AN INEQUALITY FOR CAUCHY PRODUCTS

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(Communicated by Stephan Ramon Garcia)

ABSTRACT. We show that ℓ^p is closed under the Cauchy product by ℓ^1 sequences.

1. INTRODUCTION

Let f and g be two analytic functions in a neighborhood of origin with Taylor series expansions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_n z^n.$$

The usual product of f and g has Taylor series expansion

$$(fg)(z) = \sum_{n=0}^{\infty} c_n z^n$$

where the coefficients c_n are given by

$$(1.1) \quad c_n = \sum_{k=0}^n a_k b_{n-k} \quad (n \geq 0).$$

The sequence $(c_n)_{n \geq 0}$ is called the *Cauchy product* of $(a_n)_{n \geq 0}$ and $(b_n)_{n \geq 0}$ [2, pp. 227–229]. It may also be regarded as the discrete convolution of $(a_n)_{n \geq 0}$ and $(b_n)_{n \geq 0}$ [3, pp. 13–20].

There are several questions that arise in connection with Cauchy products of infinite sequences when we study the multiplier algebra of Banach spaces of analytic functions on the open unit disc $\mathbb{D}$. The characterization of the multiplier algebra itself, or at least a subalgebra of it, is one of the main issues. In this situation, we restrict f to a Banach space of analytic functions and ask for which g the image gf stays in the space. No doubt, there is no universal answer, and the final characterization depends on the geometry of the Banach space under consideration. In this short note, we just consider the classical ℓ^p space with $1 < p < \infty$. The corresponding Banach space of analytic functions on $\mathbb{D}$ is denoted by ℓ_A^p [1]. The characterization of the multiplier algebra of ℓ^p , or equivalently of ℓ_A^p , is still an *open question*. We show that ℓ^1 continuously (even contractively) embeds in the

Received by the editors May 3, 2016 and, in revised form, August 15, 2016.

2010 *Mathematics Subject Classification*. Primary 47A30; Secondary 26D15, 47A63.

This work was supported by NSERC (Canada).

multiplier algebra of ℓ^p . More explicitly, we study sequences $(a_n)_{n \geq 0}$ for which

$$(1.2) \quad \sum_{n=0}^{\infty} \left| \sum_{k=0}^n a_k b_{n-k} \right|^p < \infty$$

for all $(b_n)_{n \geq 0} \in \ell^p$. Clearly, this holds whenever $(a_n)_{n \geq 0}$ is a compactly supported sequence, or equivalently when f is a polynomial. But, we provide a quantitative evaluation of (1.2) which works for a wide class of infinite sequences.

2. MAIN RESULT

Without loss of generality, we assume that all coefficients are positive. The passage to the more general setting of complex numbers is straightforward.

Theorem 2.1. *Let $(a_n)_{n \geq 0}$ and $(b_n)_{n \geq 0}$ be two sequences of positive numbers. Let $1 < p < \infty$, with exponent conjugate q , and let $0 < r < 1$. Then*

$$\left(\sum_{n=0}^{\infty} \left(\sum_{k=0}^n a_k b_{n-k} \right)^p \right)^{\frac{1}{p}} \leq \left\{ \left(\sum_{n=0}^{\infty} a_n^{rq} \right)^{\frac{1}{q}} \left(\sum_{n=0}^{\infty} a_n^{(1-r)p} \right)^{\frac{1}{p}} \right\} \left(\sum_{n=0}^{\infty} b_n^p \right)^{\frac{1}{p}}.$$

Proof. The proof is based on a judicious application of the classical Hölder inequality. Write $a_k b_{n-k} = a_k^r \cdot (a_k^{1-r} b_{n-k})$. Then, by Hölder's inequality,

$$\sum_{k=0}^n a_k b_{n-k} \leq \left(\sum_{k=0}^n a_k^{rq} \right)^{\frac{1}{q}} \left(\sum_{k=0}^n a_k^{(1-r)p} b_{n-k}^p \right)^{\frac{1}{p}} \quad (n \geq 0).$$

Hence,

$$\left(\sum_{k=0}^n a_k b_{n-k} \right)^p \leq \left(\sum_{k=0}^{\infty} a_k^{rq} \right)^{\frac{p}{q}} \left(\sum_{k=0}^n a_k^{(1-r)p} b_{n-k}^p \right) \quad (n \geq 0).$$

Note that in the second summation, we replaced n by ∞ . Now, we add all these inequalities and change the order of summation on the right side:

$$\begin{aligned} \sum_{n=0}^{\infty} \left(\sum_{k=0}^n a_k b_{n-k} \right)^p &\leq \left(\sum_{k=0}^{\infty} a_k^{rq} \right)^{\frac{p}{q}} \times \sum_{n=0}^{\infty} \left(\sum_{k=0}^n a_k^{(1-r)p} b_{n-k}^p \right) \\ &\leq \left(\sum_{k=0}^{\infty} a_k^{rq} \right)^{\frac{p}{q}} \times \sum_{k=0}^{\infty} \left(\sum_{n=k}^{\infty} a_k^{(1-r)p} b_{n-k}^p \right) \\ &\leq \left(\sum_{k=0}^{\infty} a_k^{rq} \right)^{\frac{p}{q}} \times \sum_{k=0}^{\infty} a_k^{(1-r)p} \times \sum_{n=0}^{\infty} b_n^p. \end{aligned}$$

Finally, take the p -th root of both sides. □

The parameter r is free to run between 0 and 1. However, at the point of intersection of lines $r \mapsto rq$ and $r \mapsto (1-r)p$, i.e., for $r = p/(p+q)$, something interesting happens.

Corollary 2.2. *Let $(a_n)_{n \geq 0}$ and $(b_n)_{n \geq 0}$ be two sequences of positive numbers. Let $1 < p < \infty$. Then*

$$\left(\sum_{n=0}^{\infty} \left(\sum_{k=0}^n a_k b_{n-k} \right)^p \right)^{\frac{1}{p}} \leq \left(\sum_{n=0}^{\infty} a_n \right) \left(\sum_{n=0}^{\infty} b_n^p \right)^{\frac{1}{p}}.$$

Briefly speaking, Corollary 2.2 says that the Cauchy product of a sequence in ℓ^1 with another in ℓ^p stays in ℓ^p .

3. EQUALITY

Let us study the case of equality in Corollary 2.2. The procedure is similar for the other inequality in Theorem 2.1. We show that equality holds just in the following cases:

- (i) $(a_n) \equiv 0$ (trivial case).
- (ii) $(b_n) \equiv 0$ (trivial case).
- (iii) There is an integer k such that $a_n = 0$ for all $n \geq 0$ except possibly for $n = k$ (interesting case).

The verification of the third case is a bit delicate and needs special care.

At two stages, we have an inequality in the proof of Corollary 2.2 which must turn into an equality. First, equality holds in

$$\sum_{k=0}^n a_k b_{n-k} \leq \left(\sum_{k=0}^n a_k \right)^{\frac{1}{q}} \left(\sum_{k=0}^n a_k b_{n-k}^p \right)^{\frac{1}{p}} \quad (n \geq 0)$$

if and only if one of the following two cases happens:

- (i) $(a_0^{1/q}, a_1^{1/q}, \dots, a_n^{1/q}) = 0$ or $(a_0^{1/p} b_n, a_1^{1/p} b_{n-1}, \dots, a_n^{1/p} b_0) = 0$.
- (ii) $(a_0^{1/q}, a_1^{1/q}, \dots, a_n^{1/q}) \neq 0$ and $(a_0^{1/p} b_n, a_1^{1/p} b_{n-1}, \dots, a_n^{1/p} b_0) \neq 0$, but

$$(a_0 b_n^p, a_1 b_{n-1}^p, \dots, a_n b_0^p) = \lambda (a_0, a_1, \dots, a_n)$$

for some nonzero constant λ .

Due to special forms of these vectors, we can summarize the above two cases as follows:

- (i) $(a_0 b_n, a_1 b_{n-1}, \dots, a_n b_0) = 0$.
 - (ii) $(a_0 b_n, a_1 b_{n-1}, \dots, a_n b_0) \neq 0$, but
- $$(3.1) \quad (a_0 b_n^p, a_1 b_{n-1}^p, \dots, a_n b_0^p) = \lambda (a_0, a_1, \dots, a_n)$$

for some nonzero constant λ .

If the condition (i) holds for all $n \geq 0$, then either $(a_n) \equiv 0$ or $(b_n) \equiv 0$, which are acceptable (trivial) cases for equality. Hence, we assume that $(a_n) \not\equiv 0$ and $(b_n) \not\equiv 0$. Let $k \geq 0$ and $\ell \geq 0$ be the smallest integers such that $a_k \neq 0$ and $b_\ell \neq 0$. Therefore, (3.1) holds for $n = k + \ell$. Since in the vector $(a_0 b_{k+\ell}^p, a_1 b_{k+\ell-1}^p, \dots, a_{k+\ell} b_0^p)$ there is just one nonzero component, i.e., $a_k b_\ell^p$, we deduce that besides $a_0 = \dots = a_{k-1} = 0$ (assumption), we also must have $a_{k+1} = \dots = a_{k+\ell} = 0$. For other indices—and this is the second instance where an inequality should turn into an equality—note that to have equality in

$$\left(\sum_{i=0}^{k+\ell} a_i \right)^{\frac{1}{q}} \left(\sum_{i=0}^{k+\ell} a_i b_{k+\ell-i}^p \right)^{\frac{1}{p}} \leq \left(\sum_{i=0}^{\infty} a_i \right)^{\frac{1}{q}} \left(\sum_{i=0}^{k+\ell} a_i b_{k+\ell-i}^p \right)^{\frac{1}{p}},$$

we must have

$$a_{k+\ell+1} = a_{k+\ell+2} = \dots = 0.$$

In short, $a_n = 0$ for all values of $n \geq 0$ except possibly $n = k$. Conversely, it is easy to verify that for any such sequence,

$$\left(\sum_{n=0}^{\infty} \left(\sum_{m=0}^n a_m b_{n-m} \right)^p \right)^{\frac{1}{p}} = a_k \left(\sum_{n=0}^{\infty} b_n^p \right)^{\frac{1}{p}} = \left(\sum_{n=0}^{\infty} a_n \right) \left(\sum_{n=0}^{\infty} b_n^p \right)^{\frac{1}{p}}$$

for all choices of (b_n) .

ACKNOWLEDGEMENT

The author thanks the anonymous referee for helpful remarks and constructive suggestions.

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