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STABILITY OF THE SOLUTION OF STOCHASTIC DIFFERENTIAL EQUATION DRIVEN BY TIME-CHANGED LÉVY NOISE

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Abstract. This paper studies stabilities of the solution of stochastic differential equations (SDE) driven by time-changed Lévy noise in both probability and moment sense. This provides more flexibility in modeling schemes in application areas including physics, biology, engineering, finance and hydrology. Necessary conditions for the solution of time-changed SDE to be stable in different senses will be established. The connection between stability of the solution to time-changed SDE and that to corresponding original SDE will be disclosed. Examples related to different stabilities will be given. We study SDEs with time-changed Lévy noise, where the time-change processes are the inverse of general Lévy subordinators. These results are an important generalization of the results of Q. Wu (2016).

1. Introduction

Recently, stochastic differential equations (SDEs) have been applied in various areas, including biology [7], physics [5], engineering [17], and finance [6]. SDEs are taken as important tools in modeling and simulating real phenomena, and the stability of SDEs has been studied widely by mathematicians in different senses, such as stochastically stable, stochastically asymptotically stable, moment exponentially stable, almost surely stable, mean square polynomial stable; see [1, 10, 16, 19]. A systematic introduction of stabilities is provided by Mao in [12].

During the last few decades, time-changed SDEs attracted lots of attention and became one of the most active areas in stochastic analysis and many applied areas of science. Their probability density functions provide solutions to fractional Fokker-Planck equations of different kinds (see [13, 15]), which are also very important in modeling and describing phenomena in applied areas; see [14].

In [8] Kobayashi discussed the relationship between time-changed SDEs

$$\begin{aligned} dX(t) &= f(E_t, X(t-))dE_t + g(E_t, X(t-))dZ_{E_t}, \\ X(0) &= x_0, \end{aligned} \tag{1.1}$$

and the corresponding non-time-changed SDEs

$$\begin{aligned} dY(t) &= f(t, Y(t-))dt + g(t, Y(t-))dZ_t, \\ Y(0) &= x_0, \end{aligned} \tag{1.2}$$

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where Z_t is an $\mathcal{F}_t$-semimartingale and E_t is an inverse of a right continuous with left limit (RCLL) nondecreasing process $\{D(t), t \geq 0\}$: if a process $Y(t)$ satisfies SDE (1.2), then $X(t) := Y(E_t)$ satisfies the time-changed SDE (1.1); if a process $X(t)$ satisfies the time-changed SDE (1.1), then $Y(t) := X(D(t))$ satisfies SDE (1.2).

Kobayashi also studied the Itô formula driven by time-changed SDE which is provided under certain conditions as below:

$$
\begin{aligned}
f(X_t) - f(x_0) = &\int_0^t f'(X_{s-})A_s ds + \int_0^{E_t} f'(X_{D(s-)-})F_{D(s-)}ds \\
&+ \int_0^{E_t} f'(X_{D(s-)-})G_{D(s-)}dZ_s \\
&+ \frac{1}{2}\int_0^{E_t} f''(X_{D(s-)-})\{G_{D(s-)}\}^2 d[Z,Z]_s^c \\
&+ \sum_{0<s\leq t}\{f(X_s) - f(X_{s-}) - f'(X_{s-})\Delta X_s\},
\end{aligned}
\tag{1.3}
$$

where $f : \mathbb{R} \to \mathbb{R}$ is a C^2 function.

In light of the time-changed Itô formula, the recent paper [18] analyzes the SDE driven by time-changed Brownian motion

$$
\begin{aligned}
&dX(t) = k(t, E_t, X(t-))dt + f(t, E_t, X(t-))dE_t + g(t, E_t, X(t-))dB_{E_t}, \\
&X(0) = x_0,
\end{aligned}
\tag{1.4}
$$

where E_t is specified as an inverse of a stable subordinator of index β in $(0,1)$, and discusses the stability of the solution to the above SDE in the probability sense, including stochastically stable, stochastically asymptotically stable and globally stochastically asymptotically stable.

The main result of this paper is to provide necessary conditions for solutions of SDEs driven by time-changed Lévy noise to be stable not only in the probability sense but also in the moment sense. Our results generalize the results of [18] in two respects. Firstly, we study SDEs with time-changed Lévy noise. Secondly, we work with time-change processes that are the inverse of the general Lévy subordinators.

In the remaining parts of this paper, further needed concepts and related background will be given in the preliminary section. In the main result section, necessary conditions for the solution of time-changed SDEs to be stable in different senses will be given. Connections between the stability of the solution to time-changed SDE and that to corresponding original SDE will be disclosed and some examples will be given. The last section will show proofs of theorems mentioned in the main result section.

2. Preliminaries

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a filtered probability space satisfying the usual hypotheses of completeness and right continuity. Let $\mathcal{F}_t$-adapted Poisson random measure N be defined on $\mathbb{R}_+ \times (\mathbb{R} - \{0\})$ with compensator $\tilde{N}$ and intensity measure ν, where ν is a Lévy measure such that $\tilde{N}(dt, dy) = N(dt, dy) - \nu(dy)dt$ and $\int_{\mathbb{R}-\{0\}}(|y|^2 \wedge 1)\nu(dy) < \infty$.

Let $\{D(t), t \geq 0\}$ be an RCLL increasing Lévy process that is called subordinator starting from 0 with Laplace transform

$$\mathbb{E}e^{-\lambda D(t)} = e^{-t\phi(\lambda)}, \tag{2.1}$$

where the Laplace exponent is $\phi(\lambda) = \int_0^\infty (1 - e^{-\lambda x})\nu(dx)$.

Define its inverse:

$$E_t := \inf\{\tau > 0 : D(\tau) > t\}. \tag{2.2}$$

This paper focuses on different stabilities of the following SDE:

$$\begin{aligned} dX(t) &= f(t, E_t, X(t-))dt + k(t, E_t, X(t-))dE_t + g(t, E_t, X(t-))dB_{E_t} \\ &\quad + \int_{|y|<c} h(t, E_t, X(t-), y)\tilde{N}(dE_t, dy), \end{aligned} \tag{2.3}$$

with $X(0) = x_0$, where f, k, g, h are real-valued functions satisfying the following Lipschitz condition (Assumption 2.1), growth condition (Assumption 2.2) and Assumption 2.3 such that there exists a unique $\mathcal{G}_t = \mathcal{F}_{E_t}$ adapted process $X(t)$ satisfying time-changed SDE (2.3); see Lemma 4.1 in [8].

Assumption 2.1 (Lipschitz condition)**.** *There exists a positive constant K_1 such that*

$$\begin{aligned} &\Big|f(t_1, t_2, x) - f(t_1, t_2, y)\Big|^2 + \Big|k(t_1, t_2, x) - k(t_1, t_2, y)\Big|^2 + \Big|g(t_1, t_2, x) - g(t_1, t_2, y)\Big|^2 \\ &+ \int_{|z|<c} \Big|h(t_1, t_2, x, z) - h(t_1, t_2, x, z)\Big|^2 \nu(dz) \leq K_1|x - y|^2, \end{aligned} \tag{2.4}$$

for all $t_1, t_2 \in \mathbb{R}_+$ and $x, y \in \mathbb{R}$.

Assumption 2.2 (Growth condition)**.** *There exists a positive constant K_2 such that, for all $t_1, t_2 \in \mathbb{R}_+$ and $x \in \mathbb{R}$,*

$$|f(t_1, t_2, x)|^2 + |g(t_1, t_2, x)|^2 + |h(t_1, t_2, x)|^2 + \int_{|y|<c} |h(t_1, t_2, x)|^2 \nu(dy) \leq K_2(1 + |x|^2). \tag{2.5}$$

Assumption 2.3. *If $X(t)$ is an RCLL and $\mathcal{G}_t$-adapted process, then*

$$f(t, E_t, X(t)), k(t, E_t, X(t)), g(t, E_t, X(t)), h(t, E_t, X(t), y) \in \mathcal{L}(\mathcal{G}_t), \tag{2.6}$$

where $\mathcal{L}(\mathcal{G}_t)$ denotes the class of RCLL and $\mathcal{G}_t$-adapted processes.

Next we give definitions of different stabilities of SDE. Consider a differential equation $dX(t) = f(X(t), t)dt$ for $t \geq 0$. Assume existence and uniqueness of solution $X(t, x_0)$ for each initial value $X(0) = x_0$; also assume that $f(0, t) = 0$ for $t \geq 0$. Then the differential equation has trivial solution $X(t) \equiv 0$. In general, the stability of the trivial solution means the insensitivity of the system to the small change in initial value. More details can be found in [12]. Two categories of stabilities will be discussed in this paper: among stabilities in probability, stochastically asymptotically stable is weaker than globally stochastically stable, while stochastically stable is weakest; among moment stabilities, p-th moment exponentially stable is stronger than p-th moment asymptotically stable.

Definition 2.4.

(1) The trivial solution of the time-changed SDE (2.3) is said to be stochastically stable or stable in probability if for every pair of $\epsilon \in (0,1)$ and $r > 0$, there exists a $\delta = \delta(\epsilon, r) > 0$ such that

$$P\{|X(t,x_0)| < r \textit{ for all } t \geq 0\} \geq 1 - \epsilon \tag{2.7}$$

whenever $|x_0| < \delta$.

(2) The trivial solution of the time-changed SDE (2.3) is said to be stochastically asymptotically stable if for every $\epsilon \in (0,1)$, there exists a $\delta_0 = \delta_0(\epsilon) > 0$ such that

$$P\{\lim_{t\to\infty} X(t,x_0) = 0\} \geq 1 - \epsilon \tag{2.8}$$

whenever $|x_0| < \delta_0$.

(3) The trivial solution of the time-changed SDE (2.3) is said to be globally stochastically asymptotically stable or stochastically asymptotically stable in the large if it is stochastically stable and for all $x_0 \in \mathbb{R}$,

$$P\{\lim_{t\to\infty} X(t,x_0) = 0\} = 1. \tag{2.9}$$

Definition 2.5.

(1) The trivial solution of the time-changed SDE (2.3) is said to be p-th moment exponentially stable if there are positive constants λ and C such that

$$\mathbb{E}[|X(t)|^p] \leq C|x_0|^p \exp(-\lambda t), \ \forall t \geq 0, \ \forall x_0 \in \mathbb{R}, \ p > 0. \tag{2.10}$$

(2) The trivial solution of the time-changed SDE (2.3) is said to be p-th moment asymptotically stable if there is a function $v(t) : [0,+\infty) \to [0,\infty)$ decaying to 0 as $t \to \infty$ and a positive constant C such that

$$\mathbb{E}[|X(t)|^p] \leq C|x_0|^p v(t), \ \forall t \geq 0, \ \forall x_0 \in \mathbb{R}, \ p > 0. \tag{2.11}$$

Let $\mathcal{K}$ denote the family of all nondecreasing functions $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\mu(r) > 0$ for all $r > 0$. Also let $S_h = \{x \in \mathbb{R} : |x| < h\}$ and $\bar{S_h} = \{x \in \mathbb{R} : |x| \leq h\}$ for all $h > 0$.

3. Main results

In this section, the time-changed Itô formula driven by SDE (2.3) will be given, then necessary conditions for different stabilities will be established, followed by some examples.

The next lemma is a version of the Itô formula in Corollary 3.4 in [8].

Lemma 3.1 (Itô formula for time-changed Lévy noise)**.** *Let $D(t)$ be an RCLL subordinator and E_t its inverse process as in* (2.2)*. Define a filtration $\{\mathcal{G}_t\}_{t\geq 0}$ by $\mathcal{G}_t = \mathcal{F}_{E_t}$. Let X be a process defined as follows:*

$$\begin{aligned} X(t) = x_0 &+ \int_0^t f(t,E_t,X(t-))dt + \int_0^t k(t,E_t,X(t-))dE_t + \int_0^t g(t,E_t,X(t-))dB_{E_t} \\ &+ \int_0^t \int_{|y|<c} h(t,E_t,X(t-),y)\tilde{N}(dE_t,dy), \end{aligned} \tag{3.1}$$

where f, k, g, h are measurable functions such that all integrals are defined. Here c is the maximum allowable jump size.

Then, for all $F : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ *in* $C^{1,1,2}(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$*, with probability one,*

(3.2)

$$
\begin{aligned}
F(t,E_t,X(t)) - F(0,0,x_0) &= \int_0^t L_1F(s,E_s,X(s-))ds + \int_0^t L_2F(s,E_s,X(s-))dE_s \\
&+ \int_0^t\int_{|y|<c}\Big[F(s,E_s,X(s-)+h(s,E_s,X(s-),y)) - F(s,E_s,X(s-))\Big]\tilde{N}(dE_s,dy) \\
&+ \int_0^t F_x(s,E_s,X(s-))g(s,E_s,X(s-))dB_{E_s},
\end{aligned}
$$

where

(3.3)

$$
\begin{aligned}
L_1F(t_1,t_2,x) &= F_{t_1}(t_1,t_2,x) + F_x(t_1,t_2,x)f(t_1,t_2,x), \\
L_2F(t_1,t_2,x) &= F_{t_2}(t_1,t_2,x) + F_x(t_1,t_2,x)k(t_1,t_2,x) + \frac{1}{2}g^2(t_1,t_2,x)F_{xx}(t_1,t_2,x) \\
&+ \int_{|y|<c}\Big[F(t_1,t_2,x+h(t_1,t_2,x,y)) - F(t_1,t_2,x) - F_x(t_1,t_2,x)h(t_1,t_2,x,y)\Big]\nu(dy).
\end{aligned}
$$

Proof. This proof is a direct application of the multidimensional Itô formula, which is established in Corollary 3.4 in [8], to $F(t,E_t,X(t))$ in $C^{1,1,2}(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$:

(3.4)

$$
\begin{aligned}
F(t,E_t,X(t)) - F(0,0,x_0) &= \int_0^t F_{t_1}(s,E_s,X(s-))ds + \int_0^t F_{t_2}(s,E_s,X(s-))dE_s \\
&+ \int_0^t F_x(s,E_s,X(s-))\Big[f(s,E_s,X(s-))ds + k(s,E_s,X(s-))dE_s \\
&+ g(s,E_s,X(s-))dB_{E_s}\Big] + \frac{1}{2}\int_0^t F_{xx}(s,E_s,X(s-))g(s,E_s,X(s-))dE_s \\
&+ \int_0^t \int_{|y|<c}\Big[F(s,E_s,X(s-)+h(s,E_s,X(s-),y)) - F(s,E_s,X(s-))\Big]\tilde{N}(dE_s,dy) \\
&+ \int_0^t \int_{|y|<c}\Big[F(s,E_s,X(s-)+h(s,E_s,X(s-),y)) - F(s,E_s,X(s-)) \\
&\qquad - F_x(s,E_s,X(s-))h(s,E_s,X(s-),y)\Big]\nu(dy)dE_s \\
&= \int_0^t L_1F(s,E_s,X(s-))ds + \int_0^t L_2F(s,E_s,X(s-))dE_s \\
&+ \int_0^t \int_{|y|<c}\Big[F(s,E_s,X(s-)+h(s,E_s,X(s-),y)) - F(s,E_s,X(s-))\Big]\tilde{N}(dE_s,dy) \\
&+ \int_0^t F_x(s,E_s,X(s-))g(s,E_s,X(s-))dB_{E_s}.
\end{aligned}
$$

□

Lemma 3.2. *Let* $D(t)$ *be an RCLL subordinator and* E_t *its inverse process as in* (2.2). *Define a filtration* $\{\mathcal{G}_t\}_{t\geq 0}$ *by* $\mathcal{G}_t = \mathcal{F}_{E_t}$. *Let* $\tilde{N}$ *be a compensated Poisson measure defined on* $\mathbb{R}_+ \times (\mathbb{R} - \{0\})$ *with intensity measure* ν*, where* ν *is a Lévy measure such that* $\tilde{N}(dt,dy) = N(dt,dy) - \nu(dy)dt$ *and* $\int_{\mathbb{R}-\{0\}}(|y|^2 \wedge 1)\nu(dy) < \infty$.

Then, for any $A \in \mathcal{B}(\mathbb{R} - \{0\})$ bounded below, the time-changed process $\tilde{N}(E_t, A)$ is a martingale.

Proof. Let $\tau_n = \inf\{t \geq 0; |\tilde{N}(t, A)| \geq n\}$; it is obvious that $\tau_n \to \infty$ as $n \to \infty$. Then $|\tilde{N}(\tau_n \wedge t, A)| \leq n+1$, for all $t \in \mathbb{R}_+$; thus $\tilde{N}(\tau_n \wedge t, A)$ is a bounded martingale.

By the optional stopping theorem, for any $0 \leq s < t$,

$$\mathbb{E}\Big[\tilde{N}(\tau_n \wedge E_t, A)|\mathcal{G}_s\Big] = \tilde{N}(\tau_n \wedge E_s, A). \tag{3.5}$$

The right hand side $\tilde{N}(\tau_n \wedge E_s, A)$ converges to $\tilde{N}(E_s, A)$, as $n \to \infty$. For the left hand side, we have

$$|\tilde{N}(\tau_n \wedge E_t, A)| \leq \sup_{0 \leq u \leq t} |\tilde{N}(E_u, A)|. \tag{3.6}$$

Thus, by Hölder's inequality and Doob's martingale inequality,

$$\begin{aligned}
\mathbb{E}\Big[\Big|\tilde{N}(\tau_n \wedge E_t, A)\Big|\Big] &\leq \mathbb{E}\Big[\Big|\sup_{0 \leq u \leq t} |\tilde{N}(E_u, A)\Big|\Big] = \mathbb{E}\Big[\Big|\sup_{0 \leq u \leq E_t} |\tilde{N}(u, A)\Big|\Big] \\
&= \int_0^\infty \mathbb{E}\Big[\Big|\sup_{0 \leq u \leq \tau} |\tilde{N}(u, A)\Big|\Big|\tau = E_t\Big] f_{E_t}(\tau) d\tau \\
&\leq \int_0^\infty \mathbb{E}\Big[\Big|\sup_{0 \leq u \leq \tau} |\tilde{N}(u, A)\Big|^2\Big|\tau = E_t\Big]^{\frac{1}{2}} f_{E_t}(\tau) d\tau \\
&\leq \int_0^\infty 2\mathbb{E}\Big[\Big||\tilde{N}(\tau, A)\Big|^2\Big|\tau = E_t\Big]^{\frac{1}{2}} f_{E_t}(\tau) d\tau \\
&= 2\int_0^\infty [\nu(A)\tau]^{\frac{1}{2}} f_{E_t}(\tau) d\tau \\
&= 2\nu(A)^{\frac{1}{2}} \mathbb{E}[E_t^{\frac{1}{2}}] \\
&\leq 2\nu(A)^{\frac{1}{2}} \mathbb{E}[E_t]^{\frac{1}{2}},
\end{aligned} \tag{3.7}$$

where the last inequality follows from Jensen's inequality.

For any $t \geq 0$ and $x > 0$, by Markov's inequality, we have

$$P(E_t > s) \leq P(D(s) < t) = P(e^{-xD(s)} \geq e^{-xt}) \leq e^{xt}\mathbb{E}[e^{-xD(s)}] = e^{xt}e^{-s\phi(x)}. \tag{3.8}$$

It follows that

$$\mathbb{E}[E_t] = \int_0^\infty P(E_t > s) ds = e^{xt}\frac{1}{\phi(x)} < \infty. \tag{3.9}$$

Then, by the dominated convergence theorem, we have

$$\mathbb{E}\Big[\tilde{N}(\tau_n \wedge E_t, A)|\mathcal{G}_s\Big] \to \mathbb{E}\Big[\tilde{N}(E_t, A)|\mathcal{G}_s\Big], \tag{3.10}$$

as $n \to \infty$. So

$$\mathbb{E}\Big[\tilde{N}(E_t, A)|\mathcal{G}_s\Big] = \tilde{N}(E_s, A). \tag{3.11}$$

Also,

$$\mathbb{E}\Big[|\tilde{N}(E_t, A)|\Big] \leq \mathbb{E}\Big[\sup_{0 \leq u \leq t} |\tilde{N}(E_u, A)|\Big] < \infty; \tag{3.12}$$

thus $\tilde{N}(E_t, A)$ is a martingale. $\square$

Theorem 3.3. *Assume there exists a function* $V(t_1,t_2,x) \in C^{1,1,2}(\mathbb{R}_+ \times \mathbb{R}_+ \times S_h, \mathbb{R})$ *with* $h \geq 2c$ *and* $\mu \in \mathcal{K}$ *such that for all* $(t_1,t_2,x) \in \mathbb{R}_+ \times \mathbb{R}_+ \times S_h$*:*

$$\begin{aligned} &1.\ V(t_1,t_2,0)=0,\\ &2.\ \mu(|x|) \leq V(t_1,t_2,x),\\ &3.\ L_1V(t_1,t_2,x) \leq 0,\\ &4.\ L_2V(t_1,t_2,x) \leq 0. \end{aligned} \tag{3.13}$$

Then the trivial solution of the time-changed SDE (2.3) *is stochastically stable or stable in probability.*

The proofs of Theorem 3.3 and other results in this section are given in Section 4.

Remark 3.4. Note that L_1 and L_2 mentioned here and in the following theorems are the same as those in Lemma 3.1, and c is the maximum allowable jump size in (2.3).

Theorem 3.5. *Assume there exists a function* $V(t_1,t_2,x) \in C^{1,1,2}(\mathbb{R}_+ \times \mathbb{R}_+ \times S_h, \mathbb{R})$ *with* $h \geq 2c$ *and* $\mu \in \mathcal{K}$ *such that for all* $(t_1,t_2,x) \in \mathbb{R}_+ \times \mathbb{R}_+ \times S_h$*:*

$$\begin{aligned} &1.\ V(t_1,t_2,0)=0,\\ &2.\ \mu(|x|) \leq V(t_1,t_2,x),\\ &3.\ L_1V(t_1,t_2,x) \leq -\gamma_1(\alpha)\ \textit{a.s. and}\ L_2V(t_1,t_2,x) \leq -\gamma_2(\alpha)\ \textit{a.s., for any}\\ &\alpha \in (0,h), \textit{where}\ \gamma_1(\alpha) \geq 0\ \textit{and}\ \gamma_2(\alpha) \geq 0\ \textit{but not equal to zero at the same}\\ &\textit{time},\ x \in S_h - \bar{S_\alpha}. \end{aligned} \tag{3.14}$$

Then the trivial solution of the time-changed SDE (2.3) *is stochastically asymptotically stable.*

Theorem 3.6. *Assume there exists a function* $V(t_1,t_2,x) \in C^{1,1,2}(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$ *and* $u \in \mathcal{K}$ *such that for all* $(t_1,t_2,x) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$*:*

$$\begin{aligned} &1.\ V(t_1,t_2,0)=0,\\ &2.\ \mu(|x|) \leq V(t_1,t_2,x),\\ &3.\ L_1V(t_1,t_2,x) \leq -\gamma_1(x)\ \textit{a.s. and}\ L_2V(t_1,t_2,x) \leq -\gamma_2(x)\ \textit{a.s.},\\ &\ \textit{where}\ \gamma_1(x) \geq 0\ \textit{and}\ \gamma_2(x) \geq 0\ \textit{but not equal to zero at the same time},\\ &4.\ \lim_{|x|\to\infty} \inf_{t_1,t_2 \geq 0} V(t_1,t_2,x) = \infty. \end{aligned} \tag{3.15}$$

Then the trivial solution of the time-changed SDE (2.3) *is globally stochastically asymptotically stable.*

Proof. This proof has similar ideas as in Theorem 4.2.4 in [12], so we omit the details here. □

Example 3.7. Consider the following SDE driven by time-changed Lévy noise:

$$\begin{aligned} dX(t) =& f(t,E_t)X(t)dt + k(t,E_t)X(t)dE_t\\ &+ g(t,E_t)X(t)dB_{E_t} + \int_{|y|<c} h(t,E_t,y)X(t)d\tilde{N}(dE_s,dy) \end{aligned} \tag{3.16}$$

with $X(0) = x_0$, where k, f, g, h are $\mathcal{G}_t$-measurable real-valued functions satisfying Lipschitz condition (Assumption 2.1), growth condition (Assumption 2.2) and Assumption 2.3. Define Lyapunov function

$$V(t_1, t_2, x) = |x|^\alpha \tag{3.17}$$

on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$ for some $\alpha \in (0, 1)$. Then

$$L_1 V(t_1, t_2, x) = \alpha f(t_1, t_2)|x|^\alpha \tag{3.18}$$

and

(3.19)

$$L_2 V(t_1, t_2, x) = \Big[\alpha k(t_1, t_2) + \frac{\alpha(\alpha-1)}{2} g^2(t_1, t_2) + \int_{|y|<c} \Big[|1 + h(t_1, t_2, y)|^\alpha - 1 - \alpha h(t_1, t_2, y)\Big]\nu(dy)\Big]|x|^\alpha.$$

Thus, if

$$\alpha f(t, E_t) \le 0 \quad a.s. \tag{3.20}$$

and

(3.21)

$$\alpha k(t, E_t) + \frac{\alpha(\alpha-1)}{2} g^2(t, E_t) + \int_{|y|<c} \Big[|1 + h(t, E_t, y)|^\alpha - 1 - \alpha h(t, E_t, y)\Big]\nu(dy) \le 0 \quad a.s.$$

for all $t, E_t \in \mathbb{R}_+$, then the trivial solution of SDE (3.16) is stochastically stable, by Theorem 3.3.

Let $\alpha = 0.5, c = 1$ and $f(t_1, t_2) = -1, k(t_1, t_2) = 0.25, g(t_1, t_2) = 1, h(t_1, t_2, y) = y$ for all $t_1, t_2 \in \mathbb{R}_+$. Then

$$L_1 V(t_1, t_2, x) = -\frac{|x|^\alpha}{2} \le 0 \tag{3.22}$$

and

$$L_2 V(t_1, t_2, x) = \int_{|y|<1} \Big[|1 + y|^{\frac{1}{2}} - 1 - \frac{1}{2} y\Big]\nu(dy) < 0. \tag{3.23}$$

Therefore, by Theorem 3.6, the trivial solution of SDE

$$dX(t) = -X(t)dt + 0.25X(t)dE_t + X(t)dB_{E_t} + \int_{|y|<1} yX(t)d\tilde{N}(ds, dy) \tag{3.24}$$

with $X(0) = x_0$ is globally stochastically asymptotically stable.

Remark 3.8. Note that $V(t_1, t_2, x) = |x|^\alpha$ with $\alpha \in (0, 1)$ is not a C^2 function with respect to x in $\mathcal{R}$, but it is sufficient in this case since $X(t) \ne 0$ if $X(0) \ne 0$ for $t \ge 0$; see the following for details. That is, $V(t_1, t_2, x) = |x|^\alpha$ is a C^2 function with respect to x in the domain of $X(t)$ for $t \ge 0$.

By the Itô formula for time-changed Lévy noise, we have, for $x_0 \neq 0$,

(3.25)

$$\begin{aligned}
\ln(|X(t)|) =& \ln(|x_0|) + \int_0^t \frac{1}{X(s-)} f(s, E_S)X(s-)ds + \int_0^t \frac{1}{X(s-)} g(s, E_S)X(s-)dB_{E_s} \\
&+ \int_0^t \Big[\frac{1}{X(s-)} k(s, E_S)X(s-) + \frac{1}{2} g(s, E_s)^2 X(s-)^2 \frac{-1}{X(s-)^2} \\
&+ \int_{|y|<c} \Big[\ln(|X(s-) + h(s, E_s, y)X(s-)|) - \ln(|X(s-)|) \\
&- \frac{1}{X(s-)} h(s, E_s, y)X(s-)\Big]\nu(dy)\Big] dE_s \\
&+ \int_0^t \int_{|y|<c} \big[\ln(|X(s-) + h(s, E_s, y)X(s-)|) - \ln(|X(s-)|)\big] \tilde{N}(dE_s, dy) \\
=& \ln(|x_0|) + \int_0^t f(s, E_S)ds + \int_0^t g(s, E_S)dB_{E_s} \\
&+ \int_0^t \int_{|y|<c} \big[\ln(|1 + h(s, E_s, y)|)\big] \tilde{N}(dE_s, dy) \\
&+ \int_0^t \Big[k(s, E_S) - \frac{1}{2} g(s, E_s)^2 \\
&+ \int_{|y|<c} \big[\ln(|1 + h(s, E_s, y)|) - h(s, E_s, y)\Big] \nu(dy)\Big] dE_s.
\end{aligned}$$

Let

(3.26)

$$\begin{aligned}
M(t) =& \int_0^t f(s, E_S)ds + \int_0^t g(s, E_S)dB_{E_s} + \int_0^t \int_{|y|<c} \big[\ln(|1 + h(s, E_s, y)|\big] \tilde{N}(dE_s, dy) \\
&+ \int_0^t \Big[k(s, E_S) - \frac{1}{2} g(s, E_s)^2 + \int_{|y|<c} \big[\ln(|1 + h(s, E_s, y)| - h(s, E_s, y)\big]\nu(dy)\Big] dE_s.
\end{aligned}$$

Then $|X(t)| = |x_0| \exp(M(t)) > 0$ for all $t \geq 0$.

A similar argument applies to Example 3.10.

Theorem 3.9. *Let $p, \alpha_1, \alpha_2, \alpha_3$ be positive constants. If $V \in C^2(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}; \mathbb{R}_+)$ satisfies*

$$\begin{aligned}
&1.\ V(t_1, t_2, 0) = 0, \qquad 2.\ \alpha_1|x|^p \leq V(t_1, t_2, x) \leq \alpha_2|x|^p, \\
&3.\ L_2V(t_1, t_2, x) \leq 0, \quad 4.\ L_1V(t_1, t_2, x) \leq -\alpha_3V(t_1, t_2, x),
\end{aligned} \tag{3.27}$$

$\forall(t_1, t_2, x) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$, *then the trivial solution of the time-changed SDE* (2.3) *is p-th moment exponentially stable with*

$$\mathbb{E}|X(t, x_0)|^p \leq \frac{\alpha_2}{\alpha_1}|x_0|^p \exp(-\alpha_3 t). \tag{3.28}$$

Example 3.10. Consider the following SDE driven by time-changed Lévy noise:

$$dX(t) = -X(t)dt + X(t)E_t^2 dB_{E_t} + \int_{|y|<1} \Big[X(t)y^2 - X(t)\Big]\tilde{N}(dE_t, dy) \tag{3.29}$$

with $X(0) = x_0$ and ν a Lévy measure. Let $V(t_1, t_2, x) = |x|$. Then

$$L_1V(t_1, t_2, x) = -|x| \tag{3.30}$$

and

$$L_2V(t_1,t_2,x) = \frac{1}{2}x^2t_2^4(-\frac{1}{x^2})+\int_{|y|<1}\Big[|x+xy^2-x|-|x|-sgn(x)(xy^2-x)\Big]\nu(dy)$$
$$= -\frac{t_2^4}{2}+\int_{|y|<1}\Big[(|y^2|-y^2)|x|\Big]\nu(dy)\le 0. \tag{3.31}$$

By Theorem 3.9, $X(t)$ is first moment exponentially stable, that is,

$$\mathbb{E}|X(t,x_0)| \le |x_0|\exp(-t), \qquad \forall t\ge 0. \tag{3.32}$$

Next, we reduce SDE (2.3) by setting $f(t,E_t,X(t-))=0$:

$$dX(t) = k(E_t,X(t-))dE_t + g(E_t,X(t-))dB_{E_t} + \int_{|y|<c} h(E_t,X(t-),y)\tilde{N}(dE_t,dy), \tag{3.33}$$

with $X(0)=x_0$.

Kobayashi [8] mentioned duality related to (3.33) and the following SDE:

$$dY(t) = k(t,Y(t-))dt+g(t,Y(t-))dB_t+\int_{|y|<c} h(t,Y(t-),y)\tilde{N}(dt,dy), Y(0)=x_0, \tag{3.34}$$

with $Y(0)=x_0$, stating that:

1. If a process $Y(t)$ satisfies SDE (3.34), then $X(t) := Y(E_t)$ satisfies the time-changed SDE (3.33).
2. If a process $X(t)$ satisfies the time-changed SDE (3.33), then $Y(t) := X(D(t))$ satisfies SDE (3.34).

Corollary 3.11. *Let $Y(t)$ be a stochastically stable (stochastically asymptotically stable, globally stochastically asymptotically stable) process satisfying SDE* (3.34). *Then the trivial solution $X(t)$ of SDE* (3.33) *is a stochastically stable (stochastically asymptotically stable, globally stochastically asymptotically stable) process, respectively.*

Proof. This proof has similar ideas as in Corollary 3.1 in [18]; thus we omit the details. Though the conclusion of Corollary 3.1 in [18] is correct, there is a minor problem in the proof. We correct it as follows:

$$\begin{aligned} P\Big\{|X(t,x_0)|<h,\forall t\ge 0\Big\} &= P\Big\{|Y(E_t,x_0)|<h,\forall t\ge 0\Big\} \\ &= P\Big\{\sup_{0\le t<\infty}|Y(E_t,x_0)|<h\Big\} \\ &= P\Big\{\sup_{\{E_t:\ 0\le t<\infty\}}|Y(E_t,x_0)|<h\Big\} \\ &= P\Big\{\sup_{0\le \tau<\infty}|Y(\tau,x_0)|<h\Big\} \\ &= P\Big\{|Y(t,x_0)|<h,\forall t\ge 0\Big\} \\ &= 1-\epsilon. \end{aligned} \tag{3.35}$$

Here, we use the fact that the image of $[0,\infty)$ under the E_t process is almost surely equal to $[0,\infty)$. $\square$

Corollary 3.12. *Let $Y(t)$ be a p-th moment exponentially stable process satisfying SDE* (3.34)*, and let $X(t)$ be a p-th moment asymptotically stable satisfying SDE* (3.33)*.*

Remark 3.13. Existence of p-th moment stability of the solution of SDE (3.34) has been proved by Theorem 3.5.1 in Siakalli [16].

Remark 3.14. Our results cannot be easily extended to time-changed stochastic differential equations with large jumps. This is because that stochastic integral against the Poisson process is not automatically local martingale. Thus, the normal method to prove stability of solutions of time-changed stochastic differential equations as used in this paper does not work. It is possible to apply stricter conditions to derive similar results for time-changed stochastic differential equations with large jumps, but the strength of the results has to be compromised.

Remark 3.15. The Lyapunov functions V in our main results above vary from case to case, but under certain conditions it is possible to construct Lyapunov functions by a general formula; see [2] as an example.

4. Proofs of main results

4.1. **Proof of Theorem 3.3.**

Proof. Let $\epsilon \in (0,1)$ and $r \in (0,h)$ be arbitrary. By continuity of $V(t_1,t_2,x)$ and the fact that $V(t_1,t_2,0)=0$, we can find a $\delta=\delta(\epsilon,r,0)>0$ such that

$$\frac{1}{\epsilon}\sup_{x\in S_\delta} V(0,0,x_0) \le \mu(r). \tag{4.1}$$

By (4.1) and condition (2), $\delta<r$. Fix initial value $x_0 \in S_\delta$ arbitrarily and define the stopping time

$$\tau_r = \inf\{t\ge 0 : |X(t,x_0)| \ge r\}, \tag{4.2}$$

where $r \le \frac{h}{2}$, and

(4.3)
$$\begin{aligned}
U_k =& k\wedge \inf\{t\ge 0; \left|\int_0^{\tau_r\wedge t} V_x(s,E_s,X(s-))g(s,E_s,X(s-))dB_{E_s}\right| \ge k\},\\
W_k =& k\wedge \inf\{t\ge 0; \left|\int_0^{\tau_r\wedge t}\int_{|y|<c}\Big[V(s,E_s,X(s-)+H(s,E_s,X(s-),y))\right.\\
&\left.\qquad - V(s,E_s,X(s-))\Big]\tilde{N}(dE_s,dy)\right| \ge k\},
\end{aligned}$$

for $k = 1, 2, \ldots$. It is easy to see that $U_k \to \infty$ and $W_k \to \infty$ as $k \to \infty$. Apply Itô formula (3.2) to $V(t_1, t_2, x)$ associated with SDE (2.3). Then for any $t \geq 0$,

$$\begin{aligned}
&V(t \wedge \tau_r \wedge U_k \wedge W_k, E_{t\wedge\tau_r\wedge U_k\wedge W_k}, X(t \wedge \tau_r \wedge U_k \wedge W_k)) - V(0, 0, x_0) \\
&= \int_0^{t\wedge\tau_r\wedge U_k\wedge W_k} L_1 V(s, E_s, X(s-))ds \\
&+ \int_0^{t\wedge\tau_r\wedge U_k\wedge W_k} L_2 V(s, E_s, X(s-))dE_s \\
&+ \int_0^{t\wedge\tau_r\wedge U_k\wedge W_k} V_x(s, E_s, X(s-))g(s, E_s, X(s-))dB_{E_s} \\
&+ \int_0^{t\wedge\tau_r\wedge U_k\wedge W_k} \int_{|y|<c} \Big[V(s, E_s, X(s-) + H(s, E_s, X(s-), y)) \\
&\qquad - V(s, E_s, X(s-))\Big]\tilde{N}(dE_s, dy).
\end{aligned} \tag{4.4}$$

By [11] and [9], both

$$\int_0^{t\wedge\tau_r\wedge U_k\wedge W_k} V_x(s, E_s, X(s-))g(s, E_s, X(s-))dB_{E_s} \tag{4.5}$$

and

$$\begin{aligned}
&\int_0^{t\wedge\tau_r\wedge U_k\wedge W_k} \int_{|y|<c} \Big[V(s, E_s, X(s-) + H(s, E_s, X(s-), y)) \\
&\qquad - V(s, E_s, X(s-))\Big]\tilde{N}(dE_s, dy)
\end{aligned} \tag{4.6}$$

are mean zero martingales.

Taking expectations on both sides, we have

$$\mathbb{E}[V(t \wedge \tau_r \wedge U_k \wedge W_k, E_{t\wedge\tau_r\wedge U_k\wedge W_k}, X(t \wedge \tau_r \wedge U_k \wedge W_k))] \leq V(0, 0, x_0).$$

Letting $k \to \infty$,

$$\mathbb{E}[V(t \wedge \tau_r, E_{t\wedge\tau_r}, X(t \wedge \tau_r))] \leq V(0, 0, x_0).$$

Now, $|X(t \wedge \tau_r)| < r$ for $t < \tau_r$. For all $w \in \{\tau_r < \infty\}$, $|X(\tau_r)(w)| \leq r + c \leq h$. Since $V(t_1, t_2, x) \geq \mu(|x|)$ for all $x \in S_h$, we have for all $w \in \{\tau_r < \infty\}$,

$$V(\tau_r, E_{\tau_r}, X(\tau_r)(w)) \geq \mu(|X(\tau_r)(w)|) \geq \mu(r). \tag{4.7}$$

Also,

$$V(0, 0, x_0) \geq \mathbb{E}[V(t\wedge\tau_r, E_{t\wedge\tau_r}, X(t\wedge\tau_r))1_{\{\tau_r<t\}}] \geq \mathbb{E}[\mu(r)1_{\{\tau_r<t\}}] = \mu(r)P(\tau_r < t). \tag{4.8}$$

Thus, combined with (4.1),

$$P(\tau_r < t) \leq \frac{V(0, 0, x_0)}{\mu(r)} \leq \frac{\epsilon\mu(r)}{\mu(r)} = \epsilon. \tag{4.9}$$

Then, letting $t \to \infty$, we have

$$P(\tau_r < \infty) \leq \epsilon; \tag{4.10}$$

equivalently,

$$P(|X(t,x_0)| < r \text{ for all } t \geq 0) \geq 1-\epsilon, \tag{4.11}$$

so $X(t,x_0)$ is stochastically stable. □

4.2. **Proof of Theorem 3.5.**

Proof. By Theorem 3.3, the trivial solution of (2.3) is stochastically stable. For any fixed $\epsilon \in (0,1)$, there exists $\delta = \delta(\epsilon) > 0$ such that

$$P(|X(t,x_0)| < h) \geq 1 - \frac{\epsilon}{5} \tag{4.12}$$

when $x_0 \in S_\delta$. Fix $x_0 \in S_\delta$ and let $0 < \alpha < \beta < |x_0|$ arbitrarily. Define the following stopping times:
(4.13)

$$\begin{aligned}
&\tau_h = \inf\{t \geq 0; |X(t,x_0)| > h\},\\
&\tau_\alpha = \inf\{t \geq 0; |X(t,x_0)| < \alpha\},\\
&U_k = k \wedge \inf\{t \geq 0; \left| \int_0^{t\wedge\tau_h\wedge\tau_\alpha} V_x(s,E_s,X(s-))g(s,E_s,X(s-))dB_{E_s} \right| \geq k\},\\
&W_k = k \wedge \inf\{t \geq 0; \left| \int_0^{t\wedge\tau_h\wedge\tau_\alpha} \int_{|y|<c} \big[V_x(s,E_s,X(s-)+h(s,E_s,X(s-),y)) \right.\\
&\qquad\qquad \left. - V_x(s,E_s,X(s-))\big]\tilde{N}(dE_s,dy) \right| \geq k\}.
\end{aligned}$$

By Itô's formula (3.2), we have
(4.14)

$$\begin{aligned}
0 &\leq \mathbb{E}\big[V(t\wedge\tau_h\wedge\tau_\alpha\wedge U_k\wedge W_k, E_{t\wedge\tau_h\wedge\tau_\alpha\wedge U_k\wedge W_k}, X(t\wedge\tau_h\wedge\tau_\alpha\wedge U_k\wedge W_k))\big]\\
&= V(0,0,x_0) + \mathbb{E}\int_0^{t\wedge\tau_h\wedge\tau_\alpha\wedge U_k\wedge W_k} L_1V(s,E_s,X(s-))ds\\
&\qquad + \mathbb{E}\int_0^{t\wedge\tau_h\wedge\tau_\alpha\wedge U_k\wedge W_k} L_2V(s,E_s,X(s-))dE_s\\
&\leq V(0,0,x_0) - \gamma_1(\alpha)\mathbb{E}[t\wedge\tau_h\wedge\tau_\alpha\wedge U_k\wedge W_k] - \gamma_2(\alpha)\mathbb{E}[E_{t\wedge\tau_h\wedge\tau_\alpha\wedge U_k\wedge W_k}].
\end{aligned}$$

Letting $k \to \infty$ and $t \to \infty$, we have

$$\gamma_1(\alpha)\mathbb{E}[\tau_h\wedge\tau_\alpha] + \gamma_2(\alpha)\mathbb{E}[E_{\tau_h\wedge\tau_\alpha}] \leq V(0,0,x_0). \tag{4.15}$$

By condition (3) and $E_t \to \infty$ a.s. as $t \to \infty$ (see proof of Theorem 3.9), we have

$$P(\tau_h\wedge\tau_\alpha < \infty) = 1. \tag{4.16}$$

Since $P(\tau_h = \infty) > 1 - \frac{\epsilon}{5}$, it follows that $P(\tau_h < \infty) \leq \frac{\epsilon}{5}$. Thus

$$1 = P(\tau_h\wedge\tau_\alpha < \infty) \leq P(\tau_h < \infty) + P(\tau_\alpha < \infty) \leq P(\tau_\alpha < \infty) + \frac{\epsilon}{5}, \tag{4.17}$$

that is,

$$P(\tau_\alpha < \infty) \geq 1 - \frac{\epsilon}{5}. \tag{4.18}$$

Choose θ sufficiently large for

$$P(\tau_\alpha < \theta) \geq 1 - \frac{2\epsilon}{5}. \tag{4.19}$$

Then

$$\begin{aligned} P(\tau_\alpha < \tau_h \wedge \theta) &\geq P(\{\tau_\alpha < \theta\} \cap \{\tau_h = \infty\}) = P(\tau_\alpha < \theta) - P(\{\tau_\alpha < \theta\} \cap \{\tau_h < \infty\}) \\ &\geq P(\tau_\alpha < \theta) - P(\tau_h < \infty) \geq 1 - \frac{2\epsilon}{5} - \frac{\epsilon}{5} = 1 - \frac{3\epsilon}{5}. \end{aligned} \tag{4.20}$$

Now define some stopping times:

$$\sigma = \begin{cases} \tau_\alpha, & if\ \tau_\alpha < \tau_h \wedge \theta, \\ \infty, & otherwise, \end{cases} \tag{4.21}$$

$$\begin{aligned} \tau_\beta &= \inf\{t \geq \sigma; |X(t, x_0)| \geq \beta\}, \\ S_i &= \inf\{t \geq \sigma; \Big| \int_\sigma^{\tau_\beta \wedge t} V_x(s, E_s, X(s-)) g(s, E_s, X(s-)) dB_{E_s} \Big| \geq i\}, \\ T_i &= \inf\{t \geq \sigma; \Big| \int_\sigma^{\tau_\beta \wedge t} \int_{|y|<c} \big[V(s, E_s, X(s-) + h(s, E_s, X(s-), y)) \\ &\qquad - V(s, E_s, X(s-))\big] \tilde{N}(dE_s, dy) \Big| \geq i\}. \end{aligned} \tag{4.22}$$

Again, by Itô's formula,

$$\begin{aligned} &\mathbb{E}\Big[V(t \wedge \tau_\beta \wedge S_i \wedge T_i, E_{t \wedge \tau_\beta \wedge S_i \wedge T_i}, X(t \wedge \tau_\beta \wedge S_i \wedge T_i))\Big] \\ \leq &\mathbb{E}\Big[V(t \wedge \sigma, E_{t\wedge\sigma}, X(t \wedge \sigma \wedge))\Big] + \mathbb{E}\Big[\int_{t\wedge\sigma\wedge}^{t \wedge \tau_\beta \wedge S_i \wedge T_i} L_1 V(s, E_s, X(s-)) ds\Big] \\ &+ \mathbb{E}\Big[\int_{t\wedge\sigma\wedge}^{t \wedge \tau_\beta \wedge S_i \wedge T_i} L_2 V(s, E_s, X(s-)) dE_s\Big] \\ \leq &\mathbb{E}\Big[V(t \wedge \sigma, E_{t\wedge\sigma}, X(t \wedge \sigma))\Big]. \end{aligned} \tag{4.23}$$

Letting $i \to \infty$,

$$\mathbb{E}\Big[V(\sigma \wedge t, E_{\sigma\wedge t}, X(\sigma \wedge t))\Big] \geq \mathbb{E}\Big[V(\tau_\beta \wedge t, E_{\tau_\beta \wedge t}, X(\tau_\beta \wedge t))\Big], \tag{4.24}$$

that is,

$$\begin{aligned} &\mathbb{E}\Big[V(\sigma \wedge t, E_{\sigma\wedge t}, X(\sigma \wedge t))[\mathbb{1}_{\{\sigma<\infty\}} + \mathbb{1}_{\{\sigma=\infty\}}]\Big] \\ &\qquad \geq \mathbb{E}\Big[V(\tau_\beta \wedge t, E_{\tau_\beta \wedge t}, X(\tau_\beta \wedge t))[\mathbb{1}_{\{\sigma<\infty\}} + \mathbb{1}_{\{\sigma=\infty\}}]\Big]. \end{aligned} \tag{4.25}$$

For $w \in \{\tau_\alpha \geq \tau_h \wedge \theta\}$, we have $\sigma = \infty$. Then $\tau_\beta = \infty$; thus

$$V(\sigma \wedge t, E_{\sigma\wedge t}, X(\sigma \wedge t)) = V(t, E_t, X(t)) \tag{4.26}$$

and

$$V(\tau_\beta \wedge t, E_{\tau_\beta \wedge t}, X(\tau_\beta \wedge t)) = V(t, E_t, X(t)). \tag{4.27}$$

Thus,

$$\mathbb{E}\Big[V(\sigma\wedge t, E_{\sigma\wedge t}, X(\sigma\wedge t))\mathbb{1}_{\{\sigma<\infty\}}\Big] \geq \mathbb{E}\Big[V(\tau_\beta\wedge t, E_{\tau_\beta\wedge t}, X(\tau_\beta\wedge t))\mathbb{1}_{\{\sigma<\infty\}}\Big]. \tag{4.28}$$

Now, focusing on the right hand side of (4.28), by definition of τ_β, $\tau_\beta \geq \sigma$, thus $\mathbb{1}_{\{\sigma<\infty\}} \geq \mathbb{1}_{\{\tau_\beta<\infty\}}$. Then

$$\mathbb{E}\Big[V(\tau_\beta\wedge t, E_{\tau_\beta\wedge t}, X(\tau_\beta\wedge t))\mathbb{1}_{\{\sigma<\infty\}}\Big] \geq \mathbb{E}\Big[V(\tau_\beta\wedge t, E_{\tau_\beta\wedge t}, X(\tau_\beta\wedge t))\mathbb{1}_{\{\tau_\beta<\infty\}}\Big]. \tag{4.29}$$

Combining (4.28) and (4.29), we have

$$\mathbb{E}\Big[V(\sigma\wedge t, E_{\sigma\wedge t}, X(\sigma\wedge t))\mathbb{1}_{\{\sigma<\infty\}}\Big] \geq \mathbb{E}\Big[V(\tau_\beta\wedge t, E_{\tau_\beta\wedge t}, X(\tau_\beta\wedge t))\mathbb{1}_{\{\tau_\beta<\infty\}}\Big]. \tag{4.30}$$

Since $P(\sigma<\infty) = P(\tau_\alpha < \tau_h\wedge\theta)$ and $P(\tau_\beta<\infty) \geq P(\{\tau_\beta<\infty\}\cap\{\tau_h=\infty\})$, it follows that

$$\mathbb{E}\big[V(\tau_\beta, E_{\tau_\beta}, X(\tau_\beta))\mathbb{1}_{\{\tau_\beta<\infty\}\cap\{\tau_h=\infty\}}\big] \leq \mathbb{E}\big[V(\tau_\alpha, E_{\tau_\alpha}, X(\tau_\alpha))\mathbb{1}_{\{\tau_\alpha<\tau_h\wedge\theta\}}\big]. \tag{4.31}$$

By condition (2)

$$0 \leq \mu(|x|) \leq V(t_1,t_2,x), \tag{4.32}$$

for all $(t_1,t_2,x)\in\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}$, and $|X(\tau_\beta)|\geq\beta>0$.

Then, for the left hand side of (4.31), we have

$$\begin{aligned}\mathbb{E}\big[V(\tau_\beta, E_{\tau_\beta}, X(\tau_\beta))\mathbb{1}_{\{\tau_\beta<\infty\}\cap\{\tau_h=\infty\}}\big] &\geq \mathbb{E}\big[\mu(|X(\tau_\beta)|)\mathbb{1}_{\{\tau_\beta<\infty\}\cap\{\tau_h=\infty\}}\big]\\ &\geq \mathbb{E}\big[\mu(\beta)\mathbb{1}_{\{\tau_\beta<\infty\}\cap\{\tau_h=\infty\}}\big]\\ &= \mu(\beta)\mathbb{E}\big[\mathbb{1}_{\{\tau_\beta<\infty\}\cap\{\tau_h=\infty\}}\big]\\ &= \mu(\beta)P(\{\tau_\beta<\infty\}\cap\{\tau_h=\infty\}).\end{aligned} \tag{4.33}$$

Let

$$B_\alpha = \sup_{t_1\times t_2\times x\in\mathbb{R}_+\times\mathbb{R}_+\times\bar{S_\alpha}} V(t_1,t_2,x). \tag{4.34}$$

Then $B_\alpha\to 0$ as $\alpha\to 0$, that is, $\frac{B_\alpha}{\mu(\beta)} < \frac{\epsilon}{5}$ for some α.

For the right hand side of (4.31),

$$\begin{aligned}\mathbb{E}\big[V(\tau_\alpha, E_{\tau_\alpha}, X(\tau_\alpha))\mathbb{1}_{\{\tau_\alpha<\tau_h\wedge\theta\}}\big] &\leq \mathbb{E}\big[B_\alpha\mathbb{1}_{\{\tau_\alpha<\tau_h\wedge\theta\}}\big]\\ &= B_\alpha\mathbb{E}\big[\mathbb{1}_{\{\tau_\alpha<\tau_h\wedge\theta\}}\big]\\ &= B_\alpha P(\tau_\alpha<\tau_h\wedge\theta).\end{aligned} \tag{4.35}$$

Combining (4.33) and (4.35), we have

$$P(\{\tau_\beta<\infty\}\cap\{\tau_h=\infty\})\mu(\beta) \leq B_\alpha P(\tau_\alpha<\tau_h\wedge\theta), \tag{4.36}$$

thus

$$P(\{\tau_\beta<\infty\}\cap\{\tau_h=\infty\}) \leq \frac{B_\alpha}{\mu(\beta)}P(\tau_\alpha<\tau_h\wedge\theta) < \frac{\epsilon}{5}. \tag{4.37}$$

Also,

$$P(\{\tau_\beta<\infty\}\cap\{\tau_h=\infty\}) \geq P(\tau_\beta<\infty) - P(\tau_h<\infty) > P(\tau_\beta<\infty) - \frac{\epsilon}{5}, \tag{4.38}$$

so

$$P(\tau_\beta<\infty) < \frac{2\epsilon}{5}. \tag{4.39}$$

Next

$$
\begin{aligned}
P(\{\sigma < \infty\} \cap \{\tau_\beta = \infty\}) &\geq P(\sigma < \infty) - P(\tau_\beta < \infty) \\
&> P(\tau_\alpha < \tau_h \wedge \theta) - \frac{2\epsilon}{5} \\
&\geq 1 - \frac{3\epsilon}{5} - \frac{2\epsilon}{5} \\
&= 1 - \epsilon.
\end{aligned} \tag{4.40}
$$

Hence,

$$
P\{\omega; \limsup_{t\to\infty} \left|X(t,x_0)\right| \leq \beta\} > 1 - \epsilon. \tag{4.41}
$$

Since β is arbitrary, we have

$$
P\{\omega; \limsup_{t\to\infty} \left|X(t,x_0)\right| = 0\} > 1 - \epsilon, \tag{4.42}
$$

as desired. □

4.3. **Proof of Theorem 3.9.**

Proof. Define a function $Z : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ by

$$
Z(t_1, t_2, x) = \exp(\alpha_3 t_1) V(t_1, t_2, x). \tag{4.43}
$$

Fix any $x_0 \neq 0$ in $\mathbb{R}$. For each $n \geq |x_0|$, define

$$
\tau_n = \inf\{t \geq 0 : |X(t)| \geq n\}
$$

and

$$
\begin{aligned}
U_k =& k \wedge \inf\{t \geq 0; \left| \int_0^{\tau_n \wedge t} V_x(s, E_s, X(s-)) g(s, E_s, X(s-)) dB_{E_s} \right| \geq k\}, \\
W_k =& k \wedge \inf\{t \geq 0; \left| \int_0^{\tau_n \wedge t} \int_{|y|<c} \Big[V(s, E_s, X(s-) + h(s, E_s, X(s-), y)) \right. \\
& \left. - V(s, E_s, X(s-)) \Big] \tilde{N}(ds, dy) \right| \geq k\},
\end{aligned} \tag{4.44}
$$

for $k = 1, 2, \ldots$. It is easy to see that $U_k \to \infty$ and $W_k \to \infty$ as $k \to \infty$.

Apply Itô formula (3.2) to $Z(\tau_n \wedge U_k \wedge W_k, E_{\tau_n \wedge U_k \wedge W_k}, X(\tau_n \wedge U_k \wedge W_k))$. Then we have

(4.45)

$$
\begin{aligned}
&Z(t \wedge \tau_n \wedge U_k \wedge W_k, E_{t\wedge\tau_n\wedge U_k\wedge W_k}, X(t \wedge \tau_n \wedge U_k \wedge W_k)) - Z(0,0,x_0)\\
&= \int_0^{t\wedge\tau_n\wedge U_k\wedge W_k} \exp(\alpha_3 s)\Big[\alpha_3 V(s, E_s, X(s-)) + V_s(s, E_s, X(s-)\Big])ds\\
&\quad + \int_0^{t\wedge\tau_n\wedge U_k\wedge W_k} \exp(\alpha_3 s) V_{E_s}(s, E_s, X(s-))dE_s\\
&\quad + \int_0^{t\wedge\tau_n\wedge U_k\wedge W_k} \exp(\alpha_3 s) V_x(s, E_s, X(s-))\Big[f(s, E_s, X(s-))dt\\
&\qquad\qquad + k(s, E_s, X(s-))dE_t + g(s, E_s, X(s-))dB_{E_t}\Big]\\
&\quad + \frac{1}{2}\int_0^{t\wedge\tau_n\wedge U_k\wedge W_k} \exp(\alpha_3 s) V_{xx}(s, E_s, X(s-))g^2(s, E_s, X(s-))dE_s\\
&\quad + \int_0^{t\wedge\tau_n\wedge U_k\wedge W_k}\int_{|y|<c} \exp(\alpha_3 s)\Big[V(s, E_s, X(s-) + h(s, E_s, X(s-), y))\\
&\qquad\qquad - V(s, E_s, X(s-))\Big]\tilde{N}(dE_s, dy)\\
&\quad + \int_0^{t\wedge\tau_n\wedge U_k\wedge W_k}\int_{|y|<c} \exp(\alpha_3 s)\Big[V(s, E_s, X(s-) + h(s, E_s, X(s-), y))\\
&\qquad\qquad - V(s, E_s, X(s-)) - V_x(s, E_s, X(s-))h(s, E_s, X(s-), y)\Big]\nu(dy)dE_s\\
&= \int_0^{t\wedge\tau_n\wedge U_k\wedge W_k} \exp(\alpha_3 s)\Big[\alpha_3 V(s, E_s, X(s-)) + V_s(s, E_s, X(s-))\\
&\qquad\qquad + V_x(s, E_s, X(s-))f(s, E_s, X(s-))\Big]ds\\
&\quad + \int_0^{t\wedge\tau_n\wedge U_k\wedge W_k} \exp(\alpha_3 s)\Big[V_{E_s}(s, E_s, X(s-)) + V_x(s, E_s, X(s-))k(s, E_s, X(s-))\\
&\quad + \frac{1}{2}V_{xx}(s, E_s, X(s-))g^2(s, E_s, X(s-))\\
&\quad + \int_{|y|<c}\Big[V(s, E_s, X(s-) + h(s, E_s, X(s-), y))\\
&\qquad\qquad - V(s, E_s, X(s-)) - V_x(s, E_s, X(s-))h(s, E_s, X(s-), y)\Big]\nu(dy)\Big]dE_s\\
&\quad + \int_0^{t\wedge\tau_n\wedge U_k\wedge W_k} \exp(\alpha_3 s) g(s, E_s, X(s-))V_x(s, E_s, X(s-))dB_{E_s}\\
&\quad + \int_0^{t\wedge\tau_n\wedge U_k\wedge W_k}\int_{|y|<c} \exp(\alpha_3 s)\Big[V(s, E_s, X(s-) + h(s, E_s, X(s-), y))\\
&\qquad\qquad - V(s, E_s, X(s-))\Big]\tilde{N}(dE_s, dy)
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^{t\wedge\tau_n\wedge U_k\wedge W_k} \exp(\alpha_3 s)\Big[\alpha_3 V(s,E_s,X(s-)) + L_1 V(s,E_s,X(s-)\Big] ds \\
&\quad + \int_0^{t\wedge\tau_n\wedge U_k\wedge W_k} \exp(\alpha_3 s) L_2 V(s,E_s,X(s-)) dE_s \\
&\quad + \int_0^{t\wedge\tau_n\wedge U_k\wedge W_k} \exp(\alpha_3 s) g(s,E_s,X(s-)) V_x(s,E_s,X(s-)) dB_{E_s} \\
&\quad + \int_0^{t\wedge\tau_n\wedge U_k\wedge W_k} \int_{|y|<c} \exp(\alpha_3 s)\Big[V(s,E_s,X(s-)+h(s,E_s,X(s-),y)) \\
&\qquad\qquad - V(s,E_s,X(s-))\Big]\tilde{N}(dE_s,dy).
\end{aligned}
$$

By similar ideas as in the proof of (4.1), we have that

$$
\int_0^{t\wedge\tau_n\wedge U_k\wedge W_k} \exp(\alpha_3 s) g(s,E_s,X(s-)) V_x(s,E_s,X(s-)) dB_{E_s}
$$

and

$$
\begin{aligned}
&\int_0^{t\wedge\tau_n\wedge U_k\wedge W_k} \int_{|y|<c} \exp(\alpha_3 s)\Big[V(s,E_s,X(s-)+h(s,E_s,X(s-),y)) \\
&\qquad\qquad - V(s,E_s,X(s-))\Big]\tilde{N}(dE_s,dy)
\end{aligned}
$$

are mean zero martingales. Taking expectations on both sides, we have

$$
\begin{aligned}
&\mathbb{E}[\exp(\alpha_3(t\wedge\tau_n\wedge U_k\wedge W_k))V(t\wedge\tau_n\wedge U_k\wedge W_k, E_{t\wedge\tau_n\wedge U_k\wedge W_k}, X(t\wedge\tau_n\wedge U_k\wedge W_k))] \\
\le& \mathbb{E}\int_0^{t\wedge\tau_n\wedge U_k\wedge W_k} \exp(\alpha_3 s)\Big[\alpha_3 V(s,E_s,X(s-)) + L_1 V(s,E_s,X(s-)\Big] ds + V(0,0,x_0) \\
\le& V(0,0,x_0).
\end{aligned} \tag{4.46}
$$

Letting $k\to\infty$ and $n\to\infty$, $\mathbb{E}[\exp(\alpha_3 t)V(t,E_t,X(t))] \le V(0,0,x_0)$. By condition (2),

$$
\alpha_1|X(t)|^p \le V(t,E_t,X(t)). \tag{4.47}
$$

Then

$$
\alpha_1\mathbb{E}(\exp(\alpha_3 t)|X(t)|^p) \le \mathbb{E}(\exp(\alpha_3 s)V(t,E_t,X(t))) \le V(0,0,x_0) \le \alpha_2|x_0|^p, \tag{4.48}
$$

that is,

$$
\mathbb{E}(|X(t)|^p) \le \frac{\alpha_2}{\alpha_1}\exp(-\alpha_3 t)|x_0|^p, \tag{4.49}
$$

as desired. □

4.4. **Proof of Corollary 3.12.**

Proof. If $Y(t)$ satisfies SDE (3.34), by Theorem 4.2 in [8], $X(t) = Y(E_t)$ satisfies (3.33).

Since $Y(t)$ is p-th moment exponentially stable, there exist two positive constants λ and C such that

$$
\mathbb{E}[|X(t)|^p] \le C|x_0|^p\exp(-\lambda t),\ \forall t\ge 0,\ \forall x_0\in\mathbb{R},\ p>0. \tag{4.50}
$$

Then

$$\begin{aligned}\mathbb{E}[|Y(t)|^p] &= \mathbb{E}[|X(E_t)|^p] \\ &= \int_0^\infty \mathbb{E}[|X(s)|^p \exp(\lambda s)\exp(-\lambda s)|E_t = s] f_{E_t}(s)ds \\ &= \int_0^\infty \mathbb{E}[|X(s)|^p \exp(\lambda s)|E_t = s]\exp(-\lambda s) f_{E_t}(s)ds \\ &\le \int_0^\infty C|x_0|^p \exp(-\lambda s) f_{E_t}(s)ds \\ &= C|x_0|^p \mathbb{E}[\exp(-\lambda E_t)].\end{aligned} \tag{4.51}$$

Since E_t is nondecreasing and $E_0 = 0$, by definition of E_t, we claim that $\lim_{t\to\infty} E_t = \infty$ a.s.. Assume to the contrary that there exists $B > 0$ such that $E_t < B$ for all $t > 0$ with positive probability. Then $D(B) > t$ for all $t > 0$ with positive probability. However, by Lemma 12.1 of [3], $D(B)$ is bounded, which results in a contradiction. Consequently, $\mathbb{E}[\exp(-\lambda E_t)] \to 0$ as $t \to \infty$, as desired. □

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