Math. (Rozprawy Mat.) **141** (1977), 73 pp. MR0431104
- [19] Alexander S. Kechris, *Classical descriptive set theory*, Graduate Texts in Mathematics, vol. 156, Springer-Verlag, New York, 1995. MR1321597
- [20] Erwin Klein and Anthony C. Thompson, *Theory of correspondences*, Including applications to mathematical economics, Canadian Mathematical Society Series of Monographs and Advanced Texts, John Wiley & Sons, Inc., New York, 1984. MR752692
- [21] K. Kuratowski, *Topology. Vol. I*, New edition, revised and augmented. Translated from the French by J. Jaworowski, Academic Press, New York-London; Państwowe Wydawnictwo Naukowe, Warsaw, 1966. MR0217751
- [22] Keye Martin, *Topological games in domain theory*, Topology Appl. **129** (2003), no. 2, 177–186, DOI 10.1016/S0166-8641(02)00147-5. MR1961398
- [23] John C. Oxtoby, *The Banach-Mazur game and Banach category theorem*, Contributions to the theory of games, vol. 3, Annals of Mathematics Studies, no. 39, Princeton University Press, Princeton, N. J., 1957, pp. 159–163. MR0093741
- [24] E. Porada, *Jeu de Choquet* (French), Colloq. Math. **42** (1979), 345–353. MR567573
- [25] Rastislav Telgársky, *Remarks on a game of Choquet*, Colloq. Math. **51** (1987), 365–372. MR891306
- [26] H. E. White Jr., *Topological spaces that are α -favorable for a player with perfect information*, Proc. Amer. Math. Soc. **50** (1975), 477–482. MR0367941
- [27] László Zsilinszky, *Polishness of the Wijsman topology revisited*, Proc. Amer. Math. Soc. **126** (1998), no. 12, 3763–3765, DOI 10.1090/S0002-9939-98-04526-2. MR1458275
- [28] László Zsilinszky, *On β -favorability of the strong Choquet game*, Colloq. Math. **125** (2011), no. 2, 233–243, DOI 10.4064/cm125-2-8. MR2871316

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, THE UNIVERSITY OF NORTH CAROLINA AT PEMBROKE, PEMBROKE, NORTH CAROLINA 28372

E-mail address: laszlo@uncp.edu