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ON SIMULTANEOUS NONVANISHING OF THE CENTRAL L-VALUES

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ABSTRACT. In this note we derive a new quantitative result on the simultaneous nonvanishing of the central L-values twisted by quadratic characters, for pairs of holomorphic cuspidal Hecke eigenforms with large weight $2k$.

1. INTRODUCTION

Let $F, G \in S_{2k}(\Gamma_0(1))$ be two Hecke eigenforms. We know by a theorem of Waldspurger [6] that there exist fundamental discriminants D_1, D_2 (i.e. 1 or the discriminants of quadratic fields) such that

$$L(1/2,\ F \otimes \chi_{D_1})\ L(1/2,\ G \otimes \chi_{D_2}) \neq 0,$$

where χ_D is the Kronecker symbol. One natural question is to ask whether it is possible to have $D_1 = D_2$ with the above nonvanishing condition, i.e., whether there exists a fundamental discriminant D such that

$$L(1/2,\ F \otimes \chi_D)\ L(1/2,\ G \otimes \chi_D) \neq 0.$$

Necessarily $(-1)^k D > 0$ since otherwise both central L-values vanish due to the negative sign in the functional equation. The purpose of this note is to investigate this problem quantitatively.

If we define $S^+_{k+1/2}(\Gamma_0(4))$ to be the Kohnen space consisting of cusp forms $h \in S_{k+1/2}(\Gamma_0(4))$ such that

$$h(z) = \sum_{n\geq 1} a_h(n)e(nz),\ a_h(n) = 0 \text{ unless } (-1)^k n \equiv 0, 1 \pmod 4,$$

and let $f,\ g \in S^+_{k+1/2}(\Gamma_0(4))$ correspond to F and G respectively under Shimura correspondence, then the above nonvanishing is equivalent to

$$a_f(|D|)\ a_g(|D|) \neq 0,$$

where $a_f(|D|),\ a_g(|D|)$ are the $|D|$-th Fourier coefficients of $f,\ g$ respectively.

Assume k is even, and denote by H_{2k} the orthonormal Hecke basis of $S_{2k}(\Gamma_0(1))$. For $F \in H_{2k}$ write

$$F(z) = \sum_{n\geq 1} a_F(n) n^{\frac{2k-1}{2}} e(nz),$$

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where

$$a_F(n) = a_F(1)\lambda_F(n)\ (\ n \geq 1),\ |a_F(1)|^2 = \frac{(4\pi)^{2k-1}}{\Gamma(2k)} \frac{2\pi^2}{L(1,\ \mathrm{sym}^2(F))},$$

and $\lambda_F(n)$ is the n-th normalized Hecke eigenvalue of F.

In [5] we have shown that

$$\sum_{F\in H_{2k},\ L(1/2,\ F)\neq 0} 1 \gg k, \tag{1}$$

i.e., as even $k \to \infty$, there are positive proportions of Hecke eigenforms in H_{2k} with nonvanishing central L-values. Equivalently there are positive proportions of Hecke eigenforms $f \in S^+_{k+1/2}(4)$ with nonvanishing first Fourier coefficient $a_f(1) \neq 0$.

This result has the following implications with regard to the nonvanishing question mentioned at the beginning. For any $f \in H_{k+1/2}$, the orthonormal Hecke basis in $S^+_{k+1/2}(\Gamma_0(4))$, we define

$$N_k(f) = \#\{g\in H_{k+1/2},\ a_g(|D|)a_f(|D|)\neq 0 \text{ for some fundamental discriminant } D\}.$$

In view of the Waldspurger formula (see §2) with $D = 1$, the bound (1) implies that for a positive proportion of $f \in H_{k+1/2}$, we have

$$N_k(f) \gg k.$$

The goal of this note is to prove, via a uniform bound for the cubic moment of the twisted central L-values [7], the following quantitative result for *individual* form.

Theorem. *For any $f \in H_{k+1/2}$ and any $\epsilon > 0$, we have*

$$N_k(f) \gg_\epsilon k^{1/2-\epsilon}.$$

2. Simultaneous nonvanishing

Let $k \geq 6$ be an integer (not necessarily even in this section) and $S^+_{k+1/2}(\Gamma_0(4))$ be the Kohnen subspace of holomorphic cusp forms of weight $k + 1/2$ for $\Gamma_0(4)$, and denote by $H_{k+1/2} = \{f_j\}_{1\leq j\leq J}$ an L^2-*normalized* (i.e. orthonormal) Hecke basis for $S^+_{k+1/2}(\Gamma_0(4))$, where $J = \dim_{\mathbf{C}} S^+_{k+1/2}(\Gamma_0(4))$. Kohnen [2] showed that $S^+_{k+1/2}(\Gamma_0(4))$ is isomorphic to $S_{2k}(\Gamma_0(1))$ as Hecke modules under Shimura correspondence. For

$$f_j(z) = \sum_{n\geq 1} b_j(n)e(nz) \in H_{k+1/2},$$

let

$$F_j(z) = \sum_{n\geq 1} a_j(n)e(nz) \in S_{2k}(\Gamma_0(1))$$

with $a_j(1) = 1$, be the corresponding Shimura lift of $f_j(z)$. Then $\{F_j\}_{1\leq j\leq J}$ is the *arithmetically normalized* Hecke basis for $S_{2k}(\Gamma_0(1))$, and moreover by the works of Waldspurger we have for any fundamental discriminant D with $(-1)^k D > 0$,

$$|b_j(|D|)|^2 = \frac{(k-1)!}{\pi^k}|D|^{k-1/2}\frac{L(1/2, F_j \otimes \chi_D)}{\langle F_j,\ F_j\rangle},$$

where $\chi_D = \left(\frac{D}{\cdot}\right)$ is the Kronecker symbol (see [4]). Since

$$\langle F_j,\ F_j\rangle \ = \ \frac{\Gamma(2k)}{(4\pi)^{2k-1}}\frac{L(1,\mathrm{sym}^2(F_j))}{2\pi^2},$$

it follows

$$\frac{\Gamma(k+1/2)}{(4\pi)^{k-1/2}}\frac{|b_j(|D|)|^2}{|D|^{k-1/2}} = \frac{2\pi^2 L(1/2, F_j\otimes\chi_D)}{L(1,\mathrm{sym}^2(F_j))}. \tag{2}$$

Fix j_0 and assume $b_{j_0}(|D_0|)$ is the first nonvanishing Fourier coefficient of $f_{j_0}(z)$. D_0 is necessarily a fundamental discriminant with $(-1)^k D_0 > 0$, in view of the identity

$$b_{j_0}(n^2|D_0|) = b_{j_0}(|D_0|)\sum_{d|n}\mu(d)\left(\frac{D_0}{d}\right)d^{k-1}a_{j_0}(n/d).$$

We need a result of Hecke [1, Satz 1, 811].

Lemma (Hecke)**.** *Let*

$$f(z) = \sum_{n\geq 0} a_f(n)\, e(nz) \in M_k(\Gamma_0(N)).$$

If $f\neq 0$, then for some n with

$$n \leq \frac{k}{12}N\prod_{p|N}\left(1+\frac{1}{p}\right)+1,$$

we have $a_f(n)\neq 0$.

Since $f_{j_0}^4(z) \in S_{4k+2}(\Gamma_0(4))$, we deduce from the above Hecke's result that $4|D_0| \leq 2k+2$, so $|D_0| < k/2+1$.

On the other hand from the uniform bound for the cubic moment of $L(1/2, F_j\otimes\chi_D)$ due to M. Young [7], we have

$$\sum_{1\leq j\leq J} L^3(1/2, F_j\otimes\chi_D) \ll (kD)^{1+\epsilon/2}, \tag{3}$$

for any $\epsilon > 0$.

For any fundamental discriminant D with $(-1)^k D > 0$, it follows from [3, Proposition 4, p. 252] that we have the following Petersson formula on $S^+_{k+1/2}(\Gamma_0(4))$:

$$\begin{aligned}&\frac{\Gamma(k+1/2)}{(4\pi)^{k-1/2}}\sum_{1\leq j\leq J}\frac{|b_j(|D|)|^2}{|D|^{k-1/2}}\\ =\ &4(k-1/2)\left(1+A_k\sum_{c\geq 1}\left(1+\left(\frac{4}{c}\right)\right)\frac{K_{2k+1}(|D|,|D|;4c)}{4c}\ J_{k-1/2}\left(\frac{\pi|D|}{c}\right)\right),\end{aligned}$$

where

$$A_k = (-1)^{[(k+1)/2]}\sqrt{2}\pi(1-(-1)^k i),$$

$$K_\kappa(m,n;c) = \sum_{d(\mathrm{mod}\, c)}\epsilon_d^{-\kappa}\left(\frac{c}{d}\right)e\left(\frac{m\overline{d}+nd}{c}\right),$$

and

$$\epsilon_d = \begin{cases} 1, & d\equiv 1 \ (\mathrm{mod}\ 4),\\ i, & d\equiv -1\ (\mathrm{mod}\ 4).\end{cases}$$

From the bounds for the Bessel function

$$J_\nu(x) \ll \left(\frac{ex}{2\nu+1}\right)^\nu, \quad J_\nu(x) \ll x^{-1/4}(|x-\nu| + \nu^{1/3})^{-1/4},$$

where $x > 0,\ \nu \to \infty$, and assuming $|D| \ll k$, we deduce that

$$\begin{aligned}
&\sum_{c\geq 1} \left(1 + \left(\frac{4}{c}\right)\right) \frac{K_{2k+1}(|D|,|D|;4c)}{4c}\ J_{k-1/2}\left(\frac{\pi|D|}{c}\right) \\
\ll\ & \sum_{c \gg |D|/k} c^{-2}k^2 2^{-k} + \sum_{c\ll 1} \frac{K_{2k+1}(|D|,|D|;4c)}{4c} k^{-1/3} \\
\ll\ & k^{-1/3}.
\end{aligned}$$

Thus in particular for $D = D_0$ we obtain

$$\frac{\Gamma(k+1/2)}{(4\pi)^{k-1/2}} \sum_{1\leq j\leq J} \frac{|b_j(|D_0|)|^2}{|D_0|^{k-1/2}} = 4(k-1/2) + O(k^{2/3}). \tag{4}$$

From (2), we have in turn

$$\sum_{1\leq j\leq J} \frac{2\pi^2 L(1/2, F_j \otimes \chi_{D_0})}{L(1, \mathrm{sym}^2(F_j))} = 4(k-1/2) + O(k^{2/3}). \tag{5}$$

In view of the bound

$$k^{-\epsilon} \ll L(1,\ \mathrm{sym}^2(F_j)) \ll k^\epsilon$$

and (3), we infer by the Hölder's inequality that

$$\begin{aligned}
k \quad \ll \quad & k^\epsilon \Big(\sum_{1\leq j\leq J} L^3(1/2, F_j\otimes\chi_{D_0})\Big)^{1/3} \\
& \times(\#\{j,\ 1\leq j\leq J,\ L(1/2,F_j\otimes\chi_{D_0}) \neq 0\})^{2/3}.
\end{aligned}$$

Thus we have

$$\#\{j,\ 1\leq j\leq J,\ L(1/2, F_j\otimes\chi_{D_0}) \neq 0\} \gg_\epsilon k^{1/2-\epsilon},$$

for any $\epsilon > 0$. In terms of $N_k(f)$ introduced in §1, the above can be reformulated as for any $f \in H_{k+1/2}$,

$$N_k(f) \gg k^{1/2-\epsilon}.$$

This completes the proof of the Theorem.

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