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SHIFTED MOMENTS OVER THE UNITARY ENSEMBLE

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Abstract. In 2000, Keating and Snaith suggested that the value distribution of the Riemann zeta function $\zeta(1/2+it)$ is related to that of the characteristic polynomials of random unitary matrices, $\Lambda_U(\theta) = \prod_{n=1}^{N}(1-e^{i(\theta_n+\theta)})$, with respect to the circular unitary ensemble. They derived the conjecture for the moment of the Riemann zeta function through computing the exact formula for the moments of the characteristic polynomials. In this paper, we compute the shifted moments of the characteristic polynomials of random unitary matrices and express them in a determinant form. When shifts are the roots of unity, we can obtain a precise formula, and this also leads to a new formula analogous to the Selberg's identity applied in Keating and Snaith's computation.

1. Introduction

For $\mathrm{Re}(s) > 1$, the Riemann zeta function $\zeta(s)$ is defined to be

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1},$$

and can be analytically continued to the whole plane except for a simple pole at $s = 1$. One of the most important conjectures for $\zeta(s)$ is the Riemann hypothesis (RH), which states that all non-trivial zeros of $\zeta(s)$ lie on the critical line $\mathrm{Re}(s) = 1/2$. This conjecture has helped shape the development of analytic number theory and has many important consequences such as the Lindelöf hypothesis (LH), which is an upper bound for $\zeta(s)$ on the critical line of the form

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \ll_\epsilon t^\epsilon.$$

One attempt to prove LH is through exploring the moments of the Riemann zeta function. More precisely, let

$$M_{2k}(T) = \frac{1}{T}\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} dt.$$

It can be shown that LH is equivalent to $M_{2k}(T) \ll_\epsilon T^\epsilon$ for all positive integers k (e.g. see [16]). However, the only known asymptotic formulas are for $k = 1$ by Hardy and Littlewood and for $k = 2$ by Ingham [16, Chapter VII]. For other values

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of k, it is conjectured that

$$\lim_{N\to\infty} \frac{1}{(\log T)^{k^2}} M_{2k}(T) = c_k a_k,$$

where c_k is some constant, and a_k is called *the arithmetic factor*, defined by

$$a_k := \prod_p \left(\left(1 - \frac{1}{p}\right)^{k^2} \sum_{m=0}^{\infty} \left(\frac{\Gamma(m+k)}{m!\Gamma(k)} \right)^2 p^{-m} \right). \tag{1.1}$$

This a_k is derived from

$$\sum_{n\le T} \frac{d_k^2(n)}{n} \sim a_k (\log T)^{k^2},$$

where $d_k(n) = \sum_{n_1 \dots n_k = n} 1$ is the k-divisor function. Later, Conrey and Ghosh obtained a conjecture for c_3 [7], and Conrey and Gonek derived a conjecture for c_4 [8]. Around the same time, Keating and Snaith predicted the value of c_k for general k [11] by relating $\zeta(s)$ to the circular unitary ensemble in random matrix theory.

To be more precise, we first define $U = (u_{ij})$ to be an $N \times N$ matrix, and we denote U^* to be a conjugate transpose matrix of U, that is, $U^* = (\overline{u_{ji}})$. U is a unitary matrix if $UU^* = I_N$, where I_N is an $N \times N$ identity matrix. We let $U(N)$ be a group of $N \times N$ unitary matrices, namely the circular unitary ensemble with Haar measure

$$d\mu_{U(N)}(U) = \frac{1}{N!(2\pi)^N} \prod_{1\le j<k\le N} |e^{i\theta_k} - e^{i\theta_j}|^2 \prod_{n=1}^{N} d\theta_n, \tag{1.2}$$

where $e^{i\theta_j}$ is an eigenvalue of U and $0 \le \theta_1, \theta_2, \dots, \theta_N \le 2\pi$. Note that the eigenvalues of U^* are $e^{-i\theta_1}, \dots, e^{-i\theta_N}$, and the characteristic polynomial of U is of the form

$$\Lambda_U(s) := \det(I - sU) = \prod_{n=1}^{N} (1 - se^{i\theta_n}). \tag{1.3}$$

The connection between $\zeta(s)$ and random matrices was first observed by Montgomery and Dyson through studying the pair correlation of zeros of the Riemann zeta function along the vertical line. Based on Montgomery's work, Keating and Snaith [11] suggested that the value distribution of $\zeta(1/2+it)$ is related to that of the characteristic polynomials of random unitary matrices, $\Lambda_U(\theta) = \prod_{n=1}^{N}(1-e^{i(\theta_n+\theta)})$, with respect to the circular unitary ensemble. Since the average spacing of nontrivial zeros of $\zeta(s)$ at height T is $\frac{2\pi}{\log(T/2\pi)}$, we expect the distribution of zeros of $\zeta(s)$ of height T to resemble the distribution of eigenvalues of the random unitary ensemble for $N = \log\left(\frac{T}{2\pi}\right)$. To derive a conjecture for $M_k(T)$, Keating and Snaith computed the moments of the characteristic polynomials and showed that for $\mathrm{Re}(k) > -1/2$,

$$g(k,N) := \int_{U(N)} |\Lambda_U(1)|^{2k}\, dU_N = \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(j+2k)}{(\Gamma(j+k))^2} \asymp \frac{G^2(k+1)}{G(2k+1)} N^{k^2}. \tag{1.4}$$

One important ingredient that they used to prove the above formula is Selberg's formula (for example, see [13, Chapter 17])

(1.5)

$$\begin{aligned} &J(a,b,\alpha,\beta,\gamma,N) \\ &= \int_{[-\infty,\infty]} \cdots \int_{[-\infty,\infty]} \left| \prod_{\substack{j,\ell=1 \\ j\neq\ell}}^{N} (x_j - x_\ell) \right|^{2\gamma} \prod_{j=1}^{N} (a+ix_j)^{-\alpha}(b-ix_j)^{-\beta}\, dx_1 \dots dx_N \\ &= \frac{2\pi}{(a+b)^{(\alpha+\beta)N-\gamma N(N-1)-N}} \prod_{j=0}^{N-1} \frac{\Gamma(1+\gamma+j\gamma)\Gamma(\alpha+\beta-(N+j-1)\gamma-1)}{\Gamma(1+\gamma)\Gamma(\alpha-j\gamma)\Gamma(\beta-j\gamma)}. \end{aligned}$$

This led them to conjecture that for all fixed positive integers k,

$$M_{2k}(T) \asymp a_k \frac{G^2(k+1)}{G(2k+1)} T(\log T)^{k^2}, \tag{1.6}$$

where a_k is defined as in (1.1), and $G(s)$ denotes the *Barnes G-function* defined by

$$G(z+1) := (2\pi)^{z/2} e^{-(z(z+1)+\gamma z^2)/2} \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right)^n e^{-z+z^2/(2n)}$$

for complex numbers z. If k is a positive integer, we obtain

$$\frac{G^2(k+1)}{G(2k+1)} = \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}.$$

As seen from the moment computation, the conjecture derived by Keating and Snaith does not explain why a_k appears in (1.6). This point was later partially justified in the work of Gonek, Hughes and Keating [9] using a hybrid model between probabilistic theory and random matrix theory.

Later, Conrey, Farmer, Keating, Rubinstein and Snaith [5] gave a general recipe for the asymptotic formula of the moment of $\zeta(s)$ by using shifted moments. This recipe predicts a more complete conjecture for asymptotic formulae of $M_{2k}(T)$, including the factor a_k and lower order terms, and it is also applicable for other L-functions. Inspired by Conrey, et al. and Keating and Snaith's work, the author studied the shifted moments of the Riemann zeta function [3]:

$$M_{\mathbf{k}}(T,\overrightarrow{\alpha}) = \frac{1}{T}\int_0^T \left|\zeta(\tfrac{1}{2}+it+i\alpha_1)\right|^{2k_1} \left|\zeta(\tfrac{1}{2}+it+i\alpha_2)\right|^{2k_2} \dots \left|\zeta(\tfrac{1}{2}+it+i\alpha_m)\right|^{2k_m} dt,$$

where $\mathbf{k} = (k_1, k_2, \dots, k_m)$ is a sequence of positive real numbers and $\overrightarrow{\alpha} = (\alpha_1, \dots, \alpha_m)$, where $\alpha_i = \alpha_i(T)$ is a real valued function in terms of T such that $\lim_{T\to\infty} \alpha_i \log T$ and $\lim_{T\to\infty}(\alpha_i - \alpha_j)\log T$ exist or equal $\pm\infty$. Specifically, the author obtained good upper bounds, lower bounds and conjectures for them. Moreover, in order to formulate conjectures for certain shifted moments, the author computed the asymptotic formula for

$$S_{2k}(N,\theta) := \int_{U(N)} |\Lambda_U(1)|^{2k} |\Lambda_U(e^{i\theta})|^{2k}\, dU_N, \tag{1.7}$$

and obtained the following proposition.

Proposition 1.1. *Let θ be a fixed function in terms of N such that $\lim_{N\to\infty}\theta N$ exists or equals $\pm\infty$ and $\theta \neq 2n\pi$, where n is an integer. As $N\to\infty$ we obtain*

$$S_{2k}(N,\theta) \asymp \begin{cases} \frac{G^2(2k+1)}{G(4k+1)} N^{4k^2} & \text{if } \lim_{N\to\infty}|\theta|N = 0, \\ C_k N^{4k^2} & \text{if } \lim_{N\to\infty}|\theta|N = c \neq 0, \\ |1-e^{i\theta}|^{-2k^2}\frac{G^4(k+1)}{G^2(2k+1)} N^{2k^2} & \text{if } \lim_{N\to\infty}|\theta|N = \infty, \end{cases}$$

where $C_k = \lim_{\substack{\mu_1,\dots,\mu_k\to 0 \\ \mu_{k+1},\dots,\mu_{2k}\to c/2\pi}} \frac{\det(b_{jl})_{j,l=1,\dots,2k}}{\Delta^2(2\pi\mu_1,\dots,2\pi\mu_{2k})}$, $b_{jl} = \frac{\sin(\pi(\mu_j-\mu_\ell))}{\pi(\mu_j-\mu_\ell)}$ *when $j\neq\ell$ and* 1 *otherwise, and $\Delta(x_1,\dots,x_N)$ is the Vandermonde determinant defined by*

$$\Delta(x_1,\dots,x_N) := \det_{N\times N}(x_k^{j-1})_{jk} = \prod_{1\le j<k\le N}(x_k - x_j). \tag{1.8}$$

We remark that the exact formula of $S_{2k}(N,\theta)$ is not proven in the above proposition, while Keating and Snaith obtained the exact formula for $g(k,N)$ as in equation (1.4). For general exact formulae of shifted moments, Conrey, Farmer, Keating, Rubinstein and Snaith [6] computed general shifted moments[1] for the characteristic polynomials over the unitary group, which are of the form

$$\int_{U(N)} \Lambda_U(z_1^{-1})\Lambda_U(z_2^{-1})\dots\Lambda_U(z_m^{-1})\Lambda_{U^*}(z_{m+1})\dots\Lambda_{U^*}(z_n)\,dU_N. \tag{1.9}$$

They expressed the shifted moments in three forms: as a determinant sum, appearing in the work of Basor and Forrester [2]; as a combinatorial sum; and as a contour integral.

In this work, we shall express non-asymptotic, exact formulae for $S_{2k}(N,\theta)$ in terms of a determinant, where θ is a fixed real number between $0<\theta<2\pi$, as in Theorem 1.2.

Theorem 1.2. *Let $S_{2k}(N,\theta)$ be defined as in* (1.7)*, and θ be a fixed real number between* 0 *and* 2π*. Then*

$$S_{2k}(N,\theta) = \frac{e^{-ikN\theta}\det\mathcal{M}_{2k}}{(e^{i\theta}-1)^{4k^2}},$$

where

$$\mathcal{M}_d = \begin{pmatrix} f_1(e^{i\theta}) & f_2(e^{i\theta}) & \dots & f_{d-1}(e^{i\theta}) & f_d(e^{i\theta}) \\ f_1'(e^{i\theta}) & f_2'(e^{i\theta}) & \dots & f_{d-1}'(e^{i\theta}) & f_d'(e^{i\theta}) \\ \frac{f_1''(e^{i\theta})}{2!} & \frac{f_2''(e^{i\theta})}{2!} & \dots & \frac{f_{d-1}''(e^{i\theta})}{2!} & \frac{f_d''(e^{i\theta})}{2!} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{f_1^{(d-1)}(e^{i\theta})}{(d-1)!} & \frac{f_2^{(d-1)}(e^{i\theta})}{(d-1)!} & \dots & \frac{f_{d-1}^{(d-1)}(e^{i\theta})}{(d-1)!} & \frac{f_d^{(d-1)}(e^{i\theta})}{(d-1)!} \end{pmatrix},$$

and $f_n(x) := f_{n,d}(x) := x^{N+d+n-1} - \sum_{\ell=1}^{d}\binom{N+d+n-1}{\ell-1}(x-1)^{\ell-1}$.

The initial methods in the proof are the same as in Conrey et al. [6], which was adapted from Basor and Forrester's work [2].[2] Specifically, we computed the shifted moments of (1.9) when $m=2k, n=4k$ in the same manner as in [6], and then

[1] They are called "autocorrelation functions" in [6].

[2] Originally, the author found these methods in Conrey's note "A guide to random matrix theory for number theorists" [4], posted on https://www.math.ethz.ch/u/sznitman/conrey_notes.pdf, but the link is no longer available. The referee suggested the same methods can be found in [6] and [2].

take the limits z_i tending to 1 for $i = 1, \dots, k$ and $2k+1, \dots, 3k$, tending to $e^{-i\theta}$ for $i = k+1, \dots, 2k$, and tending to $e^{i\theta}$ for $i = 3k+1, \dots, 4k$. The details will be provided in Section 2. Moreover, for special case $k = 1$, we obtain the following result.

Corollary 1.3. *We have $S_2(N, \theta)$ is*

$$\frac{e^{-i\theta N}}{(e^{i\theta}-1)^4}\Big(e^{i\theta(2N+4)} - (N+2)^2 e^{i\theta(N+3)} + 2(N+3)(N+1)e^{i\theta(N+2)} - (N+2)^2 e^{i\theta(N+1)} + 1\Big).$$

By a division algorithm, the formula can be rearranged as

$$S_2(N, \theta) = \sum_{|k| \le N} \binom{N+3-|k|}{3} e^{ik\theta}.$$

From the formula of $S_2(N, \theta)$, the leading term of $S_2(N, \theta)$ as $N \to \infty$ is $-\frac{e^{i\theta}}{(e^{i\theta}-1)^2}N^2$, and this corresponds to the result in Proposition 1.1 when $\lim_{N\to\infty} |\theta| N = \infty$.

Moreover, we can compute a precise formula of $S_{2k}(N, \pi)$ for general k. In fact, we will prove the result in a more general form, i.e., the shifted moments when shifted angles are distributed over the root of unity:

(1.10)
$$S_{2k}\Big(N; 1, \frac{2\pi}{\ell}, \frac{4\pi}{\ell}, \dots, \frac{2(\ell-1)\pi}{\ell}\Big) = \int_{U(N)} |\Lambda_U(1)|^{2k} |\Lambda_U(e^{\frac{2\pi i}{\ell}})|^{2k} \dots |\Lambda_U(e^{\frac{2\pi i(\ell-1)}{\ell}})|^{2k}\, dU_N.$$

We obtain that $S_{2k}\Big(N; 1, \frac{2\pi}{\ell}, \dots, \frac{2(\ell-1)\pi}{\ell}\Big)$ can be written as the product of $g(k, N)$, which will be done in Section 3.

Theorem 1.4. *Let $N = q\ell + j$; where $0 \le j \le \ell - 1$ and $q \ge 0$,*

$$S_{2k}\Big(N; 1, \frac{2\pi}{\ell}, \dots, \frac{2(\ell-1)\pi}{\ell}\Big) = g^j(k, q+1) g^{\ell-j}(k, q).$$

This result corresponds to the result from Rains [15], which states that the eigenvalues of a power of matrix U^ℓ can be described in terms of a union of ℓ independent distributions, each of which is itself the eigenvalue distribution of a random unitary matrix. To be more specific,

$$U(N)^\ell \sim \bigoplus_{0 \le i < j-1} U(q+1) \oplus \bigoplus_{j \le i < \ell} U(q),$$

where we recall $N = q\ell + j$.

As mentioned above, the key ingredient of proving the exact formula for $g(k, N)$ is the Selberg's formula in (1.5). Since we have the exact formula for the shifted moment over the root of unity, we will then derive a new integral formula analogous to the Selberg's formula.

Theorem 1.5. *Let $g(k,N)$ be defined as in* (1.4)*, and $N = q\ell + j$, where $0 \le j \le \ell - 1$ and $q \ge 0$. Then we have*

$$\begin{aligned}
&\frac{2^{2Nk+N^2}}{N!(2\pi)^N}\int\cdots\int_{[-\infty,\infty]^N}\prod_{1\le j<n\le N}|x_n-x_j|^2\\
&\cdot\prod_{m=1}^{N}\frac{\left(\sum_{t=0}^{\lfloor\frac{\ell-1}{2}\rfloor}\binom{\ell}{2t+1}(-1)^t x_m^{\ell-(2t+1)}\right)^{2k}}{(x_m^2+1)^{k\ell+N}}\,dx_1\dots dx_N\\
=\;& g^j(k,q+1)g^{\ell-j}(k,q)\\
=\;& \frac{\Gamma^j(q+1+2k)\Gamma^j(q+1)}{\Gamma^{2j}(q+1+k)}\prod_{a=1}^{q}\frac{\Gamma^\ell(a+2k)\Gamma^\ell(a)}{\Gamma^{2\ell}(a+k)}.
\end{aligned}$$

Specially, when $\ell = 2$, we obtain that

$$\begin{aligned}
&\frac{2^{2Nk+N^2}}{N!(2\pi)^N}\int\cdots\int_{[-\infty,\infty]^N}\prod_{1\le j<n\le N}|x_n-x_j|^2\prod_{m=1}^{N}\frac{(2x_m)^{2k}}{(x_m^2+1)^{2k+N}}\,dx_1\dots dx_N\\
&\quad= g^j(k,q+1)g^{2-j}(k,q),
\end{aligned}$$

where $j = 1$ when N is odd, and $j = 0$ otherwise. The details of these calculations are in Section 4.

2. Proof of Theorem 1.2: Shifted moments in terms of a determinant

As mentioned in the introduction, we start considering shifted moments of unitary matrices as below:

$$\begin{aligned}
\mathcal{S}_{2k}(N,\theta;\vec{w}) := \int_{U(N)}&\Lambda_U(w_1^{-1})\dots\Lambda_U(w_k^{-1})\Lambda_{U^*}(w_{k+1})\dots\Lambda_{U^*}(w_{2k})\\
&\cdot\Lambda_U(w_{2k+1}^{-1}e^{-i\theta})\dots\Lambda_U(w_{3k}^{-1}e^{-i\theta})\Lambda_{U^*}(w_{3k+1}e^{i\theta})\dots\Lambda_{U^*}(w_{4k}e^{i\theta})\,dU_N,
\end{aligned}$$

when $\vec{w} := (w_1,\dots,w_{4k})$. By taking $w_1,\dots,w_{4k}\to 1$, we obtain that

$$\lim_{w_1,\dots,w_{4k}\to 1}\mathcal{S}_{2k}(N,\theta;\vec{w}) = S_{2k}(N,\theta).$$

We calculate $\mathcal{S}_{2k}(N,\theta;\vec{w})$ by writing out $\Lambda_U(w)$ as in (1.3) and following the calculation in equations (2.5) - (2.8) of [6]. Then $\mathcal{S}_{2k}(N,\theta;\vec{w})$ becomes

$$\begin{aligned}
&\frac{w_1^{-N}\dots w_k^{-N}w_{2k+1}^{-N}\dots w_{3k}^{-N}e^{-ikN\theta}}{(2\pi)^N}\\
&\quad\cdot\int_{[0,2\pi]^N}\prod_{n=1}^{N}e^{-2ki\theta_n}e^{-i(n-1)\theta_n}(e^{i\theta_n}-w_1)\dots(e^{i\theta_n}-w_{2k})\\
&\quad\cdot(e^{i\theta_n}-w_{2k+1}e^{i\theta})\dots(e^{i\theta_n}-w_{4k}e^{i\theta})\prod_{1\le j<k\le N}(e^{i\theta_k}-e^{i\theta_j})\,d\theta_1\dots d\theta_N\\
=\;&\frac{w_1^{-N}\dots w_k^{-N}w_{2k+1}^{-N}\dots w_{3k}^{-N}e^{-ikN\theta}}{(2\pi)^N}\\
&\quad\times\int_{[0,2\pi]^N}\frac{\prod_{n=1}^{N}e^{-i(n+2k-1)\theta_n}\Delta(e^{i\theta_1},\dots,e^{i\theta_N},w_1,\dots,w_{2k},w_{2k+1}e^{i\theta},\dots,w_{4k}e^{i\theta})}{\Delta(w_1,\dots,w_{2k},w_{2k+1}e^{i\theta},\dots,w_{4k}e^{i\theta})}\,d\theta_1\dots d\theta_N.
\end{aligned}$$

By the Vandermonde determinant formula in (1.8), we can write

$$\Delta(e^{i\theta_1},\dots,e^{i\theta_N},w_1,\dots,w_{2k},w_{2k+1}e^{i\theta},\dots,w_{4k}e^{i\theta}) = \det M,$$

where M is an $(N+4k)\times(N+4k)$ matrix whose j^{th} row is $(1, e^{i\theta_j}, \dots, e^{i(N+4k-1)\theta_j})$ for $1 \le j \le N$ and the $(N+m)^{th}$ row is $(1, w_m, \dots, w_m^{N+4k-1})$ for $m = 1, \dots, 2k$, and $(1, w_m e^{i\theta}, \dots, (w_m e^{i\theta})^{N+4k-1})$ for $m = 2k+1, \dots, 4k$. We then multiply the nth row by $e^{-i(n+2k-1)\theta_n}$ for $1 \le n \le N$ and obtain that

$$\int_{[0,2\pi]^N} \prod_{n=1}^{N} e^{-i(n+2k-1)\theta_n} \Delta(e^{i\theta_1}, \dots, e^{i\theta_N}, w_1, \dots, w_{2k}, w_{2k+1}e^{i\theta}, \dots, w_{4k}e^{i\theta})\, d\theta_1 \dots d\theta_N$$
$$= \int_{[0,2\pi]^N} \det M_1 \, d\theta_1 \dots d\theta_N,$$

where the jth row of M_1 for $1 \le j \le N$ is

$$(e^{-i(j+2k-1)\theta_j}, e^{-i(j+2k-2)\theta_j}, \dots, e^{-i\theta_j}, 1, e^{i\theta_j}, \dots, e^{i(N-j+2k)\theta_j}),$$

and the $(N+m)^{th}$ row of M_1, where $m = 1, \dots, 4k$, is the same as that of M. We note here that the $(j, j+2k)^{th}$ entry is 1 for $1 \le j \le N$. From the determinant formula,

$$\det M_1 = \sum_{\sigma \in S_{N+4k}} \operatorname{sgn}(\sigma) \prod_{i=1}^{N+4k} m_{i,\sigma_i},$$

where $m_{r,c}$ is the $(r,c)^{th}$ entry of M_1. After integration with respect to θ_j for all $j = 1, \dots, N$, any term with $e^{ik\theta_j}$ for $k \ne 0$ becomes 0. Therefore we have that

$$S_{2k}(N,\theta) = \lim_{w_1,\dots,w_{4k}\to 1} \mathcal{S}_{2k}(N,\theta;\vec{w})$$
$$= \lim_{w_1,\dots,w_{4k}\to 1} \frac{w_1^{-N} \dots w_k^{-N} w_{2k+1}^{-N} \dots w_{3k}^{-N} e^{-ikN\theta} \det M_2}{\prod_{1\le i<j\le 2k}(w_j - w_i) \prod_{2k+1\le i<j\le 4k}(w_j e^{i\theta} - w_i e^{i\theta}) \prod_{\substack{1\le i\le 2k \\ 2k+1\le j\le 4k}} (w_j e^{i\theta} - w_i)},$$

where

(2.1)
$$M_2 = \begin{pmatrix} 1 & w_1 & \dots & w_1^{2k-1} & w_1^{N+2k} & \dots & w_1^{N+4k-1} \\ 1 & w_2 & \dots & w_2^{2k-1} & w_2^{N+2k} & \dots & w_2^{N+4k-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & w_{2k} & \dots & w_{2k}^{2k-1} & w_{2k}^{N+2k} & \dots & w_{2k}^{N+4k-1} \\ 1 & w_{2k+1}e^{i\theta} & \dots & (w_{2k+1}e^{i\theta})^{2k-1} & (w_{2k+1}e^{i\theta})^{N+2k} & \dots & (w_{2k+1}e^{i\theta})^{N+4k-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & w_{4k}e^{i\theta} & \dots & (w_{4k}e^{i\theta})^{2k-1} & (w_{4k}e^{i\theta})^{N+2k} & \dots & (w_{4k}e^{i\theta})^{N+4k-1} \end{pmatrix}.$$

We take the limit as w_1, w_{2k+1} go to 1. This gives the first row of all 1, and the $(2k+1)^{th}$ row is $1, e^{i\theta}, \dots, e^{i(2k-1)\theta}, e^{i(N+2k)\theta}, \dots, e^{i(N+4k-1)\theta}$. Then we subtract the first row from the second row to get

$$(0, w_2 - 1, \dots, w_2^{2k} - 1, w_2^{N+2k} - 1, \dots, w_2^{N+4k-1} - 1)$$
$$= (w_2 - 1)(0, 1, \dots, w_2^{N+4k-2} + \dots + 1).$$

The factor $w_2 - 1$ cancels out that in (2.1). We then take the limit as w_2 goes to 1, and the second row of M_2 becomes $0, 1, 2, \dots, 2k-1, N+2k, \dots, N+4k-1$. Similarly, we subtract the $(2k+1)^{th}$ row from the $(2k+2)^{th}$ row, cancel the factor

$w_{2k+2}-1$, and take the limit as w_{2k+2} goes to 1. The $(2k+2)^{th}$ row of M_2 becomes $0, e^{i\theta}, \dots, (2k-1)e^{i(2k-1)\theta}, (N+2k)e^{i(N+2k)\theta}, \dots, (N+4k-1)e^{i(N+4k-1)\theta}$. Next we subtract the first row and w_3-1 times the second row from the third row. Since

$$\lim_{w_3\to 1} \frac{w_3^j - j(w_3-1)-1}{(w_3-1)^2} = \binom{j}{2},$$

we can factor $(w_3-1)^2$ out, cancel it in both denominator and numerator, take the limit as $w_3 \to 1$, and obtain that the third row becomes $0, 0, \binom{2}{2}, \dots, \binom{2k-1}{2}, \binom{N+2k}{2}, \dots, \binom{N+4k-1}{2}$. Similarly, we do the same for the $(2k+3)^{th}$ row. We then proceed similarly for other rows. In particular for the ℓth row, we subtract the first row, $(w_\ell - 1)$ times the second row, $(w_\ell-1)^2$ times the third row,..., and $(w_\ell-1)^{\ell-2}$ times the $(\ell-1)^{th}$ row from the ℓth row. Moreover,

$$\lim_{w_\ell\to 1} \frac{w_\ell^j - \sum_{n=0}^{\ell-2}\binom{j}{n}(w_\ell-1)^n}{(w_\ell-1)^{\ell-1}} = \binom{j}{\ell-1}.$$

Therefore, we can factor $(w_\ell-1)^{\ell-1}$ out, cancel it in both denominator and numerator, take the limit as $w_\ell \to 1$, and obtain that the ℓth row becomes

$$0, \dots, 0, \binom{\ell-1}{\ell-1}, \dots, \binom{2k-1}{\ell-1}, \binom{N+2k}{\ell-1}, \dots, \binom{N+4k-1}{\ell-1}.$$

By similar row operations for the $(2k+\ell)^{th}$ row, we obtain that this row becomes

$$0, \dots, 0, \binom{\ell-1}{\ell-1}e^{i(\ell-1)\theta}, \dots, \binom{2k-1}{\ell-1}e^{i(2k-1)\theta},$$
$$\binom{N+2k}{\ell-1}e^{i(N+2k)\theta}, \dots, \binom{N+4k-1}{\ell-1}e^{i(N+4k-1)\theta}.$$

Hence

$$S_{2k}(N,\theta) = \frac{e^{-ikN\theta}\det M_3}{e^{ik(2k-1)\theta}(e^{i\theta}-1)^{4k^2}},$$

where

$$M_3 = \begin{pmatrix} 1 & 1 & \dots & 1 & 1 & \dots & 1 \\ 0 & 1 & \dots & 2k-1 & N+2k & \dots & N+4k-1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & \binom{N+2k}{2k-1} & \dots & \binom{N+4k-1}{2k-1} \\ 1 & e^{i\theta} & \dots & e^{i(2k-1)\theta} & e^{i(N+2k)\theta} & \dots & e^{i(N+4k-1)\theta} \\ 0 & e^{i\theta} & \dots & (2k-1)e^{i(2k-1)\theta} & (N+2k)e^{i(N+2k)\theta} & \dots & (N+4k-1)e^{i(N+4k-1)\theta} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & e^{i(2k-1)\theta} & \binom{N+2k}{2k-1}e^{i(N+2k)\theta} & \dots & \binom{N+4k-1}{2k-1}e^{i(N+4k-1)\theta} \end{pmatrix}.$$

Next, we do row operations for M_3 as follows:

(1) We factor out $e^{i(j-1)\theta}$ from row $2k+j$, where $1 \le j \le 2k$, and we derive that

$$S_{2k}(N,\theta) = \frac{e^{-ikN\theta}\det M_4}{(e^{i\theta}-1)^{4k^2}},$$

where

$$M_4 = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 & \dots & 1 \\ 0 & 1 & 2 & \dots & 2k-1 & N+2k & \dots & N+4k-1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & \binom{N+2k}{2k-1} & \dots & \binom{N+4k-1}{2k-1} \\ 1 & x & x^2 & \dots & x^{2k-1} & x^{N+2k} & \dots & x^{N+4k-1} \\ 0 & 1 & 2x & \dots & (2k-1)x^{2k-2} & (N+2k)x^{N+2k-1} & \dots & (N+4k-1)x^{N+4k-2} \\ 0 & 0 & 1 & \dots & \binom{2k-1}{2}x^{2k-3} & \binom{N+2k}{2}x^{N+2k-2} & \dots & \binom{N+4k-1}{2}x^{N+4k-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & \binom{N+2k}{2k-1}x^{N+1} & \dots & \binom{N+4k-1}{2k-1}x^{N+2k} \end{pmatrix},$$

and we let $x = e^{i\theta}$.

(2) We subtract $\frac{1}{(j-1)!}\left\{\sum_{\ell=j}^{2k} \frac{d^{j-1}(x-1)^{\ell-1}}{dx^{j-1}} \text{row } \ell\right\}$ from row $2k+j$ of M_4, where $j = 1, \dots, 2k$. After the row operations, the $(2k+j, c)$-element, where $c = 1, \dots, 2k$, becomes

$$\begin{aligned} &\binom{c-1}{j-1}x^{c-j} - \frac{1}{(j-1)!}\left\{\sum_{\ell=j}^{2k} \frac{d^{j-1}(x-1)^{\ell-1}}{dx^{j-1}}\binom{c-1}{\ell-1}\right\} \\ &= \frac{1}{(j-1)!}\frac{d^{j-1}}{dx^{j-1}}\left\{x^{c-1} - \sum_{\ell=1}^{c}\binom{c-1}{\ell-1}(x-1)^{\ell-1}\right\} = 0, \end{aligned}$$

where the first equality comes from the fact that $\binom{c-1}{\ell-1} = 0$ when $\ell > c$ and $\frac{d^{j-1}(x-1)^{\ell-1}}{dx^{j-1}} = 0$ when $\ell < j$, and the second equality is derived from the binomial theorem.

Moreover, the $(2k+j, c)$-element, for $c = 2k+1, \dots, 4k$, becomes

$$\begin{aligned} &\binom{N+c-1}{j-1}x^{N+c-j} - \frac{1}{(j-1)!}\left\{\sum_{\ell=j}^{2k} \frac{d^{j-1}(x-1)^{\ell-1}}{dx}\binom{N+c-1}{\ell-1}\right\} \\ &= \frac{1}{(j-1)!}\frac{d^{j-1}}{dx^{j-1}}\left\{x^{N+c-1} - \sum_{\ell=1}^{2k}\binom{N+c-1}{\ell-1}(x-1)^{\ell-1}\right\}. \end{aligned}$$

Combining the above, we then obtain the desired result of Theorem 1.2.

3. Proof of Theorem 1.4: Shifted moments over the root of unity

In this section, we will prove the formula for $S_{2k}\left(N; 1, \frac{2\pi}{\ell}, \frac{4\pi}{\ell}, \dots, \frac{2(\ell-1)\pi}{\ell}\right)$, defined in (1.10). We recall that Keating and Snaith showed in [11] that

$$g(k, N) = \int_{U(N)} |\Lambda_U(1)|^{2k}\, dU_N = \prod_{j=1}^{N} \frac{\Gamma(j+2k)\Gamma(j)}{\Gamma(j+k)^2}.$$

We will show that $S_{2k}\left(N; 1, \frac{2\pi}{\ell}, \dots, \frac{2(\ell-1)\pi}{\ell}\right)$ can be written as the product of $g(k, N)$.

Proof of Theorem 1.4. We will re-write $g(k,M)$ and $S_{2k}\Big(N;1,\frac{2\pi}{\ell},\dots,\frac{2(\ell-1)\pi}{\ell}\Big)$ by using the Gram's identity (e.g. see Theorem (2.1) in [1]):

$$\int_{U(N)}\prod_{n=1}^{N}h(e^{i\theta_n})\,dU_N=\det_{N\times N}\Big(\frac{1}{2\pi}\int_0^{2\pi}h(e^{i\theta})e^{i(r-c)\theta}\,d\theta\Big).$$

Hence, we obtain that

$$g(k,M)\quad=\quad\det_{M\times M}A_M, \tag{3.1}$$

where $A_M=(a_{r-c})$, r is r^{th}-row, c is c^{th}-column, and

$$a_{r-c}=\frac{1}{2\pi}\int_0^{2\pi}|1-e^{i\theta}|^{2k}e^{i(r-c)\theta}\,d\theta.$$

It is easy to see that $a_t=a_{-t}$. Moreover, we recall that the shifted moments over the root of unity is

$$\begin{aligned}
&S_{2k}\Big(N;1,\frac{2\pi}{\ell},\frac{4\pi}{\ell},\dots,\frac{2(\ell-1)\pi}{\ell}\Big)\\
&\quad=\int_{U(N)}\prod_{n=1}^{N}|1-e^{i\theta_n}|^{2k}|1-e^{i\theta_n}e^{i2\pi/\ell}|^{2k}\dots|1-e^{i\theta_n}e^{i2\pi(\ell-1)/\ell}|^{2k}\,dU_N\\
&\quad=\int_{U(N)}\prod_{n=1}^{N}|e^{-i\theta_n}-1|^{2k}|e^{-i\theta_n}-e^{i2\pi/\ell}|^{2k}\dots|e^{-i\theta_n}-e^{i2\pi(\ell-1)/\ell}|^{2k}\,dU_N\\
&\quad=\int_{U(N)}\prod_{n=1}^{N}|e^{-i\ell\theta_n}-1|^{2k}\,dU_N=\int_{U(N)}\prod_{n=1}^{N}|1-e^{i\ell\theta_n}|^{2k}\,dU_N,
\end{aligned}$$

where the last line follows from the identity $x^\ell-1=(x-1)(x-e^{2\pi i/\ell})\dots(x-e^{2\pi i(\ell-1)/\ell})$. By Gram's identity, we then derive that

$$S_{2k}\Big(N;1,\frac{2\pi}{\ell},\dots,\frac{2(\ell-1)\pi}{\ell}\Big)\quad=\quad\det_{N\times N}B,$$

where $B=(\beta_{r,c})=(b_{r-c})$, and

$$b_{r-c}=\frac{1}{2\pi}\int_0^{2\pi}|1-e^{\ell i\theta}|^{2k}e^{i(r-c)\theta}\,d\theta.$$

By the change of variable, we can write b_{r-c} as

$$b_{r-c}=\frac{1}{2\pi\ell}\int_0^{2\ell\pi}|1-e^{i\theta}|^{2k}e^{\frac{i(r-c)\theta}{\ell}}\,d\theta=\frac{1}{2\pi\ell}\int_0^{2\pi}|1-e^{i\theta}|^{2k}e^{\frac{i(r-c)\theta}{\ell}}\sum_{a=0}^{\ell-1}e^{\frac{i2\pi a(r-c)}{\ell}}\,d\theta.$$

Because $\sum_{a=0}^{\ell-1}e^{\frac{i2\pi a(r-c)}{\ell}}=\ell$ if $\ell\mid r-c$ and 0 otherwise, we conclude that

$$b_{r-c}=\begin{cases}0 & \text{if } \ell\nmid r-c,\\ a_{(r-c)/\ell} & \text{otherwise.}\end{cases}$$

We recall that $N=q\ell+j$, where $0\le j\le\ell-1$. For $0\le k\le j-1$, we then obtain that $\beta_{m\ell+k+1,n\ell+k+1}=a_{m-n}$, where $0\le m,n\le q$, and for $j\le k\le\ell-1$,

$\beta_{m\ell+k+1,n\ell+k+1} = a_{m-n}$, where $0 \le m, n \le q-1$. Moreover, $\beta_{r\ell+k,t} = \beta_{t,c\ell+k} = 0$ when $t \not\equiv k \pmod{\ell}$. In particular, B is of the form

$$\begin{pmatrix} a_0 I_\ell & a_{-1} I_\ell & \dots & a_{-q+1} I_\ell & a_{-q} D_{\ell,j} \\ a_1 I_\ell & a_0 I_\ell & \dots & a_{-q+2} I_\ell & a_{-q+1} D_{\ell,j} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{q-1} I_\ell & a_{q-2} I_\ell & \dots & a_0 I_\ell & a_{-1} D_{\ell,j} \\ a_q E_{j,\ell} & a_{q-1} E_{j,\ell} & \dots & a_1 E_{j,\ell} & a_0 I_j \end{pmatrix},$$

where I_k is the $k \times k$ identity matrix, $D_{\ell,j} = \begin{pmatrix} I_j \\ O_{\ell-j,j} \end{pmatrix}$, $E_{j,\ell} = \begin{pmatrix} I_j & O_{j,\ell-j} \end{pmatrix}$, and $O_{m,n}$ is the $m \times n$ zero matrix.

We then permute rows and columns of B so that row/column $k+1, \ell+k+1, 2\ell+k+1, \dots, q\ell+k+1$, where $0 \le k \le j-1$, become row/column $k(q+1)+1, k(q+1)+2, \dots, (k+1)(q+1)$, and row/column $k+1, \ell+k+1, 2\ell+k+1, \dots, (q-1)\ell+k+1$, where $j \le k \le \ell-1$, become row/column $(q+1)j+(k-j)q+1, (q+1)j+(k-j)q+2, \dots, (q+1)j+(k-j+1)q$. Hence $\det_{N\times N} B$ is

$$\det_{N\times N} \begin{pmatrix} A_{q+1} & O & \dots & O & O & \dots & O \\ O & A_{q+1} & \dots & O & O & \dots & O \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ O & O & \dots & A_{q+1} & O & \dots & O \\ O & O & \dots & O & A_q & \dots & O \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ O & O & \dots & O & O & \dots & A_q \end{pmatrix},$$

where O is the zero matrix. The diagonal of the first j blocks of the above matrix is A_{q+1}, and the diagonal of the last $\ell - j$ blocks is A_q, where A_M is defined as in (3.1). Therefore the determinant of the matrix is $g^j(k, q+1) g^{\ell-j}(k, q)$. This proves Theorem 1.4. □

4. Proof of Theorem 1.5: Selberg's formula

To prove Theorem 1.5, we need the following lemmas. The first one expresses $\sin \ell\theta$ in terms of $\cot \theta$.

Lemma 4.1. *If $x = \cot\theta$, then*

$$\sin \ell\theta = \frac{1}{(x^2+1)^{\ell/2}} \sum_{k=0}^{\lfloor \frac{\ell-1}{2} \rfloor} \binom{\ell}{2k+1} (-1)^k x^{\ell-(2k+1)}. \tag{4.1}$$

Proof. We will prove the lemma by induction. First, the expression is true when $\ell = 1, 2$ because $\sin\theta = \frac{1}{(x^2+1)^{1/2}}$, and $\sin 2\theta = \frac{2x}{1+x^2}$. Assume that equation (4.1)

is true for all $\ell \leq q$. From the relation

$$\sin(q+1)\theta = 2\cos\theta\sin q\theta - \sin(q-1)\theta,$$

we have

$$\begin{aligned}
&\sin(q+1)\theta = 2\cos\theta\sin q\theta - \sin(q-1)\theta \\
= \quad & \frac{2x}{(x^2+1)^{\frac{q+1}{2}}}\sum_{k=0}^{\lfloor\frac{q-1}{2}\rfloor}\binom{q}{2k+1}(-1)^k x^{q-(2k+1)} \\
& -\frac{1}{(x^2+1)^{\frac{q-1}{2}}}\sum_{k=0}^{\lfloor\frac{q-2}{2}\rfloor}\binom{q-1}{2k+1}(-1)^k x^{q-1-(2k+1)} \\
= \quad & \frac{1}{(x^2+1)^{\frac{q+1}{2}}}\Bigg[\sum_{k=0}^{\lfloor\frac{q-1}{2}\rfloor}2\binom{q}{2k+1}(-1)^k x^{q+1-(2k+1)} \\
& \qquad -\sum_{k=0}^{\lfloor\frac{q-2}{2}\rfloor}\binom{q-1}{2k+1}(-1)^k x^{q+1-(2k+1)} \\
& \qquad -\sum_{k=0}^{\lfloor\frac{q-2}{2}\rfloor}\binom{q-1}{2k+1}(-1)^k x^{q-1-(2k+1)}\Bigg].
\end{aligned}$$

To compute the expression above, we consider two cases depending on q.

Case 1: *q is even.* For this case, $\lfloor\frac{q-1}{2}\rfloor = \lfloor\frac{q-2}{2}\rfloor = \frac{q-2}{2}$. Rearranging terms, we obtain that $\sin(q+1)\theta$ is

$$\begin{aligned}
& \frac{1}{(x^2+1)^{\frac{q+1}{2}}}\Bigg[\binom{q+1}{1}x^q + \sum_{k=1}^{\lfloor\frac{q-1}{2}\rfloor}(-1)^k x^{q+1-(2k+1)} \\
& \qquad \cdot\left(2\binom{q}{2k+1} - \binom{q-1}{2k+1} + \binom{q-1}{2k-1}\right) \\
& \qquad +(-1)^{q/2}\binom{q+1}{q+1}\Bigg] \\
= \quad & \frac{1}{(x^2+1)^{\frac{q+1}{2}}}\Bigg[\binom{q+1}{1}x^q + \sum_{k=1}^{\lfloor\frac{q-1}{2}\rfloor}(-1)^k x^{q+1-(2k+1)}\binom{q+1}{2k+1} \\
& \qquad +(-1)^{q/2}\binom{q+1}{q+1}\Bigg] \\
= \quad & \frac{1}{(x^2+1)^{\frac{q+1}{2}}}\sum_{k=0}^{\lfloor\frac{q}{2}\rfloor}\binom{q+1}{2k+1}(-1)^k x^{q+1-(2k+1)}\ .
\end{aligned}$$

Case 2: *q is odd.* For this case, $\lfloor \frac{q-1}{2} \rfloor = \lfloor \frac{q-2}{2} \rfloor + 1 = \frac{q-1}{2}$. Rearranging terms, we obtain that $\sin(q+1)\theta$ is

$$\begin{aligned}
&\frac{1}{(x^2+1)^{\frac{q+1}{2}}}\Bigg[\binom{q+1}{1}x^q + \sum_{k=1}^{\lfloor \frac{q-3}{2} \rfloor}(-1)^k x^{q+1-(2k+1)} \\
&\qquad \cdot\left(2\binom{q}{2k+1} - \binom{q-1}{2k+1} + \binom{q-1}{2k-1}\right) \\
&\qquad + (-1)^{\frac{q-1}{2}}\binom{q+1}{q}x\Bigg] \\
= \;& \frac{1}{(x^2+1)^{\frac{q+1}{2}}}\Bigg[\binom{q+1}{1}x^q + \sum_{k=1}^{\lfloor \frac{q-3}{2} \rfloor}(-1)^k x^{q+1-(2k+1)}\binom{q+1}{2k+1} \\
&\qquad + (-1)^{\frac{q-1}{2}}\binom{q+1}{q}x\Bigg] \\
= \;& \frac{1}{(x^2+1)^{\frac{q+1}{2}}}\sum_{k=0}^{\lfloor \frac{q}{2} \rfloor}\binom{q+1}{2k+1}(-1)^k x^{q+1-(2k+1)} .
\end{aligned}$$

□

Next, we express $S_{2k}\left(N; 1, \frac{2\pi}{\ell}, \ldots, \frac{2(\ell-1)\pi}{\ell}\right)$ as the integral forms analogous to Selberg's formula.

Lemma 4.2. *Let $S_{2k}\left(N; 1, \frac{2\pi}{\ell}, \ldots, \frac{2(\ell-1)\pi}{\ell}\right)$ be defined as in* (1.10). *Then we have*

$$\begin{aligned}
&S_k\left(N; 1, \frac{2\pi}{\ell}, \ldots, \frac{2(\ell-1)\pi}{\ell}\right) \\
&\quad = \frac{2^{2Nk+N^2}}{N!(2\pi)^N}\int\cdots\int_{[-\infty,\infty]^N} \\
&\quad \cdot \prod_{1\le j<n\le N}|x_n - x_j|^2 \prod_{m=1}^{N}\frac{\left(\sum_{t=0}^{\lfloor \frac{\ell-1}{2} \rfloor}\binom{\ell}{2t+1}(-1)^t x_m^{\ell-(2t+1)}\right)^{2k}}{(x_m^2+1)^{k\ell+N}}\,dx_1\ldots dx_N.
\end{aligned}$$

Proof. From the definition of S_{2k}, we have

$$\begin{aligned}
&S_{2k}\left(N;1,\frac{2\pi}{\ell},\dots,\frac{2(\ell-1)\pi}{\ell}\right)\\
=\;&\frac{1}{N!(2\pi)^N}\int_{[0,2\pi]^N}\prod_{m=1}^{N}\left(\left|1-e^{i\theta_m}\right|\left|1-e^{i(\theta_m+\frac{2\pi}{\ell})}\right|\dots\left|1-e^{i(\theta_m+\frac{2(\ell-1)\pi}{\ell})}\right|\right)^{2k}\\
&\qquad\cdot\prod_{1\le j<n\le N}|e^{i\theta_n}-e^{i\theta_j}|^2\,d\theta_1\dots d\theta_N\\
=\;&\frac{1}{N!(2\pi)^N}\int_{[0,2\pi]^N}\prod_{1\le j<n\le N}|e^{i\theta_n}-e^{i\theta_j}|^2\prod_{m=1}^{N}|1-e^{i\ell\theta_m}|^{2k}\,d\theta_1\dots d\theta_N\\
=\;&\frac{2^{2Nk+N^2}}{N!(2\pi)^N}\int_{[0,\pi]^N}\prod_{1\le j<n\le N}|\sin(\theta_n-\theta_j)|^2\prod_{m=1}^{N}|\sin(\ell\theta_m)|^{2k}\,d\theta_1\dots d\theta_N.
\end{aligned}$$

Now using $\sin(\alpha-\beta)=(\cot\beta-\cot\alpha)\sin\alpha\sin\beta$, we find that

$$\begin{aligned}
&S_{2k}\left(N;1,\frac{2\pi}{\ell},\dots,\frac{2(\ell-1)\pi}{\ell}\right)\\
&\quad=\frac{2^{2Nk+N^2}}{N!(2\pi)^N}\int_{[0,\pi]^N}\prod_{1\le j<n\le N}|\cot\theta_n-\cot\theta_j|^2\\
&\qquad\cdot\prod_{m=1}^{N}|\sin(\ell\theta_m)|^{2k}|\sin\theta_m|^{2N-2}\,d\theta_1\dots d\theta_N.
\end{aligned}$$

Let $x_m=\cot\theta_m$. Applying Lemma 4.1, we obtain that

$$\begin{aligned}
&S_{2k}\left(N;1,\frac{2\pi}{\ell},\dots,\frac{2(\ell-1)\pi}{\ell}\right)\\
&\quad=\frac{2^{2Nk+N^2}}{N!(2\pi)^N}\int\cdots\int_{[-\infty,\infty]^N}\prod_{1\le j<n\le N}|x_n-x_j|^2\\
&\qquad\cdot\prod_{m=1}^{N}\frac{\left(\sum_{t=0}^{\lfloor\frac{\ell-1}{2}\rfloor}\binom{\ell}{2t+1}(-1)^t x_m^{\ell-(2t+1)}\right)^{2k}}{(x_m^2+1)^{k\ell+N}}\,dx_1\dots dx_N,
\end{aligned}$$

as desired. □

Finally, Theorem 1.5 follows easily from Theorem 1.4 and Lemma 4.2.

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