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DIOPHANTINE PROPERTIES OF ORTHOGONAL POLYNOMIALS AND RATIONAL FUNCTIONS

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Abstract. Calogero and his collaborators recently observed that some hypergeometric polynomials can be factored as a product of two polynomials, one of which is factored into a product of linear terms. Chen and Ismail showed that this property prevails through all polynomials in the Askey scheme. We show that this factorization property is also shared by the associated Wilson and Askey-Wilson polynomials and some biorthogonal rational functions. This is applied to a specific model of an isochronous system of particles with small oscillations around the equilibrium position.

1. Introduction

Consider a mechanical system with N degrees of freedom and time independent constraints which is acted upon by forces derivable from a potential U. The coordinates are $q_i, 1 \leq i \leq N$. We denote $\frac{dx}{dt}$ by $\dot{x}$. The Lagrange equations of such a system take the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = \frac{\partial U}{\partial q_k}, k = 1, 2, \cdots, N, \tag{1.1}$$

where T is the kinetic energy given by the quadratic form

$$T = \sum_{j,k=1}^{N} a_{j,k}\dot{q}_j\dot{q}_k. \tag{1.2}$$

In applications T and U below are assumed to be real symmetric, and T is assumed to be positive definite. Our analysis only requires that T and U be real quadratic forms. We assume that the equilibrium is at $q_1 = q_2 = \cdots = q_N = 0$ and $\frac{\partial U}{\partial q_k} = 0, 1 \leq k \leq N$ at equilibrium. This makes

$$-U = \sum_{j,k=1}^{N} b_{j,k}q_jq_k \tag{1.3}$$

around the equilibrium. Now we assume that the small oscillations around the equilibrium are isochronous; that is, each component is periodic with the same

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period (hence have the same frequency). Then the displacements take the form

$$q_k = A_k \cos(\lambda t + \phi), 1 \le k \le N, \tag{1.4}$$

and we transform (1.1) to the eigenvalue equation

$$\det\,(U + U^T - \lambda^2(T + T^T)) = 0, \tag{1.5}$$

where $T = (a_{j,k})$ and $U = (b_{j,k})$ are the matrices of the quadratic forms T and U, respectively, and A^T is the transpose of A. On the other hand, the simultaneous diagonalization of the quadratic forms T and U reduces (1.1) to $\frac{d^2}{dt^2}q_k + \lambda_k^2 q_k = 0$, whose solution is of the form (1.4); see [21].

When we assume next neighbor interaction the matrices $\tilde{T}$ and $\tilde{U}$ become tridiagonal.

Calogero and his collaborators investigated this problem when T is diagonal and $U + U^T$ tridiagonal; see for example [2], [3], and [4]. In this case (1.5) becomes

$$\det(\lambda\, \mathsf{I}_N - \mathsf{U}_N) = 0.$$

We may take N to be the number of particles in the many-body problem. See [7] for a detailed treatment. Hence

$$P_N(\lambda) := \det(\lambda \mathsf{I}_N - \mathsf{A}_N)$$

may be interpreted as orthogonal polynomials if A_N is symmetric and the super-diagonal elements of A_N are real and none of them vanishes, [12]. We show that the Diophantine property occurs in degenerate cases when the parameters of the orthogonal polynomials are suitably chosen. Thus the factorization occurs when the polynomials, although still characteristic polynomials of tridiagonal matrices, are no longer orthogonal.

A sequence of monic orthogonal polynomials satisfies a three term recurrence relation

$$xP_n(x) = P_{n+1}(x) + \alpha_n P_n(x) + \beta_n P_{n-1}(x), n > 0, \tag{1.6}$$

with $P_0(x) := 1, P_1(x) := x - \alpha_0$. The monic polynomials $\{P_n(x)\}$ have the determinant representation

$$P_n(x) = \begin{vmatrix} x-\alpha_0 & -a_1 & 0 & \cdots & 0 & 0 & 0 \\ -a_1 & x-\alpha_1 & -a_2 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & & -a_{n-2} & x-\alpha_{n-2} & -a_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & -a_{n-1} & x-\alpha_{n-1} \end{vmatrix}, \tag{1.7}$$

where $a_n^2 = \beta_n, n > 0$. It is clear that if $\beta_k = 0$ for some $k < n$, then $P_n(x)$ factors into a product of two polynomials of the same type, that is, a product of two characteristic polynomials of tridiagonal matrices. All the factorizations in the work of Bruschi, Calogero, and Droghei are of this type. What is surprising is that one of the two characteristic polynomials has equi-spaced zeros in the cases studied by the Calogero team. One referee pointed out that some interesting cases of (1.6) occur when some β_n's are negative and the P's are no longer orthogonal. In such cases we take $a_j = \pm\sqrt{-\beta_j}$, and use one sign in the super-diagonal and the opposite sign in the diagonal below the main diagonal to make the tridiagonal matrix Hermitian.

Consider a sequence of monic polynomials satisfying (1.6) with initial values $P_0(x) = 1, P_1(x) = x - \alpha_0$. We shall consider the case when $\beta_n < 0, 0 < n < m, \beta_m = 0$ and $\beta_n > 0, n > m$. It is easy to see by induction that

$$P_n(x) = P_m(x)Q_{n-m}(x), n \geq m, \tag{1.8}$$

where $\{Q_n(x)\}$ are generated by

$$\begin{gathered} Q_0(x) = 1, Q_1(x) = x - \alpha_m, \\ Q_{j+1}(x) = (x - \alpha_{m+j})Q_j(x) - \beta_{m+j}Q_{j-1}(x). \end{gathered} \tag{1.9}$$

Instances of this factorization have appeared in many references and one goes back to Szegő's book [24, (4.22.2)] and [24, (5.2.1)]. In [5] and [10] it was shown that when the P_n's are Wilson polynomials with parameters $t_1, t_2, 1-m-t_4, t_4$ the Q_n's are Wilson polynomials with parameters $t_1, t_2, 1-t_4, t_4+m$ and all the zeros of P_m can be explicitly found. It seems unlikely that the P_n's and Q_n's will come from the same family of polynomials with different parameters, but this is exactly what happens for the classical polynomials, that is, the polynomials from the Askey scheme [17]. It is also remarkable that the zeros of P_m can be explicitly found. Additional relevant references are [6], [8], and [9].

Chen and Ismail [10] showed that the factorizations encountered by Calogero and his team are instances of the Whipple transformation [20] or special cases of the so-called Carlsson-Minton formulas, [11]. They also proved q-analogues of the results of Calogero et al. using the Sears transformation, [11]. This covers all the special polynomials in the Askey scheme [17]. They also pointed out that the zeros of the explicit polynomials in the factorization are precisely interpolation points in the q-Taylor expansions in [16].

The present work goes beyond the Askey scheme and shows that the factorization phenomenon continues to hold. In §2 the results in [10] are extended to the ${}_{10}\phi_9$ biorthogonal rational functions. The limiting case $q \to 1$ establishes the corresponding results for the ${}_9F_8$ biorthogonal rational functions. In §3 we use the three term recurrence relation to establish a factorization of any polynomial solution to the recurrence relation of our associated Askey-Wilson polynomials ([15]) or the associated Wilson polynomials [14]. In §4 we explicitly give the factorization alluded to in §3 through the use of transformations and identities of basic hypergeometric series. Section 5 contains the factorization of the Rahman biorthogonal rational functions. We use a three term recurrence relation for the Rahman functions and postpone its technical proof to §6, in an appendix. The recursion relation is stated in Theorem 5.1.

2. Biorthogonal rational functions

Recall that the q-shifted factorial is

$$(a;q)_n = \prod_{k=1}^{n}(1 - aq^{k-1}), \quad (a_1, a_2, \cdots, a_m; q)_n = \prod_{j=1}^{m}(a_j;q)_n, \tag{2.1}$$

and a basic hypergeometric function is

$${}_{r+1}\phi_r \left(\begin{matrix} a_1, a_2, \cdots, a_{r+1} \\ b_1, b_2, \cdots, b_r \end{matrix} \middle| q, z \right) = \sum_{n=0}^{\infty} \prod_{k=1}^{r+1} \frac{(a_k;q)_n}{(b_{k-1};q)_n} z^n, \tag{2.2}$$

where $b_0 := q$. This is the notation in [1] and [11], which we shall follow. We also use the notation

$$
\begin{aligned}
&{}_{10}W_9(A; A_1, \cdots, A_7; q, Z) \\
(2.3) \qquad &= {}_{10}\phi_9\left(\left.\begin{matrix} A, \; q\sqrt{A}, \; -q\sqrt{A}, \; A_1, \; A_2, \cdots, \; A_7 \\ \sqrt{A}, -\sqrt{A}, qA/A_1, qA/A_2, \cdots, qA/A_7 \end{matrix}\right| q, Z\right).
\end{aligned}
$$

Consider the rational functions [11, Ex. 8.3.1]

$$
\begin{aligned}
(2.4) \qquad R_N(\cos\theta; a, b, c, d, f) &:= \frac{(a^2bcdf, bcdf, de^{i\theta}, de^{-i\theta}; q)_N}{(ad, d/a, abcdfe^{i\theta}; abcdfe^{-i\theta}; q)_N} \\
&\times {}_{10}W_9(aq^{-N}/d; q^{1-N}/bd, q^{1-N}/cd, 1/df, abcf, ae^{i\theta}, ae^{-i\theta}, q^{-N}; q, q).
\end{aligned}
$$

Case I. We choose $d = q^{1-N}/c$. In this case the ${}_{10}W_9$ becomes 1, and we find the complete factorization

$$
\begin{aligned}
(2.5) \qquad R_N(\cos\theta; a, b, q^{1-N}/c, d, f) &= \frac{(a^2bcdf, bcdf, de^{i\theta}, de^{-i\theta}; q)_N}{(ad, d/a, abcdfe^{i\theta}; abcdfe^{-i\theta}; q)_N} \\
&= \frac{(1/bf, a^{-2}/bf; q)_N (de^{i\theta}, de^{-i\theta}; q)_N}{(ad, d/a; q)_N (e^{i\theta}/abf, e^{-i\theta}/abf; q)_N}.
\end{aligned}
$$

Case II. This is a case of partial factorization. We set $d = q^{1-m}/c$ for $0 \leq m \leq N$. In this case we apply the Bailey ${}_{10}\phi_9$ transformation formula, [11, (III.28)], and conclude that the ${}_{10}W_9$ equals

$$
\begin{aligned}
&\frac{(aq^{1-N}/d, fq^{1-N}e^{i\theta}, fq^{1-N}e^{-i\theta}, q^{1-N}/ad; q)_{N-m}}{(afq^{1-n}, q^{1-N}e^{-i\theta}/d, q^{1-N}e^{i\theta}/d, fq^{1-N}/ad; q)_{N-m}} \\
&\times {}_{10}W_9(aq^{m-1}/f; q^m/bf, dq^{m-1}/f, abq^N, 1/df, ae^{i\theta}, ae^{-i\theta}, q^{m-N}; q, q).
\end{aligned}
$$

The factors in front of the ${}_{10}W_9$ simplify to

$$
\frac{(dq^m/a, adq^m, fq^m e^{i\theta}, fq^m e^{-i\theta}; q)_{N-m}}{(q^m/af, dq^m e^{i\theta}, dq^m e^{-i\theta}, aq^m/f; q)_{N-m}}.
$$

This proves the following theorem.

Theorem 2.1. *We have the partial factorization*

$$
\begin{aligned}
R_N(\cos\theta; a, b, q^{1-m}/c, d, f) \; &= \frac{(a^2bfq^{1-m}, bfq^{1-m}; q)_N}{(abfq^{1-m}e^{i\theta}, abfq^{1-m}e^{-i\theta}; q)_N} \\
&\times \frac{(a/f, 1/af; q)_m}{(a/f, 1/af; q)_N} \frac{(de^{i\theta}, de^{-i\theta}; q)_m}{(ad, d/a; q)_m} \frac{(fe^{i\theta}, fe^{-i\theta}; q)_N}{(fe^{i\theta}, fe^{-i\theta}; q)_m} \\
&\times {}_{10}W_9(aq^{m-1}/f; q^m/bf, dq^{m-1}/f, abq^N, 1/df, ae^{i\theta}, ae^{-i\theta}, q^{m-N}; q, q).
\end{aligned}
$$

Ismail and Masson [13] introduced the R_{II} continued fractions which are associated with the three term recurrence relation

$$
(2.6) \qquad (x - c_n)P_n = (x - a_{n+1})P_{n+1} + \beta_n(x - b_n)P_{n-1}.
$$

They also showed that there are rational functions $\{P_n\}$ that satisfy the orthogonality relation $\mathcal{L}(P_n\pi_m) = 0$, for all polynomials π_m of degree $m, m < n$, where $\mathcal{L}$ is a linear functional associated with the three term recurrence relation (2.6). They showed how to construct a second family which is biorthogonal to the P_n's. Zhedanov [23] observed that (2.6) can be written as the generalized eigenvalue equation $A\mathbf{X} = xT\mathbf{X}$, where A and T are matrices and $\mathbf{X}$ is a vector. He also showed that the second family comes from the adjoint problem; see also [22].

Our thesis is that the general Calogero problem indeed fits the R_{II} model and that each specific set of R_{II} biorthogonal rational functions corresponds to a tridiagonal isochronous model.

3. Factorization of Wilson and Askey-Wilson polynomials

Before we consider the associated-Wilson polynomials, introduced in [14], we say a few words about the Wilson polynomials

$$\begin{aligned} W_n(x;\mathbf{t}) &= \prod_{j=2}^{4}(t_1+t_j)_n \\ &\times {}_4F_3\left(\left.\begin{matrix} -n, n+t_1+t_2+t_3+t_4-1, t_1+i\sqrt{x}, t_1-i\sqrt{x} \\ t_1+t_2, t_1+t_3, t_1+t_4, \end{matrix}\right| 1\right), \end{aligned} \tag{3.1}$$

where $\mathbf{t}$ stands for the vector (t_1, t_2, t_3, t_4). The Wilson polynomials are symmetric in the four parameters $t_j, 1 \le j \le 4$.

The associated Wilson polynomials contain an additional parameter α, which appears in the three term recurrence relation after we replace the degree parameter n by $n+\alpha$ and adjust the initial conditions. We let

$$\begin{aligned} A_n(\alpha) &= \frac{(n+\alpha-1+\sum_{j=1}^{4} t_j)\prod_{j=2}^{4}(n+\alpha+t_1+t_j)}{(2n-1+\sum_{j=1}^{4} t_j)(2n+\sum_{j=1}^{4} t_j)}, \\ C_n(\alpha) &= \frac{(n+\alpha)\prod_{2\le j<k\le 4}(n+\alpha+t_j+t_k-1)}{(2n-2+\sum_{j=1}^{4} t_j)(2n-1+\sum_{j=1}^{4} t_j)}. \end{aligned} \tag{3.2}$$

The monic polynomials $\{P_n(x;\mathbf{t},\alpha)\}$ satisfy

$$\begin{aligned} (x+t_1^2)P_n(x;\mathbf{t},\alpha) &= P_{n+1}(x;\mathbf{t},\alpha) - [A_n(\alpha)+C_n(\alpha)]P_n(x;\mathbf{t},\alpha) \\ &\quad + A_{n-1}(\alpha)C_n(\alpha)P_{n-1}(x;\mathbf{t},\alpha). \end{aligned} \tag{3.3}$$

Ismail, Letessier, Valent and Wimp [14] introduced two families of associated Wilson polynomials. Both satisfy (3.3) and $P_0(x) = 1$. The difference is that $P_1(x)$ is either $x + t_1^2 + A_0(\alpha) + C_0(\alpha)$ or $x + t_1^2 + A_0(\alpha)$.

When $\alpha = 0$, that is, in the Wilson case, the monic polynomials are

$$P_n(x:\mathbf{t}) := \frac{(-1)^n}{(n-1+\sum_{j=1}^{4} t_j)_n}\, W_n(x;t_1,t_2,t_3,t_4). \tag{3.4}$$

For both families of associated Wilson polynomials $\beta_n = A_{n-1}(\alpha)C_n(\alpha)$. The choice $t_3 = 1-m-t_4-\alpha$, for a positive integer m, makes $\beta_m = 0$. If $t_1 > 0, t_2 > 0, \alpha \ge 0, t_4 \in (0,1)$, then $\beta_n > 0$ for $n > m$. Moreover it follows that

$$\begin{aligned} \beta_{n+m} = &\frac{(n+m+\alpha)(n+t_1+t_2-1)(n+m+\alpha+t_1+t_2-1)}{(2n+m+\alpha+t_1+t_2-2)_3(2n+m+\alpha+t_1+t_2-1)} \\ &\times n(n+m+\alpha+t_1+t_4-1)(n+m+\alpha+t_2+t_4-1) \\ &\times (n+t_1-t_4)(n+t_2-t_4). \end{aligned}$$

Let A'_n and C'_n be the A_n and C_n for a Wilson polynomial with parameters $t_1, t_2, 1-t_4, m+\alpha+t_4$. A simple calculation shows that $A'_n = A_{n+\alpha}$ and $C'_n = C_{n+\alpha}$. This establishes the following theorem.

Theorem 3.1. *Let* $\{\phi_n(x)\}$ *be a sequence of monic polynomials generated by* (3.3) *for* $n > 0$ *and* $\phi_0(x) = 1, \phi_1(x) = Ax + B, A \neq 0$. *Assume that* $t_3 = 1 - t_4 - \alpha - m$. *Then* $\phi_{n+m}(x)$ *is divisible by* $W_n(x, t_1, t_2, 1 - t_4, m + \alpha + t_4)$.

We now come to the Askey-Wilson and associated Askey-Wilson polynomials. The Askey-Wilson polynomials are

$$P_n(x; \mathbf{t}) = t_1^{-n} \prod_{j=2}^{4} (t_1 t_j; q)_n \times {}_4\phi_3 \left(\begin{matrix} q^{-n}, q^{n-1} t_1 t_2 t_3 t_4, t_1 e^{i\theta}, t_1 e^{-i\theta} \\ t_1 t_2, \quad t_1 t_3, \quad t_1 t_4, \end{matrix} \middle| q, q \right). \tag{3.5}$$

The associated Askey-Wilson polynomials were introduced by the present authors in [15] as a one-parameter generalization of the Askey-Wilson polynomials. As we pointed out in our above-mentioned paper there are two natural families of associated Askey-Wilson polynomials defined by different initial conditions. The three term recurrence relation for associated Askey-Wilson polynomials is

$$\begin{aligned} 2x y_n(x) &= y_{n+1}(x) + A_{n-1}(\alpha) C_n(\alpha) y_{n-1}(x) \\ &+ [t_1 + 1/t_1 - (A_n(\alpha) + C_n(\alpha))]\, y_n(x), \end{aligned} \tag{3.6}$$

and the coefficients are given by

$$\begin{aligned} A_n(\alpha) &= \frac{(1 - t_1 t_2 t_3 t_4 q^{n+\alpha-1}) \prod_{j=2}^{4} (1 - q^{n+\alpha} t_1 t_j)}{t_1 (1 - t_1 t_2 t_3 t_4 q^{2n+2\alpha-1})(1 - t_1 t_2 t_3 t_4 q^{2n+2\alpha})}, \\ C_n(\alpha) &= \frac{t_1 (1 - q^{n+\alpha}) \prod_{2 \leq j < k \leq 4} (1 - t_j t_k q^{n+\alpha-1})}{(1 - t_1 t_2 t_3 t_4 q^{2n+2\alpha-2})(1 - t_1 t_2 t_3 t_4 q^{2n+2\alpha-1})}. \end{aligned} \tag{3.7}$$

Instead of the specific initial conditions in our paper [15] we choose the general initial conditions $\psi_0(x; \mathbf{t}, \alpha) = 1, \psi_1(x; \mathbf{t}, \alpha) = Ax + B, A \neq 0$.

We now take $t_3 = q^{1-m-\alpha}/t_4$. Therefore

$$\begin{aligned} A_{n+m}(\alpha) &= \frac{(1 - t_1 t_2 q^n)(1 - q^{n+m+\alpha} t_1 t_2)(1 - q^{n+m+\alpha} t_1 t_4)(1 - q^{n+1} t_1/t_4)}{t_1 (1 - t_1 t_2 q^{2n+\alpha+m})(1 - t_1 t_2 q^{n+\alpha+m+1})}, \\ C_{n+m}(\alpha) &= \frac{t_1 (1 - q^n)(1 - q^{n+m+\alpha})(1 - q^n t_2/t_4)(1 - t_2 t_4 q^{n+m+\alpha-1})}{(1 - t_1 t_2 q^{2n+m+\alpha-1})(1 - t_1 t_2 q^{2n+m+\alpha})}. \end{aligned}$$

It is evident that $A_{m-1}(\alpha) C_m(\alpha) = 0$; hence $\psi_{n+m}(x; \mathbf{t}, \alpha)$ factors and we establish the following factorization theorem.

Theorem 3.2. *A general associated Askey-Wilson polynomial*

$$\psi_{n+m}(x; t_1, t_2, q^{1-m-\alpha}/t_4, t_4, \alpha)$$

is divisible by the Askey-Wilson polynomial $P_n(x; t_1, t_2, q/t_4, q^{m+\alpha} t_4)$.

4. Associated Askey–Wilson polynomials

In this section we first consider the case $t_3 t_4 = q^{1-m-\alpha}$. Later we further assume that $t_1 t_2 = q^{m+1}$. Recall that the two families of the associated Askey-Wilson polynomials of [15] have the explicit representations

$$
\begin{aligned}
&p_n^{(\alpha)}(\cos\theta, \mathbf{t}\,|\,q) = (t_1t_2q^\alpha, t_1t_3q^\alpha, t_1t_4q^\alpha; q)_n t_1^{-n} \\
&\times \sum_{k=0}^{n} \frac{(q^{-n}, t_1t_2t_3t_4q^{2\alpha+n-1}, t_1t_2t_3t_4q^{2\alpha-1}, t_1e^{i\theta}, t_1e^{-i\theta}; q)_k}{(q, t_1t_2q^\alpha, t_1t_3q^\alpha, t_1t_4q^\alpha, t_1t_2t_3t_4q^{\alpha-1}; q)_k} q^k \\
&\times {}_{10}W_9\big(t_1t_2t_3t_4q^{2\alpha+k-2}; q^\alpha, t_2t_3q^{\alpha-1}, t_2t_4q^{\alpha-1}, t_3t_4q^{\alpha-1}, \\
&\qquad q^{k+1}, q^{k-n}, t_1t_2t_3t_4q^{2\alpha+n+k-1}; q, t_1^2\big)
\end{aligned}
\tag{4.1}
$$

and

$$
\begin{aligned}
&q_n^{(\alpha)}(\cos\theta, \mathbf{t}\,|\,q) = (t_1t_2q^\alpha, t_1t_3q^\alpha, t_1t_4q^\alpha; q)_n t_1^{-n} \\
&\times \sum_{k=0}^{n} \frac{(q^{-n}, t_1t_2t_3t_4q^{2\alpha+n-1}, t_1t_2t_3t_4q^{2\alpha-1}, t_1e^{i\theta}, t_1e^{-i\theta}; q)_k}{(q, t_1t_2q^\alpha, t_1t_3q^\alpha, t_1t_4q^\alpha, t_1t_2t_3t_4q^{\alpha-1}; q)_k} q^k \\
&\times {}_{10}W_9\big(t_1t_2t_3t_4q^{2\alpha+k-2}; q^\alpha, t_2t_3q^{\alpha-1}, t_2t_4q^{\alpha-1}, t_3t_4q^{\alpha-1}, \\
&\qquad q^{k}, q^{k-n}, t_1t_2t_3t_4q^{2\alpha+n+k-1}; q, qt_1^2\big).
\end{aligned}
\tag{4.2}
$$

Theorem 4.1. *For* $n \ge m$ *we have the factorization*

$$
\begin{aligned}
p_n^\alpha(x; t_1, t_2, q^{-m+1-\alpha}/t_4, t_4) &= \frac{(t_1t_2q^{\alpha-m}; q)_n (t_4q^\alpha e^{i\theta}, t_4q^\alpha e^{-i\theta}; q)_m}{(q, t_1t_2q^{-m}, t_1t_2q^\alpha; q)_n} q^{-\alpha m} \\
&\times {}_4\phi_3\left(\left.\begin{matrix} q^{-m}, q^\alpha, t_1t_4q^{\alpha-1}, t_2t_4q^{\alpha-1} \\ t_1t_2q^{\alpha-m-1}, t_4q^\alpha e^{i\theta}, t_4q^\alpha e^{-i\theta}\end{matrix}\,\right|\, q, q\right) \\
&\times p_{n-m}(x, t_1, t_2, q/t_4, q^{m+\alpha}t_4).
\end{aligned}
\tag{4.3}
$$

Proof. In the proof we will repeatedly use the identity

$$
\frac{(A;q)_i}{(q^{1-s}/A;q)_s} = (-A)^s q^{\binom{s}{2}} (q^sA;q)_{i-s}, \quad i \ge s. \tag{4.4}
$$

With $t_3 = q^{1-m-\alpha}/t_4$, Exercise 2.20 in [11] shows that the ${}_{10}W_9$ in (4.1) has the representation

$$
\begin{aligned}
&\frac{(t_1t_2q^{\alpha+k-m}, q^{-k}; q)_m}{(q^{-n}, t_1t_2q^{\alpha+n-m}; q)_m} \sum_{i=0}^{m} \frac{(q^{-m}, q^{k-n}, t_1t_2q^{\alpha+n-m+k}, t_1q^{k+1}/t_2; q)_i}{(q, t_1q^{k-m+1}/t_4, t_1t_4q^{\alpha+k}, q^{k+1-m}; q)_i} q^i \\
&\qquad \times {}_4\phi_3\left(\left.\begin{matrix} q^{-i}, t_2t_4q^{\alpha-1}, t_2q^{-m}/t_4, t_1t_2q^{-m-1} \\ t_1t_2q^{\alpha-m-1}, t_1t_2q^{k-m}, t_2q^{-k-i}/t_1\end{matrix}\,\right|\, q, q\right) \\
&= \frac{(t_1t_2q^{\alpha+k-m}; q)_m (-1)^m}{(q^{-n}, t_1t_2q^{\alpha+n-m}; q)_m} q^{\binom{m}{2}} \sum_{i \ge j} \frac{(q^{-m}, q^{k-n}, t_1t_2q^{\alpha+n-m+k}; q)_i q^i}{(t_1q^{k-m+1}/t_4, t_1t_4q^{\alpha+k}; q)_i (q;q)_{i+k-m}} \\
&\times \frac{(t_1q^{k+1}/t_2; q)_{i-j}}{(q;q)_{i-j}} \left(\frac{qt_1}{t_2}\right)^j q^{k(j-m)} \frac{(t_2t_4q^{\alpha-1}, t_2q^{-m}/t_4, t_1t_2q^{-m-1}; q)_j}{(q, t_1t_2q^{\alpha-m-1}, t_1t_2q^{k-m}; q)_j}.
\end{aligned}
$$

We now replace the ${}_{10}W_9$ in (4.1) by the above expression. The restrictions on the summation indices are: $m \ge i, k+i-m \ge 0, i \ge j$, so we set $k = r+s, i-j = m-s-j$ and we demand that $r \ge 0, s \ge 0$. Therefore $p_n^{(\alpha)}(\cos\theta, t_1, t_2, q^{1-m-\alpha}/t_4, t_4)$ is given by

$$\frac{(t_1t_2q^{\alpha-m};q)_m(-1)^m q^{\binom{m}{2}}}{(t_1t_4q^{\alpha},t_1q^{1-m}/t_4;q)_m}\sum_{j,r,s}\frac{(q^{m-n},t_1t_2q^{n+\alpha};q)_r(q^{-m};q)_{m-s}}{(q,t_1t_4q^{\alpha+m},t_1q/t_4;q)_r}$$
$$\times\frac{(qt_1/t_2;q)_{r+m-j}(t_1e^{i\theta},t_1e^{-i\theta};q)_{r+s}}{(t_1t_2q^{-n};q)_{r+s+j}(qt_1/t_2;q)_{r+s}(q;q)_{m-r-s}}\frac{(t_2t_4q^{\alpha-1},q^{-m}t_2/t_4,t_1t_2q^{-m-1};q)_j}{(q,t_1t_2q^{\alpha-m-1};q)_j}$$
$$\times\left(\frac{t_1}{t_4}\right)^j q^{(r+s)(1+j-m)}q^{m-s+j}.$$

The s-sum is

$$\sum_{s=0}^{m-j}\frac{(q^{-m};q)_{m-s}(t_1e^{i\theta},t_1e^{-i\theta};q)_{r+s}q^{m-s+s(1-m+j)}}{(q;q)_{m-j-s}(t_1t_2q^{-m};q)_{r+s+j}(qt_1/t_2;q)_{r+s}}$$
$$=\frac{(q^{-m};q)_m}{(q;q)_{m-j}}\frac{(t_1e^{i\theta},t_1e^{-i\theta};q)_r}{(t_1t_2q^{-m};q)_{r+j}(t_1q/t_2;q)_r}\,q^m$$
$$\times{}_3\phi_2\left(\begin{matrix}q^{j-m},t_1q^re^{i\theta},t_1q^re^{-i\theta}\\ t_1t_2q^{r-m+j},t_1q^{r+1}/t_2\end{matrix}\,\middle|\,q,q\right).$$

The ${}_3\phi_2$ can be summed by the q-analogue of the Pfaff-Saalschütz theorem [11, (II.12)], and after some simplifications the s-sum reduces to

$$q^{-\binom{m}{2}}\frac{(-1)^m(q;q)_m(t_1e^{i\theta},t_1e^{-i\theta};q)_r(t_2q^{j-m}e^{i\theta},t_2q^{j-m}e^{-i\theta};q)_{m-j}}{(q;q)_{m-j}(t_1t_2q^{-m};q)_{m+r}(qt_1/t_2;q)_r(t_2q^{j-m-r}/t_1;q)_{m-j}}.$$

Thus the polynomial $p_n^{\alpha}(x;t_1,t_2,q^{-m+1-\alpha}/t_4,t_4)$ is given by

$$\frac{(t_1t_2q^{\alpha-m};q)_m}{(t_1t_2q^{-m},t_1t_4q^{\alpha},t_1q^{1-m}/t_4;q)_m}\sum_{j,r\geq 0}\frac{(q^{m-n},t_1t_2q^{n+\alpha},t_1e^{i\theta},t_1e^{-i\theta};q)_r}{(t_1t_2,t_1t_4q^{\alpha+m},qt_1/t_4;q)_r(q,q)_{m-j}}$$
$$\times\frac{(t_2t_4q^{\alpha-1},t_2q^{-m}/t_4,t_1t_2q^{-m-1};q)_j}{(q,t_1t_2q^{\alpha-m-1};q)_j}\frac{(t_2q^{j-m}e^{i\theta},t_2q^{j-m}e^{-i\theta},t_1q^{r+1}/t_2;q)_{m-j}}{(t_2q^{j-m-r}/t_1;q)_{m-j}}$$
$$\times q^{r(1-m+j)}\,(qt_1/t_2)^j$$
$$=\frac{(t_1t_2q^{\alpha-m},t_2q^{-m}e^{i\theta},t_2q^{-m}e^{-i\theta};q)_m}{(t_1t_2q^{-m},t_1q^{1-m}/t_4,t_1t_4q^{\alpha};q)_m}(-1)^m q^{\binom{m+1}{2}}\left(\frac{t_1}{t_2}\right)^m$$
$$\times\sum_{r=0}^{n-m}\frac{(q^{m-n},t_1t_2q^{\alpha+n},t_1e^{i\theta},t_1e^{-i\theta};q)_r}{(q,t_1t_2,qt_1/t_4,t_1t_4q^{\alpha+m};q)_r}\,q^r$$
$$\times\sum_{j=0}^{m}\frac{(q^{-m},t_2t_3q^{\alpha-1},t_2q^{-m}/t_4,t_1t_2q^{-m-1};q)_j}{(q,t_1t_2q^{\alpha-m-1},t_2q^{-m}e^{i\theta},t_2q^{-m}e^{-i\theta};q)_j}\,q^j.$$

It is easy to reduce the above expression to the right-hand side of (4.3). ☐

Corollary 4.2. *When $t_3t_4=q^{1-m-\alpha}$ and $t_1t_2=q^{m+1}$ we have the factorization*

$$\begin{aligned}&p_n^{\alpha}(\cos\theta;t_1,q^{m+1}/t_1,q^{-m+1-\alpha}/t_4,t_4)\\ &\quad=\frac{(q^{\alpha+1},qe^{i\theta}/t_1,qe^{-i\theta}/t_1;q)_m}{(q,t_4/t_1,t_1t_4q^{\alpha};q)_m}(t_1t_4/q)^m\\ &\qquad\times p_{n-m}(\cos\theta;t_1,q^{m+1}/t_1,q/t_4,t_4q^{\alpha+m}).\end{aligned}\tag{4.5}$$

Theorem 4.3. *For* $n \ge m$ *the polynomials* $q_n^{\alpha}(x; t_1, t_2, q^{1-m-\alpha}/t_4, t_4)$ *have the factorization*

$$q_n^{\alpha}(x; t_1, t_2, q^{-m+1-\alpha}/t_4, t_4) = \frac{t_1^m (t_1t_2q^{\alpha-m}, t_2q^{-m}e^{i\theta}, t_2q^{-m}e^{-i\theta}; q)_m}{(-t_2)^m (t_1t_2q^{-m}, t_1t_4q^{\alpha}, t_1q^{1-m}/t_4; q)_n}$$
$$\times q^{\binom{m+1}{2}} {}_4\phi_3\left(\left.\begin{matrix} q^{-m}, t_2t_4q^{\alpha-1}, t_2q^{-m}/t_4, t_1t_2q^{-m} \\ t_1t_2q^{\alpha-m}, t_2q^{-m}e^{i\theta}, t_2q^{-m}e^{-i\theta} \end{matrix} \right| q, q\right) \tag{4.6}$$
$$\times p_{n-m}(cos\theta; t_1, t_2, q/t_4, q^{m+\alpha}t_4).$$

Proof. As in the proof of Theorem 4.1 we apply Exercise 2.20 in [11] to see that the ${}_{10}W_9$ in (4.2) has the representation

$$\frac{(t_1t_2q^{\alpha+k-m}, q^{-k}; q)_m}{(q^{-n}, t_1t_2q^{\alpha+n-m}; q)_m} \sum_{i=0}^{m} \frac{(q^{-m}, q^{k-n}, t_1t_2q^{\alpha+n-m+k}, t_1q^{k+1}/t_2; q)_i}{(q, t_1q^{k-m+1}/t_4, t_1t_4q^{\alpha+k}, q^{k+1-m}; q)_i} q^i$$
$$\times {}_4\phi_3\left(\left.\begin{matrix} q^{-i}, t_2t_4q^{\alpha-1}, t_2q^{-m}/t_4, t_1t_2q^{-m} \\ t_1t_2q^{\alpha-m}, t_1t_2q^{k-m}, t_2q^{-k-i}/t_1 \end{matrix} \right| q, q\right)$$
$$= \frac{(t_1t_2q^{\alpha+k-m}; q)_m(-1)^m}{(q^{-n}, t_1t_2q^{\alpha+n-m}; q)_m} q^{\binom{m}{2}} \sum_{i \ge j} \frac{(q^{-m}, q^{k-n}, t_1t_2q^{\alpha+n-m+k}; q)_i q^i}{(t_1q^{k-m+1}/t_4, t_1t_4q^{\alpha+k}; q)_i (q; q)_{i+k-m}}$$
$$\times \frac{(t_1q^{k+1}/t_2; q)_{i-j}}{(q; q)_{i-j}} \left(\frac{qt_1}{t_2}\right)^j q^{k(j-m)} \frac{(t_2t_4q^{\alpha-1}, t_2q^{-m}/t_4, t_1t_2q^{-m}; q)_j}{(q, t_1t_2q^{\alpha-m}, t_1t_2q^{k-m}; q)_j}.$$

We then replace the ${}_{10}W_9$ in (4.2) by the above expression. As in the proof of Theorem 4.1 the restrictions on the summation indices are: $m \ge i, k+i-m \ge 0, i \ge j$, and as before we set $k = r+s, i-j = m-s-j$ and we demand that $r \ge 0, s \ge 0$. Therefore $q_n^{(\alpha)}(\cos\theta, t_1, t_2, q^{1-m-\alpha}/t_4, t_4)$ is given by

$$\frac{(t_1t_2q^{\alpha-m}; q)_m(-1)^m q^{\binom{m}{2}}}{(t_1t_4q^{\alpha}, t_1q^{1-m}/t_4; q)_m} \sum_{j,r,s} \frac{(q^{m-n}, t_1t_2q^{n+\alpha}; q)_r (q^{-m}; q)_{m-s}}{(q, t_1t_4q^{\alpha+m}, t_1q/t_4; q)_r}$$
$$\times \frac{(qt_1/t_2; q)_{r+m-j}(t_1e^{i\theta}, t_1e^{-i\theta}; q)_{r+s}}{(t_1t_2q^{-n}; q)_{r+s+j}(qt_1/t_2; q)_{r+s}(q; q)_{m-r-s}} \frac{(t_2t_4q^{\alpha-1}, q^{-m}t_2/t_4, t_1t_2q^{-m}; q)_j}{(q, t_1t_2q^{\alpha-m}; q)_j}$$
$$\times \left(\frac{t_1}{t_4}\right)^j q^{(r+s)(1+j-m)} q^{m-s+j}.$$

The s-sum is

$$\sum_{s=0}^{m-j} \frac{(q^{-m}; q)_{m-s}(t_1e^{i\theta}, t_1e^{-i\theta}; q)_{r+s} q^{m-s+s(1-m-j)}}{(q; q)_{m-j-s}(t_1t_2q^{-m}; q)_{r+s+j}(qt_1/t_2; q)_{r+s}}$$
$$= \frac{(q^{-m}; q)_m}{(q; q)_{m-j}} \frac{(t_1e^{i\theta}, t_1e^{-i\theta}; q)_r}{(t_1t_2q^{-m}; q)_{r+j}(t_1q/t_2; q)_r} q^m$$
$$\times {}_3\phi_2\left(\left.\begin{matrix} q^{j-m}, t_1q^re^{i\theta}, t_1q^re^{-i\theta} \\ t_1t_2q^{r-m+j}, t_1q^{r+1}/t_2 \end{matrix} \right| q, q\right).$$

Again the ${}_3\phi_2$ can be summed by [11, (II.12)], and after some simplifications the s-sum becomes

$$q^{-\binom{m}{2}} \frac{(-1)^m (q; q)_m (t_1e^{i\theta}, t_1e^{-i\theta}; q)_r (t_2q^{j-m}e^{i\theta}, t_2q^{j-m}e^{-i\theta}; q)_{m-j}}{(q; q)_{m-j}(t_1t_2q^{-m}; q)_{m+r}(qt_1/t_2; q)_r (t_2q^{j-m-r}/t_1; q)_{m-j}}.$$

Thus the polynomial $q_n^\alpha(x; t_1, t_2, q^{-m+1-\alpha}/t_4, t_4)$ is given by

$$\frac{(q, t_1t_2q^{\alpha-m}; q)_m}{(t_1t_2q^{-m}, t_1t_4q^\alpha, t_1q^{1-m}/t_4; q)_m} \sum_{j,r\geq 0} \frac{(q^{m-n}, t_1t_2q^{n+\alpha}, t_1e^{i\theta}, t_1e^{-i\theta}; q)_r}{(q, t_1t_2, t_1t_4q^{\alpha+m}, qt_1/t_4; q)_r}$$
$$\times \frac{(t_2t_4q^{\alpha-1}, t_2q^{-m}/t_4, t_1t_2q^{-m}; q)_j}{(q,q)_{m-j}(q, t_1t_2q^{\alpha-m}; q)_j} \frac{(t_2q^{j-m}e^{i\theta}, t_2q^{j-m}e^{-i\theta}, t_1q^{r+1}/t_2; q)_{m-j}}{(t_2q^{j-m-r}/t_1; q)_{m-j}}$$
$$\times q^{r(1-m+j)} (qt_1/t_2)^j$$
$$= \frac{(t_1t_2q^{\alpha-m}, t_2q^{-m}e^{i\theta}, t_2q^{-m}e^{-i\theta}; q)_m}{(t_1t_2q^{-m}, t_1q^{1-m}/t_4, t_1t_4q^\alpha; q)_m} q^{\binom{m+1}{2}} \left(-\frac{t_1}{t_2}\right)^m$$
$$\times \sum_{r=0}^{n-m} \frac{(q^{m-n}, t_1t_2q^{\alpha+n}, t_1e^{i\theta}, t_1e^{-i\theta}; q)_r}{(q, t_1t_2, qt_1/t_4, t_1t_4q^{\alpha+m}; q)_r} q^r$$
$$\times \sum_{j=0}^{m} \frac{(q^{-m}, t_2t_3q^{\alpha-1}, t_2q^{-m}/t_4, t_1t_2q^{-m}; q)_j}{(q, t_1t_2q^{\alpha-m}, t_2q^{-m}e^{i\theta}, t_2q^{-m}e^{-i\theta}; q)_j} q^j.$$

It is easy to reduce the above expression to the right-hand side of (4.6). □

5. R_{II}-RATIONAL FUNCTIONS

Rahman [18] studied the biorthogonal rational functions

$$R_n(\cos\theta; \mathbf{t}) := {}_{10}\phi_9 \left(\begin{matrix} t_1T/q, \sqrt{qt_1T}, -\sqrt{qt_1T}, & T/t_2, T/t_3, T/t_4, \\ \sqrt{t_1T/q}, -\sqrt{t_1T/q}, & t_1t_2,\ t_1t_3,\ t_1t_4, \end{matrix} \right. \tag{5.1}$$
$$\left. \begin{matrix} t_1e^{i\theta}, t_1e^{-i\theta}, t_1t_2t_3t_4q^{n-1}, q^{-n} \\ Te^{-i\theta}, Te^{i\theta}, t_1t_5q^{1-n}, q^nTt_1 \end{matrix} \middle| q, q \right),$$

where $\mathbf{t} = (t_1, t_2, t_3, t_4, t_5)$ and $T := t_1t_2t_3t_4t_5$. Note that the functions studied in §2 correspond to the case when the functions in (5.1) form a finite set.

Theorem 5.1. *The rational functions $\{R_n\}$ satisfy the three term recurrence relation*

$$(2x - t_1 - 1/t_1)y_n = A_ny_{n+1} + C_ny_{n-1} - (A_n + C_n)y_n, \tag{5.2}$$

where

$$A_n = \frac{t_1^{-1}(1 - q^{n-1}T/t_5)\prod_{j=2}^4(1 - t_1t_jq^n)(q^{-n}t_1t_5; q)_2}{(1 - q^{2n-1}T/t_5)(1 - q^{2n}T/t_5)(1 - qt_1t_5)\prod_{2\leq j<k\leq 4}(1 - t_1t_jt_kt_5)} \times[1 - 2q^nxT + q^{2n}T^2],$$
$$C_n = \frac{t_1(1 - q^n)(q^{n-1}t_1T; q)_2\prod_{2\leq j<k\leq 4}(1 - t_jt_kq^{n-1})}{(1 - q^{2n-2}T/t_5)(1 - q^{2n-1}T/t_5)(1 - qt_1t_5)} \times \frac{1 - 2q^{1-n}t_5x + t_5^2q^{2-2n}}{\prod_{2\leq j<k\leq 4}(1 - t_1t_jt_kt_5)}. \tag{5.3}$$

The proof of this theorem is rather lengthy and uses contiguous relations for ${}_{10}W_9$ functions and will be presented in the next section, §6. Now set

$$A_n = [1 - 2q^nxT + q^{2n}T^2]\tilde{A}_n, \quad C_n = [1 - 2q^{1-n}t_5x + t_5^2q^{2-2n}]\tilde{C}_n. \tag{5.4}$$

Thus the three term recurrence relation (5.3) reduces (5.2) to

$$\begin{aligned} &2x\left[y_n + Tq^n\tilde{A}_n\, y_{n+1} + t_5q^{1-n}\tilde{C}_n\, y_{n-1} - (Tq^n\tilde{A}_n + t_5q^{1-n}\tilde{C}_n)\, y_n\right] \\ &\qquad = (1+q^{2n}T)\tilde{A}_n y_{n+1} + (1+q^{2-2n}t_5^2)\tilde{C}_n y_{n-1} \\ &\qquad\quad - \left[(1+q^{2n}T^2)\tilde{A}_n + (1+q^{2-2n}t_5^2)\tilde{C}_n\right] y_n. \end{aligned} \tag{5.5}$$

It is clear from the above form that this leads to an isochronous system.

It must be noted that the three term recurrence relation in (5.5) is of the form (2.6), and hence the factorization property in (1.8) holds, but now for rational functions, under a degenerate condition like $b_m = 0$.

6. Appendix

In this section we prove Theorem 5.1.

Let $q^2a^3 = bcdefgh$ and $\lambda = qa^2/cde$. Then, by Bailey's transformation formula [11, (III.39)],

$$\begin{aligned} &{}_{10}V_9\,(a; b, c, d, e, f, g, h; q, q) \\ &= \frac{(aq, b/a, \lambda q/f, \lambda q/g, \lambda q/h, bf/\lambda, bg/\lambda, bh/\lambda; q)_\infty}{(\lambda q, b/\lambda, aq/f, aq/g, aq/h, bf/a, bg/a, bh/a; q)_\infty} \\ &\times {}_{10}V_9\,(\lambda; b, \lambda c/a, \lambda d/a, \lambda e/a, f, g, h; q, q)\,, \end{aligned} \tag{6.1}$$

where

(6.2)

$$\begin{aligned} &{}_{10}V_9\,(a; b, c, d, e, f, g, h; q, q) \\ &:= {}_{10}W_9\,(a; b, c, d, e, f, g, h; q, q) \\ &+ \frac{(aq, b/a, c, d, e, f, g, h; q)_\infty\,(bq/c, bq/d, bq/e, bq/f, bq/g, bq/h; q)_\alpha}{(qb^2/a, a/b, aq/c, aq/d, aq/e, aq/f, aq/g, aq/h; q)_\infty\,(bc/a, bd/a, be/a, bf/a, bg/a, bh/a; q)_\alpha} \\ &\times {}_{10}W_9\left(b^2/a; b, bc/a, b/da, be/a, bf/a, bg/a, bh/a; q, q\right). \end{aligned}$$

For an unspecified α, let

$$p_\mu(x) := {}_{10}V_9\left(\alpha q^{-1}; az, a/z, \alpha/ab, \alpha/ac, \alpha/ad, abcdq^{\mu-1}, q^{-\mu}; q, q\right). \tag{6.3}$$

Then, by a straightforward calculation and by applying the transformation (6.1) twice, we obtain

$$\begin{aligned} &\triangle_\mu p_\mu(x) := p_{\mu+1}(x) - p_\mu(x) \\ &= E_\mu\,(\alpha; a, b, c, d)\, F_\mu(x) \\ &\times {}_{10}V_9\left(a^2b^2cd/\alpha q; bz, b/z, abcd/\alpha q, a^2bc/\alpha, a^2bd/\alpha, abcdq^\mu, q^{-\mu}; q, q\right), \end{aligned} \tag{6.4}$$

where

$$\begin{aligned} &E_\mu\,(\alpha; a, b, c, d) \\ &= -q^{-k}\frac{(1-\alpha/ac)\,(1-\alpha/ad)\left(1-abcdq^{2\mu}\right)}{(1-ab)\,(1-ac)} \\ &\times \frac{\left(\alpha, 1/ac, bd, \alpha/ab, \alpha q/abcd, adq^{\mu+1}, q^{1-\mu}/bc, \alpha^2cdq^\mu/\alpha, \alpha q^{1-\mu}/a^2cd; q\right)_\infty}{(q/bc, ad, \alpha q/a^2cd, a^2b^2cd/\alpha, a^2cd/\alpha, \alpha q^\mu, bdq^\mu, \alpha q^{\mu+1}/ab, q^{-\mu}/ac, \alpha q^{-\mu}/abcd; q)_\infty} \end{aligned}$$

and

$$F_\mu(x) = (1-az)(1-a/z)\frac{\left(\frac{a^2bcdz}{\alpha}, \frac{a^2bcd}{\alpha z}, \frac{\alpha z q^{\mu+1}}{a}, \frac{\alpha q^{\mu+1}}{az}; q\right)_\infty}{\left(\alpha z/a, \alpha/az, \frac{a^2bcdzq^\mu}{\alpha}, \frac{a^2bcdq^\mu}{\alpha z}; q\right)_\infty}.$$

Denoting

$$\begin{aligned} &C_\mu/A_\mu \\ &:= \frac{a^2\left(1-abcdq^{2\mu}\right)\left(1-bcq^{\mu-1}\right)\left(1-bdq^{\mu-1}\right)}{\left(1-abcdq^{2\mu-2}\right)\left(1-abq^\mu\right)\left(1-acq^\mu\right)\left(1-adq^\mu\right)\left(1-abcdq^{\mu-1}\right)} \\ &\times \frac{\left(1-\alpha q^{\mu-1}\right)\left(1-\alpha q^\mu\right)\left(1-q^\mu\right)\left(1-\frac{\alpha q^{1-\mu}}{a^2bcdz}\right)\left(1-\frac{\alpha z q^{1-\mu}}{a^2bcd}\right)}{\left(1-cdq^{\mu-1}\right)\left(1-\frac{\alpha q^{-\mu}}{abcd}\right)\left(1-\frac{\alpha q^{1-\mu}}{abcd}\right)\left(1-\alpha z q^\mu/a\right)\left(1-\alpha q^\mu/az\right)} \end{aligned} \tag{6.5}$$

and

$$D_\mu(x) := \triangle_\mu p_\mu(x) - \frac{C_\mu}{A_\mu}\triangle_\mu p_{\mu-1}(x), \tag{6.6}$$

we do the long and tedious calculations to finally obtain the functional relation

$$2xp_\mu(x) = A_\mu p_{\mu+1}(x) - B_\mu p_\mu(x) - C_\mu p_{\mu-1}(x), \tag{6.7}$$

$$B_\mu = A_\mu + C_\mu - \left(a + a^{-1}\right). \tag{6.8}$$

Note that in (6.5) and (6.6) A_μ and C_μ are unspecified. However, in (6.7) a choice has been made:

$$\begin{aligned} A_\mu =& a^{-1}\frac{\left(1-abq^\mu\right)\left(1-acq^\mu\right)\left(1-adq^\mu\right)\left(1-abcdq^{\mu-1}\right)\left(1-\alpha q^{-\mu}/abcd\right)}{\left(1-abcdq^{2\mu-1}\right)\left(1-abcdq^{2\mu}\right)\left(1-\alpha/ab\right)\left(1-\alpha/ac\right)} \\ &\times \frac{\left(1-\alpha q^{1-\mu}/abcd\right)}{\left(1-\alpha/ad\right)\left(1-\alpha q/abcd\right)}\left(1-\frac{\alpha z q^{1-\mu}}{a^2bcd}\right)\left(1-\frac{\alpha q^{1-\mu}}{a^2bcdz}\right). \end{aligned} \tag{6.9}$$

This completes the proof of Theorem 5.1.

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