QUANTITATIVE GEOMETRIC AND ARITHMETIC RESULTS ON PROJECTIVE SURFACES

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ABSTRACT. In this paper, we improve Ru's defect relation and the height inequality in the case when X is a normal projective surface and D_j , $1 \leq j \leq q$, are big and asymptotic free divisors without irreducible common components on X . As a consequence, we derive a sharp result in the qualitative statement.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

In recent years, there has been some significant progress in the study of qualitative and quantitative aspects of geometric and arithmetic properties of the complement of divisor in an algebraic projective variety. Ru (see [Ru15]) introduced the notion of Nevanlinna constant, denoted by $\text{Nev}(D)$, for an effective Cartier divisor D on a normal projective variety X . He then derived a new defect relation $\delta_f(D) \leq \text{Nev}(D)$ for any Zariski dense holomorphic mapping $f : \mathbb{C} \rightarrow X$. As a consequence, he obtained the following result.

Theorem A ([Ru15, Theorem 5.6]). *Let X be a complex normal projective variety of dimension ≥ 2 , and $D_1, \dots, D_q$ effective and big Cartier divisors in l -subgeneral position on X . Let $r_i > 0$ be real numbers such that $D := \sum_{i=1}^q r_i D_i$ is equidegree (such numbers exist due to Lemma 2.2). We further assume that the linear system $|ND_i|$ ($i = 1, \dots, q$) is base-point free for $N \geq N_0$. Let $f : \mathbb{C} \rightarrow X$ be a Zariski dense holomorphic map. Then, for $\epsilon_0 > 0$ small enough, which depends only on the given divisors,*

$$\sum_{j=1}^q r_j m_f(r, D_j) < \left(\frac{2l \dim X}{q} - \epsilon_0 \right) \left(\sum_{j=1}^q r_j T_{f, D_j}(r) \right) \parallel,$$

where $\parallel$ means the inequality holds for all $r \in (0, \infty)$ except for a set of finite Lebesgue measure.

Recall that the divisors $D_1, \dots, D_q$ on X are said to be in l -subgeneral position on X if for any subset of $l + 1$ elements $\{i_0, \dots, i_l\} \subset \{1, \dots, q\}$,

$$\text{supp} D_{i_0} \cap \dots \cap \text{supp} D_{i_l} = \emptyset.$$

When $l = \dim X$, then we say that the divisors $D_1, \dots, D_q$ are in general position on X . The purpose of this paper is to improve Theorem A in the case when

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$\dim X = 2$ with an additional condition that the divisors $D_1, \dots, D_q$ have no common irreducible components. The precise statement is as follows.

Main Theorem (Complex part). *Let X be a normal complex projective surface. Let $D_1, \dots, D_q$ be effective, big Cartier divisors on X , and let the linear system $|ND_i|$ ($i = 1, \dots, q$) be base-point free for $N \geq N_0$. Assume that $D_1, \dots, D_q$ have no irreducible components in common and are in l -subgeneral position. We further assume that $D := \sum_{j=1}^q r_j D_j$ is equidegree for some positive real numbers r_j (such r_j always exist by Lemma 2.2). Let $f : \mathbb{C} \rightarrow X$ be holomorphic and Zariski dense. Then*

$$\sum_{j=1}^q r_j m_f(r, D_j) \leq \frac{4[(l+1)/2]}{q(1+\alpha)} \left(\sum_{j=1}^q r_j T_{f, D_j}(r) \right) \parallel,$$

where $\alpha = \frac{\min_{1 \leq j \leq q} (r_j^2 D_j^2)}{384qD^2}$, $[x]$ denotes the greatest integer less than or equal to x , and $\parallel$ means the inequality holds for all $r \in (0, \infty)$ except for a set with finite Lebesgue measure.

Define the defect

$$\delta_f(D) := \liminf_{r \rightarrow +\infty} \frac{m_f(r, D)}{T_{f, D}(r)}.$$

Then, under the assumptions in the Main Theorem, we have the defect relation

$$\delta_f(D) \leq \frac{4[(l+1)/2]}{q(1+\alpha)}.$$

Note that our techniques used in this paper to prove the above theorem are similar to Ru's method (see [Ru15]). The main contribution of this paper is to use the joint filtrations lemma due to Corvaja and Zannier [CZ04a] (see Lemma 3.2) to lower the upper bound of the defect, under the additional assumption that $D_1, \dots, D_q$ have no irreducible components in common. Furthermore, we give the explicit computation of the ϵ_0 appearing in Theorem A. Note that our result also holds for any dimension of X . The reason we only focus on the case when $\dim X = 2$ is the following sharp corollary of our Main Theorem, which is due to Levin [Lev09].

Corollary 1.1 ([Lev09, Theorem 11.5B]). *Let X be a nonsingular projective surface and $D_1, D_2, \dots, D_q$ be effective and big divisors on X . Assume that $D_1, \dots, D_q$ have no irreducible components in common and are in general position. If $q \geq 4$, then every holomorphic mapping $f : \mathbb{C} \rightarrow X \setminus \bigcup_{j=1}^q D_j$ must be algebraically degenerate.*

Proof. According to Theorem 11.5 in [Lev09], we can reduce to the case when $|D_i|$ is base-point free (in particular, $D_i, 1 \leq i \leq q$, are nef) and D_i is big for all i . If f is not algebraically degenerate, then the above Main Theorem implies that $\delta_f(D) < 1$, where $D = D_1 + D_2 + \dots + D_q$. On the other hand, from the first main theorem, we have $\delta_f(D) = 1$. This gives a contradiction. Therefore, f must be algebraically degenerate. $\square$

On the arithmetic side, the counterpart of Theorem A in Diophantine approximation is stated as follows. For standard notation in Diophantine approximation, see [Ru16].

Theorem B ([Ru16, Theorem 4.1]). *Let k be a number field and $S \subset M_k$ be a finite set containing all archimedean places. Let X be a normal projective variety with*

$\dim X \geq 2$ and let $D_1, \dots, D_q$ be effective and big Cartier divisors in l -subgeneral position on X , both defined over k . Let $r_i > 0$ be real numbers such that $D := \sum_{i=1}^q r_i D_i$ is equidegree (such numbers exist due to Lemma 2.2). We further assume that the linear system $|ND_i|$ ($i = 1, \dots, q$) is base-point free for $N \geq N_0$. Let $\lambda_{D_j, v}(x)$, $1 \leq j \leq q$, be the Weil function associated to D_j for $v \in S$.

Then, for $\epsilon_0 > 0$ small enough,

$$\sum_{j=1}^q \sum_{v \in S} r_j \lambda_{D_j, v}(x) < \left(\frac{2l \dim X}{q} - \epsilon_0 \right) \left(\sum_{j=1}^q r_j h_{D_j}(x) \right)$$

holds for all $x \in X$ outside a Zariski closed subset Z of X , where $h_{D_j}(x) = \sum_{v \in M_k} \lambda_{D_j, v}(x)$.

Our theorem on the arithmetic side which improves Theorem B is as follows.

Main Theorem (Arithmetic Part). *Let k be a number field and $S \subset M_k$ a finite set containing all archimedean places. Let X be a normal projective surface, and let $D_1, \dots, D_q$ be effective and big Cartier divisors on X , both defined over k . Assume that the linear system $|ND_i|$ ($i = 1, \dots, q$) is base-point free for $N \geq N_0$. Assume that $D_1, \dots, D_q$ have no irreducible components in common and are in l -subgeneral position. We further assume that $D := \sum_{i=1}^q r_i D_i$ is equidegree for some positive real numbers r_j (such r_j always exist by Lemma 2.2). Let $\lambda_{D_j, v}(x)$, $1 \leq j \leq q$, be the Weil function associated to D_j for $v \in S$. Then*

$$\sum_{j=1}^q \sum_{v \in S} r_j \lambda_{D_j, v}(x) \leq \frac{4[(l+1)/2]}{q(1+\alpha)} \left(\sum_{j=1}^q r_j h_{D_j}(x) \right)$$

holds for all $x \in X$ outside a Zariski closed subset Z of X , where $\alpha = \frac{\min_{1 \leq j \leq q} (r_j^2 D_j^2)}{384qD^2}$, $[x]$ denotes the greatest integer less than or equal to x , and $h_{D_j}(x) = \sum_{v \in M_k} \lambda_{D_j, v}(x)$.

The above theorem gives the following corollary.

Corollary 1.2 ([Lev09, Theorem 11.5A]). *Let k be a number field and $S \subset M_k$ be a finite set containing all archimedean places. Let X be a nonsingular projective surface and $D_1, \dots, D_q$ effective and big Cartier divisors on X , both defined over a number field k . Assume that $D_1, \dots, D_q$ have no irreducible components in common and are in general position. If $q \geq 4$, then any set of (D, S) -integral points of $X \setminus D$ is contained in a proper subvariety of X .*

The proof of the result of the arithmetic case is similar to the complex case (see [Ru16]), so the rest of the paper will only focus on the complex part.

2. NOTATION AND PREPARATION LEMMAS

We recall some notation. Let X be a complex projective variety. For a Cartier divisor D on X , the Weil function for D is given by

$$(1) \quad \lambda_D(x) = -\log \|s_D(x)\|,$$

where s_D is the canonical section of the line bundle $\mathcal{O}_X(D)$, i.e. $(s_D) = D$, and $\|\cdot\|$ is any continuous metric on $\mathcal{O}_X(D)$. It is well defined, up to a bounded term, independent of the choices of the metric. In the case when $X = \mathbb{P}^n(\mathbb{C})$ and

$D = \{Q = 0\} \subset \mathbb{P}^n(\mathbb{C})$ where Q is a homogeneous polynomial of degree d , λ_D can be chosen as, for $x = [x_0 : \cdots : x_n] \in \mathbb{P}^n(\mathbb{C}) \setminus \text{supp } D$,

$$\lambda_D(x) = \log \frac{(\max_{0 \leq i \leq n} |x_i|^d) \cdot \|Q\|}{|Q(x)|},$$

where $\|Q\|$ is the maximum of the norm of the coefficients of Q .

Let $f : \mathbb{C} \rightarrow X$ be a holomorphic map whose image is not contained in the support of D . The proximity function of f with respect to D is defined by

$$(2) \quad m_f(r, D) = \int_0^{2\pi} \lambda_D(f(re^{i\theta})) \frac{d\theta}{2\pi}.$$

The counting function of f is defined by

$$(3) \quad N_f(r, D) = \int_1^r \frac{n_f(t, D)}{t} dt,$$

where $n_f(t, D)$ is the number of zeros of $\rho \circ f$ inside $\{|z| < t\}$, counting multiplicities, and ρ is a local defining function of D (note that $n_f(t, D)$ is independent of the choice of ρ). We define the characteristic function by

$$(4) \quad T_f(r, D) := m_f(r, D) + N_f(r, D).$$

The notions of the Weil function and the height function in the arithmetic (Diofantine approximation) side are defined in a similar way (see, for example, [Lan87] or [Vojcm]).

Next, we recall some definitions and lemmas from Levin's paper [Lev09].

Definition 2.1 ([Lev09, Definition 9.6]). Suppose that X is a projective variety of dimension n . Let $D = D_1 + D_2 + \cdots + D_q$ be a divisor on X with D_i effective. D is said to have equidegree with respect to $D_1, D_2, \dots, D_q$ if $D_i \cdot D^{n-1} = \frac{1}{q} D^n$ for $1 \leq i \leq q$.

Lemma 2.2 ([Lev09, Lemma 9.7]). Let X be a projective variety of dimension n . If $D_j, 1 \leq j \leq q$, are big and nef, then there exist positive real numbers r_j such that $D = \sum_{j=1}^q r_j D_j$ is of equidegree with respect to $r_1 D_1, \dots, r_q D_q$.

Proof. We follow the simple proof given by Autissier [Aut1]. Let

$$\Delta := \{(t_1, \dots, t_q) \in \mathbb{R}_+^q \mid t_1 + \cdots + t_q = 1\}.$$

Define a map $g : \Delta \rightarrow \Delta$ by letting, for $t = (t_1, \dots, t_q) \in \Delta$,

$$g(t) = \left(\frac{\phi(t)}{(\sum_{j=1}^q t_j D_j)^{n-1} \cdot D_1}, \dots, \frac{\phi(t)}{(\sum_{j=1}^q t_j D_j)^{n-1} \cdot D_q} \right),$$

where $\phi(t) := \left(\sum_{i=1}^q \frac{1}{(\sum_{j=1}^q t_j D_j)^{n-1} \cdot D_i} \right)^{-1}$. By Brouwer's fixed point theorem, there exists a point $x = (r_1, \dots, r_q)$ such that $g(x) = x$, i.e. $\phi(x) = (\sum_{j=1}^q r_j D_j)^{n-1} \cdot (r_i D_i)$ for $i = 1, \dots, q$. This implies, by summing all i , that $q\phi(x) = (\sum_{j=1}^q r_j D_j)^n$. Thus

$$\frac{1}{q} \left(\sum_{j=1}^q r_j D_j \right)^n = \phi(x) = (r_i D_i) \cdot \left(\sum_{j=1}^q r_j D_j \right)^{n-1},$$

which proves the lemma. □

Lemma 2.2 tells us that we can always make the given big and nef divisors to be of equidegree without changing their supports since the divisors $r_j D_j$ and D_j are of the same supports. This means the notion of equidegree, rather than the condition of linear equivalence for the divisors $D = D_1 + \cdots + D_q$, would be a correct (or proper) notion in the study of degeneracy of holomorphic mappings $f : \mathbb{C} \rightarrow X \setminus D$. In the surface case, we denote by (D_i, D_j) the intersection number of D_i and D_j . We also denote (D, D) by D^2 .

We also need the following results in algebraic geometry. Let X be a complex projective algebraic manifold of dimension two. Let D be a divisor on X . Let $\mathcal{O}(D)$ be the invertible sheaf associated to the divisor D on X . For $i = 0, 1, 2$, let $h^i(D) = \dim H^i(X, \mathcal{O}(D))$ (we sometimes also write $H^i(X, \mathcal{O}(D))$ as $H^i(X, D)$ for simplicity).

Lemma 2.3 ([Laz04, Corollary 1.4.41]). *Suppose that D is a nef divisor on a projective surface on X . Then*

$$h^0(ND) = \frac{D^2 N^2}{2} + O(N) .$$

In particular, D^2 is positive if and only if D is big.

Lemma 2.4 ([Aut1, Lemma 4.2]). *Let X be a projective surface. Let F be a big and base-point free Cartier divisor and D a Cartier divisor such that $D - F$ is also nef. Let $\beta > 0$ be a positive real number. Then for any positive integer N, k with $1 \leq k \leq \beta N$, we have*

$$h^0(ND - kF) \geq \frac{D^2 N^2}{2} - (D.F)Nk + \frac{F^2}{2} \min\{k^2, N^2\} + O(N) ,$$

where $O(N)$ depends on β .

Lemma 2.5. *Let D and F be the same as above on a projective surface X . We further assume that $\frac{D^2}{(D.F)} \geq 1$. Then*

$$\sum_{k=1}^{\infty} h^0(ND - kF) \geq \left(\frac{D^2}{4(D.F)} + \frac{F^2}{24D^2} \right) N h^0(ND) + O(N^2) .$$

Proof. Using Lemma 2.4 with $\beta = \frac{D^2}{2(D.F)} \geq \frac{1}{2}$, we get, by noticing that $\min\{k^2, N^2\} \geq \min\{k^2, N^2/4\}$,

$$\begin{aligned} & \sum_{k=1}^{\infty} h^0(ND - kF) \\ & \geq \sum_{k=1}^{[\beta N]} \left(\frac{D^2 N^2}{2} - (D.F)Nk + \frac{F^2}{2} \min\{k^2, \frac{N^2}{4}\} \right) + O(N^2) \\ & \geq \frac{(D^2)^2 N^3}{4(D.F)} - (D.F) \frac{(D^2)^2 N^3}{8(D.F)^2} + \sum_{k=1}^{[N/2]-1} \frac{F^2}{2} k^2 + \sum_{[N/2]}^{[\beta N]} \frac{F^2}{8} N^2 + O(N^2) \\ & = \frac{(D^2)^2}{8(D.F)} N^3 + \frac{F^2}{48} N^3 + \frac{F^2}{8} \beta N^3 - \frac{F^2}{16} N^3 + O(N^2) \\ & = \left(\frac{D^2}{4(D.F)} + \frac{F^2}{D^2} \left(\frac{\beta}{4} - \frac{1}{12} \right) \right) \frac{D^2}{2} N^3 + O(N^2) \end{aligned}$$

$$\begin{aligned}
&\geq \left(\frac{D^2}{4(D.F)} + \frac{F^2}{24D^2} \right) \frac{D^2}{2} N^3 + O(N^2) \\
&= \left(\frac{D^2}{4(D.F)} + \frac{F^2}{24D^2} \right) N h^0(ND) + O(N^2).
\end{aligned}$$

□

3. PROOF OF THE MAIN THEOREM

The proof of the Main Theorem relies on the notion of Nevanlinna constant, $\text{Nev}(D)$, introduced by Ru (see [Ru15]), and his results in terms of $\text{Nev}(D)$. Let X be a normal projective variety and D an effective Cartier divisor on X . Note that the condition of normality of X is assumed so that $\text{ord}_E D$ (called the *coefficient of D in E*) is defined for any prime divisor E and any effective Cartier divisor D on X (see [Laz04, Remark 1.1.4]). For any section $s \in H^0(X, D)$, we use $\text{ord}_E s$ or $\text{ord}_E(s)$ to denote the coefficients of (s) in E , where (s) is the divisor on X associated to s .

Definition 3.1. Let X be a normal complex projective variety and D an effective Cartier divisor on X . The *Nevanlinna constant* of D , denoted by $\text{Nev}(D)$, is given by

$$(5) \quad \text{Nev}(D) := \inf_N \left(\inf_{\{\mu_N, V_N\}} \frac{\dim V_N}{\mu_N} \right),$$

where the infimum “ $\inf_N$ ” is taken over all positive integers N and the infimum “ $\inf_{\{\mu_N, V_N\}}$ ” is taken over all pairs $\{\mu_N, V_N\}$, where μ_N is a positive real number and $V_N \subseteq H^0(X, ND)$ is a linear subspace with $\dim V_N \geq 2$ such that, for all $P \in \text{supp} D$, there exists a basis B of V_N with

$$\sum_{s \in B} \text{ord}_E(s) \geq \mu_N \text{ord}_E(ND),$$

for all irreducible components E of D passing through P . If $h^0(ND) \leq 1$ for all positive integers N , we define $\text{Nev}(D) = +\infty$.

Theorem C ([Ru15, Main Theorem]). *Let X be a complex normal projective variety and D an effective Cartier divisor on X . Then, for every $\epsilon > 0$,*

$$m_f(r, D) \leq (\text{Nev}(D) + \epsilon) T_f(r, D) \parallel$$

holds for any Zariski dense holomorphic mapping $f : \mathbb{C} \rightarrow X$.

The following joint filtrations lemma is crucial to the proof of our Main Theorem.

Lemma 3.2 ([CZ04a, Lemma 3.2]). *Let V be a vector space of finite dimension d over a field k . Let $V = W_1 \supset W_2 \supset W_3 \supset \dots \supset W_h$ and $V = W_1^* \supset W_2^* \supset W_3^* \supset \dots \supset W_{h^*}^*$ be two filtrations on V . Then there exists a basis of V that contains a basis of each W_j and W_j^* .*

Proof of the Main Theorem (complex part). The proof uses Theorem C, so we need to compute the Nevanlinna constant. By taking $N \geq N_0$, we can assume that $D_j, 1 \leq j \leq q$, are base-point free. By the assumption that D has equidegree with respect to $r_1 D_1, \dots, r_q D_q$, we have, with $D := \sum_{j=1}^q r_j D_j$,

$$(r_i D_i, D) = \frac{1}{q} D^2.$$

To simplify the notation, we write

$$(6) \quad \alpha := \frac{\min_{1 \leq j \leq q} (r_j^2 D_j^2)}{384qD^2}.$$

Choose positive rational numbers $a_j, 1 \leq j \leq q$, such that

$$(7) \quad |a_j - r_j| \leq \frac{\delta_1 (\min_{1 \leq j \leq q} r_j)}{2} \min \left\{ 1, \frac{q(1+2\alpha)}{4[(l+1)/2]} \right\},$$

and, for $i = 1, \dots, q$,

$$(8) \quad \left| \frac{(a_i D_i, D')}{D'^2} - \frac{1}{q} \right| < \delta_2,$$

where $D' := \sum_{j=1}^q a_j D_j$ and δ_1, δ_2 will be chosen below (see (17) and (13)). Note that with our choice of δ_1 and $|a_j - r_j| \leq \frac{r_j}{2}$, we have

$$(9) \quad D'^2 \geq \frac{1}{4} D^2 \text{ and } D^2 \geq \frac{1}{4} D'^2.$$

Now, for $P \in \text{supp} D$, let $D'_P := \sum_{i: P \in \text{supp} D_i} a_i D_i$. Since $D_1, \dots, D_q$ are in l -subgeneral position and any two of $D_1, \dots, D_q$ have no common components, we can write

$$D'_P := D'_{P,1} + D'_{P,2},$$

where $D'_{P,1}$ and $D'_{P,2}$ are each a sum of no more than $[(l+1)/2]$ terms of the $a_i D_i$, and $D'_{P,1}$ and $D'_{P,2}$ have no irreducible components in common. Let d be the product of the denominators of $a_1, \dots, a_q$ and consider $V_N := H^0(X, NdD')$. Note that D' is only a $\mathbb{Q}$ -divisor, so we need to multiply d by D' to make dD' be an (integral) divisor. We consider the following two filtrations of V_N :

$$W_m := H^0(X, NdD' - mdD'_{P,1}), \quad W_m^* := H^0(X, NdD' - mdD'_{P,2}), \quad m = 0, 1, \dots$$

Using the filtration lemma above, we obtain a basis B that contains a basis for each W_m and W_m^* . Let E be an irreducible component of D which contains P . Then E is contained either in $dD'_{P,1}$ or $dD'_{P,2}$, but not both. Without loss of generality, we assume that E is an irreducible component of $dD'_{P,1}$, and thus $\text{ord}_E(dD') = \text{ord}_E(dD'_{P,1})$. We also note that $\text{ord}_E s \geq m \text{ord}_E(dD'_{P,1})$ for any $s \in H^0(X, NdD' - mdD'_{P,1})$ (regarded as a subspace of $H^0(X, NdD')$). Hence

$$\begin{aligned} & \frac{1}{\text{ord}_E(NdD')} \sum_{s \in B} \text{ord}_E s = \frac{1}{\text{ord}_E(NdD'_{P,1})} \sum_{s \in B} \text{ord}_E s \\ & \geq \frac{1}{N} \sum_{m=0}^{\infty} m(h^0(NdD' - mdD'_{P,1}) - h^0(NdD' - (m+1)dD'_{P,1})) \\ & = \frac{1}{N} \sum_{k=1}^{\infty} h^0(NdD' - kdD'_{P,1}). \end{aligned}$$

Noticing that $\frac{D'^2}{(D'.D'_{P,1})} \geq 1$, by Lemma 2.5, we get

$$\begin{aligned} & \sum_{k=1}^{\infty} h^0(NdD' - kdD'_{P,1}) \\ & \geq \left(\frac{(dD')^2}{4(dD'.dD'_{P,1})} + \frac{(dD'_{P,1})^2}{24(dD')^2} \right) Nh^0(NdD') + O(N^2) \\ & = \left(\frac{D'^2}{4(D'.D'_{P,1})} + \frac{D_{P,1}'^2}{24D'^2} \right) Nh^0(NdD') + O(N^2). \end{aligned}$$

Therefore,

$$(10) \quad \frac{1}{\text{ord}_E(NdD')} \sum_{s \in B} \text{ord}_E s \geq \left(\frac{D'^2}{4(D'.D'_{P,1})} + \frac{D_{P,1}'^2}{24D'^2} \right) h^0(NdD') + O(N).$$

We now estimate each term appearing above. Since $D'_{P,1}$ is a sum of no more than $[(l+1)/2]$ terms of the $a_i D_i$, by using (8),

$$(11) \quad \frac{(D'.D'_{P,1})}{D'^2} \leq \frac{[(l+1)/2] \max_{i,P \in D_i} \{(D'.(a_i D_i))\}}{D'^2} \leq [(l+1)/2] \left(\frac{1}{q} + \delta_2 \right).$$

Also notice that, using (6), (7) and (9), we get

$$\begin{aligned} (12) \quad (D'_{P,1}.D'_{P,1}) & \geq \min_{1 \leq i \leq q} (a_i D_i . a_i D_i) \geq \frac{1}{4} \min_{1 \leq i \leq q} (r_i D_i . r_i D_i) \\ & = 96q(D.D)\alpha \geq 24q(D'.D')\alpha. \end{aligned}$$

Combining (10), (11), and (12), we get

$$\begin{aligned} & \frac{1}{\text{ord}_E(NdD')} \sum_{s \in B} \text{ord}_E s \\ & \geq \left(\frac{q}{4[(l+1)/2](1+q\delta_2)} + q\alpha \right) \frac{d^2 D'^2}{2} N^2 + O(N) \\ & \geq \frac{q}{4[(l+1)/2]} \left(\frac{1}{1+q\delta_2} + 4\alpha \right) h^0(NdD') + O(N) \\ & = \frac{q(1+3\alpha)}{4[(l+1)/2]} h^0(NdD') + O(N), \end{aligned}$$

where the last equality holds when choosing

$$(13) \quad \delta_2 = \frac{\alpha}{q(1-\alpha)}.$$

Hence, from the definition of Nevanlinna's constant (see (5)), we obtain

$$\text{Nev}(dD') \leq \frac{4[(l+1)/2]}{q(1+3\alpha)}.$$

Applying Theorem C with $\epsilon = \frac{4[(l+1)/2]\alpha}{q(1+3\alpha)(1+2\alpha)}$, we get

$$\sum_{j=1}^q a_j dm_f(r, D_j) \leq \left(\frac{4[(l+1)/2]}{q(1+2\alpha)} \right) \left(\sum_{j=1}^q a_j dT_f(r, D_j) \right) \parallel.$$

By canceling d on both sides, we have

$$(14) \quad \sum_{j=1}^q a_j m_f(r, D_j) \leq \left(\frac{4[(l+1)/2]}{q(1+2\alpha)} \right) \left(\sum_{j=1}^q a_j T_f(r, D_j) \right) \parallel.$$

We now use this result to derive our desired result. Notice that (7) gives us

$$(15) \quad \sum_{j=1}^q r_j m_f(r, D_j) \leq \sum_{j=1}^q a_j m_f(r, D_j) + \frac{\delta_1}{2} \left(\min_{1 \leq j \leq q} r_j \right) \sum_{j=1}^q m_f(r, D_j)$$

and

$$(16) \quad \sum_{j=1}^q a_j T_{f, D_j}(r) \leq \sum_{j=1}^q r_j T_{f, D_j}(r) + \frac{\delta_1}{2} \left(\min_{1 \leq j \leq q} r_j \right) \frac{q(1+2\alpha)}{4[(l+1)/2]} \sum_{j=1}^q T_{f, D_j}(r).$$

Using (14), (15), (16), together with the first main theorem, we get

$$\begin{aligned} & \sum_{j=1}^q r_j m_f(r, D_j) \\ & \leq \frac{4[(l+1)/2]}{q(1+2\alpha)} \left(\sum_{j=1}^q a_j T_f(r, D_j) \right) + \frac{(\min r_j) \delta_1}{2} \left(\sum_{j=1}^q T_f(r, D_j) \right) \parallel \\ & \leq \frac{4[(l+1)/2]}{q(1+2\alpha)} \left(\sum_{j=1}^q r_j T_f(r, D_j) + \frac{(\min r_j) \delta_1}{2} \frac{q(1+2\alpha)}{4[(l+1)/2]} \sum_{j=1}^q T_f(r, D_j) \right) \\ & \quad + \frac{\delta_1}{2} \sum_{j=1}^q r_j T_f(r, D_j) \parallel \\ & \leq \left(\frac{4[(l+1)/2]}{q(1+2\alpha)} + \delta_1 \right) \sum_{j=1}^q r_j T_f(r, D_j) \parallel \\ & \leq \left(\frac{4[(l+1)/2]}{q(1+\alpha)} \right) \sum_{j=1}^q r_j T_f(r, D_j) \parallel, \end{aligned}$$

with

$$(17) \quad \delta_1 = \min \left\{ 1, \frac{4[(l+1)/2]}{q} \frac{\alpha}{(1+\alpha)(1+2\alpha)} \right\}.$$

This finishes the proof of the Main Theorem.

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