A CONTAINMENT RESULT IN P^n AND THE CHUDNOVSKY CONJECTURE

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ABSTRACT. In this paper we prove the containment $I^{(nm)} \subset M^{(n-1)m}I^m$, for a radical ideal I of s general points in $\mathbb{P}^n$, where $s \geq 2^n$. As a corollary we get that the Chudnovsky Conjecture holds for a very general set of at least 2^n points in $\mathbb{P}^n$.

1. INTRODUCTION

Given any subscheme $Z \subset \mathbb{P}^n$ and its homogenous ideal $I = I_Z$ in $\mathbb{F}[\mathbb{P}^n] = \mathbb{F}[x_0, \dots, x_n]$, we define $\alpha(I)$ as the minimal degree of a non-zero element in I . We will assume that $\text{char } \mathbb{F} = 0$.

In general $\alpha(I)$ is hard to compute and it behaves quite unpredictably. However there is an asymptotic counterpart of the α -invariant, which is the *Waldschmidt constant* ([17]) defined as

$$(1) \quad \widehat{\alpha}(I) = \lim_{m \rightarrow \infty} \frac{\alpha(I^{(m)})}{m},$$

where $I^{(m)}$ denotes the m -th symbolic power of the ideal I (for the definition and basic properties of $I^{(m)}$ see [13]). It turns out that this constant is well defined and satisfies the inequality

$$(2) \quad \alpha(I^{(m)}) \geq m\widehat{\alpha}(I)$$

for all m . In fact $\widehat{\alpha}(I) = \inf_{m \geq 1} \frac{\alpha(I^{(m)})}{m}$.

For an ideal I we have $\alpha(I^n) = n\alpha(I)$, but the behaviour of $\alpha(I^{(m)})$ is much more complicated and less understood. Skoda in 1977 [16] showed that $\alpha(I^{(m)}) \geq m\alpha(I)/n$ for an ideal I of points in $\mathbb{P}^n$ (over complex numbers). Chudnovsky [4], in 1981, improved that bound for $n = 2$ to $\alpha(I^{(m)}) \geq m(\alpha(I) + 1)/2$ and conjectured the following.

Chudnovsky Conjecture. *For an ideal I of points in $\mathbb{P}^n$ the following inequality holds:*

$$\frac{\alpha(I^{(m)})}{m} \geq \frac{\alpha(I) + n - 1}{n}.$$

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In particular

$$\widehat{\alpha}(I) \geq \frac{\alpha(I) + n - 1}{n}.$$

Esnault and Viehweg [11], in 1983 showed that $\alpha(I^{(m)}) \geq m(\alpha(I) + 1)/n$ for any set of points in $\mathbb{P}^n$.

In 2001 Ein, Lazarsfeld, Smith in [10] and in 2002 Hochster and Huneke in [14] showed that, for any homogeneous ideal I in $\mathbb{F}[\mathbb{P}^n]$, the containment $I^{(nm)} \subset I^m$ holds, thus recovering the Skoda bound more generally — for all homogeneous ideals (since $I^{(nm)} \subset I^m$, $\alpha(I^{(nm)}) \geq \alpha(I^m) = nm\alpha(I)/n$; passing with m to infinity gives $\widehat{\alpha}(I) \geq \alpha(I)/n$). Harbourne and Huneke in [13, Lemma 3.2], observed that the Chudnovsky Conjecture would follow from a more general containment

$$(3) \quad I^{(nm)} \subset M^{(n-1)m} I^m,$$

where by $M = (x_0, \dots, x_n)$ we denote the irrelevant maximal ideal.

The containment (3) holds (for a given m) for general points in $\mathbb{P}^2$ ([13], Proposition 3.10), for at most $n + 1$ general points in $\mathbb{P}^n$ and also for general points in $\mathbb{P}^3$ ([7], [8]). As a corollary, the Chudnovsky Conjecture holds for very general points in $\mathbb{P}^3$.

The main result of the present paper is the following theorem.

Theorem 1. *For a non-negative integer m , and for a radical ideal I of s general points in $\mathbb{P}^n$, where $s \geq 2^n$, the containment*

$$I^{(nm)} \subset M^{(n-1)m} I^m$$

holds.

As a corollary, the Chudnovsky Conjecture holds for a very general set of at least 2^n points in $\mathbb{P}^n$.

We will work on filling the gap between $n + 1$ and 2^n in our future project.

2. A BOUND FOR $\widehat{\alpha}(I)$

In this section we give a bound for the Waldschmidt constant of an ideal of s very general points in $\mathbb{P}^n$. The bound in fact easily follows from the much stronger result of Evain [12], who showed that for an ideal I of general k^n points $\alpha(I)$ is “expected”. Since the methods of Evain are highly non-trivial and very delicate, we give a short proof of our bound here to make the paper more self-contained.

Theorem 2. *For a radical ideal I of s very general points $P_1, \dots, P_s$ in $\mathbb{P}^n$ we have*

$$\widehat{\alpha}(I) \geq \lfloor \sqrt[n]{s} \rfloor.$$

Proof. To prove the bound we have to show that the system of divisors of degree $dm - 1$ passing through $P_1, \dots, P_s$ with multiplicity m is empty if $d = \lfloor \sqrt[n]{s} \rfloor$. By $(d, m^{\times s})$ we denote a system of divisors of degree d , passing through s (general) points with multiplicity m .

Without loss of generality we may suppose that $s = k^n, k \in \mathbb{N}$, as the emptiness of the system $(km - 1; m^{\times k^n})$, implies the emptiness of $(km - 1, m^{\times r})$, $k^n \leq r < (k + 1)^n$.

For $n = 1$ the non-existence of the system $(km - 1; m^{\times k})$ is trivial. Then we proceed by induction. (Note that for $n = 2$ the non-existence of the system $(km - 1; m^{\times k^2})$ was also proved by Nagata in [15].)

For $n \geq 2$, suppose there exists a divisor in $\mathbb{P}^n$ of degree $km - 1$, passing through $P_1, \dots, P_s$ with multiplicity m . Take k general hyperplanes in $\mathbb{P}^n$ and put k^{n-1} points on each such hyperplane (in general position). Then on the hyperplane we have the system of divisors of degree $km - 1$ which have to pass through k^{n-1} points with multiplicity m . By inductive assumption, this system is empty. Thus, all hyperplanes must be components of the system. Repeating the procedure (of checking that all hyperplanes must be the components of the system) m times we get that the system has to have degree km , not $km - 1$, a contradiction.

3. A COMBINATORIAL LEMMA

Here we prove an auxiliary combinatorial lemma, which we will use in the proof of Theorem 4.

Lemma 3. (1) For any integers $k \geq 4$ and $n \geq 3$ the following inequality holds:

$$k^n \leq \binom{kn - n}{n}.$$

(2) Moreover, if $n \geq 5$,

$$3^n \leq \binom{2n}{n}$$

holds.

Proof. First we prove (1). Observe that it is enough to prove the inequality for $k = 4$ only. Indeed, our inequality may be written as

$$(nk - n) \cdot \dots \cdot (nk - 2n + 1) \geq n!k^n.$$

The left-hand side is greater than or equal to

$$\left(n - \frac{n}{4}\right) k \cdot \dots \cdot \left(n - \frac{2n-1}{4}\right) k,$$

as $k \geq 4$. Dividing both sides by k^n and multiplying by 4^n we see that this is enough to prove

$$(4n - n) \cdot \dots \cdot (4n - 2n + 1) \geq n!4^n,$$

i.e., our inequality with $k = 4$.

Thus, we have to prove

$$\binom{3n}{n} \geq 4^n.$$

Since

$$\frac{\binom{3(n+1)}{n+1}}{\binom{3n}{n}} = \frac{(3n+3)(3n+2)(3n+1)}{(n+1)(2n+2)(2n+1)} > 4$$

for all $n \geq 1$ and

$$\binom{9}{3} > 4^3,$$

the claim follows by induction.

As for the second claim of the lemma, we proceed analogously, observing that

$$\frac{\binom{2(n+1)}{n+1}}{\binom{2n}{n}} = \frac{(2n+2)(2n+1)}{(n+1)(n+1)} > 3$$

for all $n \geq 1$ and

$$\binom{10}{5} > 3^5.$$

□

4. A CONTAINMENT RESULT

Now we are able to prove our containment theorem.

Theorem 4. *For a radical ideal I of s very general points in $\mathbb{P}^n$, where $s \geq 2^n$ and $n \geq 3$ the containment*

$$I^{(nm)} \subset M^{(n-1)m} I^m$$

holds, for any integer $m \geq 1$.

Proof. Dumnicki in [7] showed the containment for any number of very general points in $\mathbb{P}^3$, so we assume $n \geq 4$.

Let $\text{reg}(I)$ denote the Castelnuovo-Mumford regularity of I , see e.g. [5], and let $\sigma(I)$ denote the maximal degree of an element in a minimal set of generators of I . Since in general $\sigma(I) \leq \text{reg}(I)$, by [13, Proposition 2.3.] it is enough to show that

$$\alpha(I^{(mn)}) \geq (n-1)m + m \text{reg}(I).$$

By (2) the above inequality follows from

$$n\hat{\alpha}(I) \geq n-1 + \text{reg}(I).$$

From Theorem 2 we know that

$$\hat{\alpha}(I) \geq \lfloor \sqrt[n]{s} \rfloor.$$

For the ideal I as in our theorem we have that $\text{reg}(I) = r+1$, where r is such an integer that

$$\binom{r-1+n}{n} < s \leq \binom{r+n}{n},$$

see [2, p. 1176].

Thus, it is enough to prove that

$$(4) \quad n \lfloor \sqrt[n]{s} \rfloor \geq n-1 + r+1,$$

for r satisfying

$$\binom{r-1+n}{n} < s.$$

Suppose to the contrary, that $n \lfloor \sqrt[n]{s} \rfloor < n-1 + r+1$. If we can prove that then

$$(5) \quad \binom{r+n-1}{n} \geq s,$$

we get a contradiction, and we are done.

As $\lfloor \sqrt[n]{s} \rfloor$ is constant (equals $k-1$) for all $s = (k-1)^n, \dots, k^n-1$, we may assume that the right-hand side of inequality (5) equals k^n-1 , which is the worst possible case.

Thus, we reduced the problem to proving that for $r \geq nk-2n+1$ we have

$$k^n-1 \leq \binom{r+n-1}{n}.$$

This will follow if we prove that

$$k^n \leq \binom{kn - n}{n}.$$

This last inequality is proved in Lemma 3 for $n = 4$ and $k \geq 4$ and for $n \geq 5$ and $k \geq 3$. This proves our theorem for all $n \geq 5$ and $s \geq 2^n$ (remember, that we have $s \geq (k - 1)^n$) and also when $n = 4$ and $s \geq 3^n$. Moreover, by direct computation inequality (4) holds for $n = 4$ with $2^4 \leq s \leq 70$, so it remains only to prove our theorem only for $n = 4$ and $71 \leq s < 3^4$. This is done in the lemma below.

Lemma 5. *The Waldschmidt constant for an ideal of at least 71 very general points in $\mathbb{P}^4$ is bounded below by $9/4$.*

Before the proof, let us observe that using the bound $9/4$ instead of $\lfloor \sqrt[4]{s} \rfloor$ in (4) finishes the proof for $n = 4$ and $s = 71, \dots, 80$.

Proof. It is enough to show that for each m the system of hypersurfaces of degree $9m - 1$ with 71 $4m$ -fold points is empty. Since the system of degree $8m - 1$ with 16 $4m$ -fold points is empty, by [6] (Theorem 1) it is enough to show that $9m - 1$ with four $8m$ -fold points and seven $4m$ -fold points is empty. The last one, by Theorem 3 in [6], is equivalent to a system of degree 2 with at least one point of multiplicity $4m$, which is empty. $\square$

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