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KUMMER SUBSPACES OF TENSOR PRODUCTS OF CYCLIC ALGEBRAS

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Abstract. We discuss the Kummer subspaces of tensor products of cyclic algebras, focusing mainly on the case of cyclic algebras of degree 3. We present a family of maximal Kummer spaces in any tensor product of cyclic algebras of prime degree, classify all the monomial Kummer spaces in tensor products of cyclic algebras of degree 3, and provide an upper bound for the dimension of Kummer spaces in the generic tensor product of cyclic algebras of degree 3.

1. Introduction

Let p be a prime integer and F be an infinite field of characteristic not p containing a primitive pth root of unity ρ. A cyclic algebra of degree p over F is an algebra that can be presented as

$$F[x, y : x^p = \alpha, y^p = \beta, yx = \rho xy]$$

for some $\alpha, \beta \in F^\times$. We denote this presentation by $(\alpha, \beta)_{p,F}$. A given algebra can have more than one presentation. Fixing a presentation, we call the elements of the form cx^iy^j, where i and j are integers between 0 and $p-1$ and $c \in F$, "monomials". The same goes for tensor products of cyclic algebras; i.e. if we fix presentations $F[x_k, y_k : x_k^p = \alpha_k, y_k^p = \beta_k, y_kx_k = \rho x_ky_k] = (\alpha_k, \beta_k)_{p,F}$ for $k \in \{1, \dots, n\}$, then the monomials in the tensor product $\bigotimes_{k=1}^n (\alpha_k, \beta_k)_{p,F}$ are of the form $c\prod_{k=1}^n x_k^{i_k} y_k^{j_k}$ where $i_1, j_1, \dots, i_n, j_n$ are integers between 0 and $p-1$ and $c \in F$.

Let A be a tensor product of n cyclic algebras of degree p. Assume A is a division algebra. An element $x \in A$ is called Kummer if $x^p \in F$ and $x^k \notin F$ for all $k \in \{1, \dots, p-1\}$. An F-vector subspace of A is called Kummer if all its nonzero elements are Kummer. A necessary and sufficient condition for a space $Fx_1 + \cdots + Fx_m$ to be Kummer is that $x_1^{d_1} * \cdots * x_m^{d_m} \in F$ for every m-tuple of nonnegative integers $d_1, \dots, d_m$ satisfying $d_1 + \cdots + d_m = p$ [CV12, Corollary 2.2]. The expression $x_1^{d_1} * \cdots * x_m^{d_m}$ stands for the sum of all the words in which each x_k appears d_k times, e.g. $x_1^2 * x_2 = x_1^2x_2 + x_1x_2x_1 + x_2x_1^2$. This notation was introduced in [Rev77]. Fixing the presentations of the cyclic algebras, a Kummer space is called "monomial" if it is spanned by monomials.

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Any central simple algebra of exponent p over F is Brauer equivalent to a tensor product of cyclic algebras of degree p by the renowned Merkurjev-Suslin theorem [MS82]. Its symbol length is the minimal number of cyclic algebras required in order to express its Brauer class. In [Mat16] Kummer spaces were used in bounding the symbol length from above when F is a C_m field. This motivates the study of such spaces, and in particular their structure and possible dimensions.

We conjecture that the maximal dimension of a Kummer subspace of A is $np+1$. The conjecture is known to hold true in certain special cases: for $p = 2$ and any positive integer n (an immediate result of the theory of Clifford algebras of quadratic forms; see [Lam73, Chapter 5, Section 2]), and for $p = 3$ and $n = 1$. In the latter case, the Kummer spaces were completely classified in [Rac09] and [MV12]. A classification of the Kummer spaces in the case of $p = n = 2$ can be found in [CV13, Section 5].

In [CGM16] the monomial Kummer spaces in division cyclic algebras of prime degree were classified. Furthermore, an upper bound of $p + 1$ was provided for the dimension of the Kummer subspaces of the generic cyclic algebra, i.e. the algebra $(\alpha, \beta)_{p,K}$ where $K = F(\alpha, \beta)$ is the function field in two algebraically independent variables over F.

In this paper we present some new results on the classification of Kummer spaces and their maximality. In Section 2 we present a family of maximal Kummer subspaces in any tensor product of cyclic algebras. This section is based on results from [Cha09, Chapter 2, Section 1] and [Cha13, Chapter 2, Section 2]. In Section 3 we classify the monomial Kummer subspaces of tensor products of cyclic algebras of degree 3. This section is based on results from [Cha13, Chapter 2, Section 6]. In Section 4 we provide an upper bound for the dimension of a Kummer subspace of the generic tensor product of cyclic algebras of degree 3.

2. Maximal Kummer subspaces of tensor products of cyclic algebras

Fix $A = \bigotimes_{k=1}^{n}(\alpha_k, \beta_k)_{p,F} = \bigotimes_{k=1}^{n} F[x_k, y_k : x_k^p = \alpha_k, y_k^p = \beta_k, y_k x_k = \rho x_k y_k]$. Assume A is a division algebra. Let $V_0 = F$ and $V_k = F[x_k]y_k + V_{k-1}x_k$ for every $k \in \{1, \dots, n\}$. The dimension of each V_k is $pk + 1$. These spaces appeared also in [Mat16, Section 3].

Proposition 2.1. *For each $k \in \{1, \dots, n\}$, V_k is a Kummer space.*

Proof. Assume V_{k-1} is either Kummer or F for a certain k. Every element of V_k is of the form $f(x_k)y_k + vx_k$ for some $f(x_k) \in F[x_k]$ and $v \in V_{k-1}$. Since v commutes with x_k and y_k, and $y_k x_k = \rho x_k y_k$, $(f(x_k)y_k + vx_k)^p = (f(x_k)y_k)^p + v^p x_k^p = \mathrm{N}_{F[x_k]/F}(f(x_k))\beta_k + v^p\alpha_k \in F$ (see [CV12, Remark 2.5]). For any $1 \le m \le p-1$, if $f(x_k) \neq 0$, then the eigenvector of $(f(x_k)y_k+vx_k)^m$ corresponding to the eigenvalue ρ^m with respect to conjugation by x_k is $(f(x_k)y_k)^m$, which is not zero, and therefore $(f(x_k)y_k + vx_k)^m \notin F$. If $f(x_k) = 0$, then what is left is vx_k, and of course $v^m x_k^m \notin F$. Consequently, V_k is Kummer. Since $V_0 = F$, by induction V_k is Kummer for every $k \in \{1, \dots, n\}$. □

Remark 2.2. If $uv = \rho^k vu$ for some $u, v \in A \setminus \{0\}$ and $k \in \{0, \dots, p-1\}$, then $u^{p-1} * v = 0$ if $k \neq 0$ and $u^{p-1} * v = pu^{p-1}v$ if $k = 0$. Hence, if $uv = vu$ and $Fu + Fv$ is a Kummer subspace of A, then v is a scalar multiple of u.

Theorem 2.3. *For each $k \in \{1, \dots, n\}$, V_k is a maximal Kummer subspace of A with respect to inclusion.*

Proof. Fix some $k \in \{1, \dots, n\}$. The Kummer space V_k has a basis of monomials

$$B_k = \left\{ x_i^\ell y_i \left(\prod_{j=i+1}^{k} x_j \right) : 1 \le i \le k, 0 \le \ell \le p-1 \right\} \cup \{x_1 x_2 \dots x_k\}.$$

Note that when $i = k$, $\prod_{j=i+1}^{k} x_j$ is the empty product and so it is 1, and therefore $x_i^\ell y_i \left(\prod_{j=i+1}^{k} x_j \right) = x_k^\ell y_k \in B_k$. Let z be an element in A and write $z = \sum_\alpha m_\alpha$ for some monomials m_α. Assume $V_k + Fz$ is Kummer. Let w be an element in B_k. Since $w^{p-1} * z \in F$, for every α either m_α is a scalar multiple of w or it does not commute with w. Therefore, by subtracting from z an element of V_k, we may assume that none of the monomials m_α commute with any w in B_k.

It remains to show that every monomial commutes with some element in B_k, which implies that $z = 0$: Consider a monomial $t = c x_1^{d_1} y_1^{e_1} \dots x_n^{d_n} y_n^{e_n}$. If $e_1 = e_2 = \dots = e_n = 0$, then t commutes with $x_1 x_2 \dots x_k \in B_k$. Otherwise, let i be the maximal integer in $\{1, \dots, n\}$ with $e_i \neq 0$. Then t commutes with $x_i^\ell y_i x_{i+1} \dots x_k \in B_k$ where ℓ is the unique integer in $\{0, \dots, p-1\}$ with $\ell \equiv d_i e_i^{-1} \pmod p$. □

3. Monomial Kummer subspaces of tensor products of cyclic algebras of degree 3

Keep A as before and assume $p = 3$. Let $\mathcal{X}$ be the set of all Kummer elements in A. We build a directed graph $(\mathcal{X}, E)$ by drawing an edge from y to x,

$$y \longrightarrow x,$$

if $yxy^{-1} = \rho x$. For any subset $B \subset \mathcal{X}$, (B, E_B) is the subgraph obtained by taking the vertices in B and all the edges between them. The set B is called ρ-commuting if it is linearly independent over F and for every two distinct elements $x, y \in B$, $yxy^{-1} = \rho^k x$ for $k \in \{0, 1, 2\}$. In particular, any set of linearly independent monomials is ρ-commuting.

Remark 3.1. If $\{x, y\}$ is a ρ-commuting set spanning a Kummer space, then either $x \longrightarrow y$ or $x \longleftarrow y$.

Proof. If $xy = yx$, then $x^2 * y = 3x^2 y \in F$, which means that $y \in Fx$, a contradiction. □

According to [CV12, Corollary 2.2], a set $\{x_1, \dots, x_m\}$ spans a Kummer space if and only if every subset of cardinality 3 $\{x_i, x_j, x_k\}$ spans a Kummer space. Therefore we will start with sets of cardinality 3.

Lemma 3.2. *Given a ρ-commuting set $\{x, y, z\}$, $Fx + Fy + Fz$ is Kummer if and only if (up to some permutation on $\{x, y, z\}$) either*

$$\begin{tikzcd} x \arrow[r] \arrow[d] & y \\ z \arrow[ur] & \end{tikzcd}$$

or $xyz \in F$*, in which case*

$$\begin{tikzcd} x & y \arrow[l] \\ z \arrow[u] & \end{tikzcd}$$

Proof. From Remark 3.1, the only possible graphs (up to permutation of the vertices) are the two graphs above. In the first case, $x * y * z = 0$, so there are no extra conditions. In the second case, $x * y * z = -3\rho^{-1}xyz \in F$. The opposite direction is a straightforward computation. □

Let B be a ρ-commuting set spanning a Kummer space. We will now study the properties of the directed graph (B, E_B). By a cycle we always mean a **simple directed cycle**.

Proposition 3.3. *If* (B, E_B) *contains a cycle of length* 3*:*

$$\begin{tikzcd} x_0 & x_1 \arrow[l] \\ x_2 \arrow[u] & \end{tikzcd}$$

then for every $y \in B \setminus \{x_0, x_1, x_2\}$*, either* $x_k \longrightarrow y$ *for all* $k \in \{0, 1, 2\}$ *or* $x_k \longleftarrow y$ *for all* $k \in \{0, 1, 2\}$*.*

Proof. If $x_0 \longrightarrow y$ and $x_1 \longleftarrow y$, then

$$\begin{tikzcd} x_0 & x_1 \arrow[l] \\ y \arrow[u] & \end{tikzcd}$$

which means that $yx_0x_1 \in F$. Since $x_0x_1x_2 \in F$, we get $y \in Fx_2$, which contradicts the linear independence. □

Proposition 3.4. *The cycles in* (B, E_B) *are vertex-disjoint.*

Proof. First assume that

$$\begin{tikzcd} x_0 & x_1 \arrow[l] \arrow[d] \\ x_2 \arrow[u] & y \arrow[l] \end{tikzcd}$$

Then $yx_1x_2 \in F$ and $x_0x_1x_2 \in F$, which means that $y \in Fx_0$, and that contradicts the linear independence.

Assume that

$$\begin{tikzcd} x_0 & x_1 \arrow[l] \arrow[d] & y_1 \arrow[l] \\ x_2 \arrow[u] & y_2 \arrow[ur] & \end{tikzcd}$$

From Proposition 3.3 we have $x_0 \longrightarrow y_2$ and $y_1 \longrightarrow x_0$. But then

$$\begin{tikzcd} x_0 & y_1 \arrow[l] \arrow[d] \\ y_2 \arrow[u] & x_1 \arrow[l] \end{tikzcd}$$

and we already saw that this is impossible. □

Proposition 3.5. *There are no cycles of length greater than* 3 *in* (B, E_B)*.*

Proof. Assume

$$\begin{tikzcd} x_1 \arrow[dr] & x_2 \arrow[l] & \dots \arrow[l] & x_{r-1} \arrow[l] \\ & & x_r \arrow[ur] & \end{tikzcd}$$

for some $r \geq 4$. Let i be the maximal integer between 1 and r such that $x_i \longrightarrow x_1$. Now, $x_1 \longrightarrow x_{i+1}$. Therefore

$$\begin{tikzcd} x_1 \arrow[d] & x_i \arrow[l] \\ x_{i+1} \arrow[ur] & \end{tikzcd}$$

If $i \geq 3$, then according to Proposition 3.3, $x_1 \longrightarrow x_{i-1}$, which implies that $i \neq 3$, or in other words $i \geq 4$. Let j be the minimal index for which $x_1 \longrightarrow x_{j+1}$. In particular $x_j \longrightarrow x_1$. Now, $j + 1 \leq i - 1$, which means that

$$\begin{tikzcd} x_{i+1} \arrow[d] & x_1 \arrow[l] \arrow[d] & x_j \arrow[l] \\ x_i \arrow[ur] & x_{j+1} \arrow[ur] & \end{tikzcd}$$

But this is impossible. If $i = 2$, then according to Proposition 3.3, $x_4 \longrightarrow x_1$, which contradicts the maximality of i. □

As a consequence we obtain the following theorem:

Theorem 3.6. *A ρ-commuting subset B of $\mathcal{X}$ spans a Kummer space if and only if the graph (B, E_B) satisfies the following axioms:*

(1) *For every two distinct elements $x, y \in B$, either $x \longrightarrow y$ or $x \longleftarrow y$.*
(2) *All cycles are of length 3.*
(3) *The product of all the elements in a cycle is in F.*
(4) *The cycles are vertex-disjoint.*

Proof. The straightforward direction is an immediate result of what we've done so far. The opposite direction is a result of the fact that every three elements in this set span a Kummer space according to Lemma 3.2. □

The following remark may help the reader get an idea of what the graph (B, E_B) looks like:

Remark 3.7. Assume B is a ρ-commuting set spanning a Kummer space. Let $\sim$ be the following equivalence relation: $x \sim y$ if and only if $x = y$ or x and y belong to the same cycle in (B, E_B). As we already saw, this equivalence relation is also direction preserving in the sense that, assuming $x \not\sim z$, if $x \longrightarrow z$ and $x \sim y$, then $y \longrightarrow z$, and if $z \longrightarrow x$ and $x \sim y$, then $z \longrightarrow y$. Define an order $\leq$ on the equivalence classes: $[x] \leq [z]$ if $[x] = [z]$ or $z \longrightarrow x$. Then the set of equivalence classes is a totally ordered set.

One can therefore visualize the graph as graded into levels, where in each level we have either one element or a cycle, and each element has edges going from it to all the elements in the lower levels.

Corollary 3.8. *Given a ρ-commuting set B spanning a Kummer space, if $\#B = m$, then the longest path $x_1 \longrightarrow x_2 \longrightarrow \cdots \longrightarrow x_r$ in the graph (B, E_B) satisfying $x_i \longrightarrow x_j$ for any $1 \leq i < j \leq r$ is of length no less than $m - \lfloor \frac{m}{3} \rfloor$.*

Proof. Take B and take off exactly one element from each cycle. The number of elements taken off is at most $\lfloor \frac{m}{3} \rfloor$, and what is left satisfies the required condition. $\square$

Corollary 3.9. *The maximal ρ-commuting set spanning a Kummer space in $A = \bigotimes_{k=1}^{n}(\alpha_k, \beta_k)_{3,F}$ is of cardinality $3n+1$.*

Proof. We already know of the existence of monomial Kummer spaces of dimension $3n+1$. According to the previous corollary, if we have a ρ-commuting set B of size $3n+2$ spanning a Kummer space, then we have a path in (B, E_B) of length $2n+2$,

$$x_1 \longrightarrow x_2 \longrightarrow \cdots \longrightarrow x_{2n+2},$$

satisfying $x_i \longrightarrow x_j$ for any $1 \leq i < j \leq 2n+2$. Then the set B generates over F a tensor product of $n+1$ cyclic algebras of degree 3,

$$\begin{aligned} &F[x_1, x_2] \otimes F[x_1 x_2^{-1} x_3, x_1 x_2^{-1} x_4] \\ &\quad \otimes \cdots \otimes F[(\prod_{k=1}^{n} x_{2k-1} x_{2k}^{-1}) x_{2n+1}, (\prod_{k=1}^{n} x_{2k-1} x_{2k}^{-1}) x_{2n+2}], \end{aligned}$$

a contradiction. $\square$

4. The generic tensor product of cyclic algebras

Let $K = F(\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n)$ be the function field in $2n$ algebraically independent variables over F. Fix $A = \bigotimes_{k=1}^{n} K[x_k, y_k : x_k^p = \alpha_k, y_k^p = \beta_k, y_k x_k = \rho x_k y_k]$.

Theorem 4.1. *For any Kummer subspace of A there exists a monomial Kummer space of the same dimension.*

Proof. Write $V = Kv_1 + Kv_2 + \cdots + Kv_m$, where m is the dimension of V. Each v_k is the sum of monomials of the form $c x_1^{i_1} y_1^{j_1} \ldots x_n^{i_n} y_n^{j_n}$ where $i_1, j_1, \ldots, i_n, j_n$ are integers between 0 and $p-1$ and the coefficient c is a quotient of two polynomials in the variables $\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n$ over F. By multiplying by all the denominators, we can assume that the coefficients are polynomials.

Since $\alpha_i = x_i^p$ and $\beta_i = y_i^p$, each v_k is also in the ring of twisted polynomials

$$\begin{aligned} R \quad = \quad & F[x_1, y_1, \ldots, x_n, y_n : y_i x_i = \rho x_i y_i \ \ \forall i \in \{1, \ldots, n\}, \\ & \qquad y_i x_j = x_j y_i, x_i x_j = x_j x_i, y_i y_j = y_j y_i \ \ \forall i \neq j] \end{aligned}$$

over F. Consider the left-to-right lexicographical grading on R: the degree of a nonzero monomial $a x_1^{i_1} y_1^{j_1} \ldots x_n^{i_n} y_n^{i_n}$ is $(i_1, j_1, \ldots, i_n, j_n)$. There is a left-to-right lexicographical ordering on the degrees, i.e. $(i_1, j_1, \ldots, i_n, j_n) > (r_1, s_1, \ldots, r_n, s_n)$ if and only if either $i_1 = r_1, j_1 = s_1, \ldots, i_{k-1} = r_{k-1}, j_{k-1} = s_{k-1}$ and $i_k > r_k$ or $i_1 = r_1, j_1 = s_1, \ldots, i_{k-1} = r_{k-1}, j_{k-1} = s_{k-1}, i_k = r_k$ and $j_k > s_k$ for some $k \in \{1, \ldots, n\}$. Every twisted polynomial v in R has a "leading term", i.e. a term of highest degree. We define $\deg(v)$ to be the degree of its leading term.

We perform a Gram-Schmidt-like process on the basis $v_1, \ldots, v_m$ to obtain a new basis whose leading terms are linearly independent over K: Let $\deg(v_1)$ be $(a_1, b_1, \ldots, a_n, b_n)$. Let $a_1', b_1', \ldots, a_n', b_n'$ be the unique integers between 0 and $p-1$ such that $(a_1, b_1, \ldots, a_n, b_n) \equiv (a_1', b_1', \ldots, a_n', b_n') \pmod{p}$. Recall that each v_k is the sum of monomials of the form $c x_1^{i_1} y_1^{j_1}, \ldots, x_n^{i_n} y_n^{j_n}$ where $i_1, j_1, \ldots, i_n, j_n$ are integers between 0 and $p-1$ and the coefficient c is a polynomial in $F[\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n]$. For each $k \in \{1, \ldots, n\}$, let c_k be the coefficient of $x_1^{a_1'} y_1^{b_1'}, \ldots, x_n^{a_n'} y_n^{b_n'}$ in v_k.

For each $i \in \{2, \dots, m\}$ we replace v_i with $c_1 v_i - c_i v_1$. Now the coefficient of $x_1^{a_1'} y_1^{b_1'}, \dots, x_n^{a_n'} y_n^{b_n'}$ in v_k is zero for every $k \in \{2, \dots, m\}$. We continue in this manner by fixing v_2 and changing $v_3, \dots, v_m$ similarly, and so on. In the end, we obtain a basis $v_1, \dots, v_m$ satisfying $\deg(v_i) \not\equiv \deg(v_j) \pmod p$ for any $i \neq j$. Consequently, the leading terms of $v_1, \dots, v_m$ are linearly independent over K.

For every $k \in \{1, \dots, m\}$, let w_k be the leading term of v_k. The K-vector space $Kw_1 + \cdots + Kw_m$ is of dimension m and spanned by monomials, so it remains to prove that it is Kummer. Let $d_1, \dots, d_m$ be nonnegative integers satisfying $d_1 + \cdots + d_m = p$. It is easy to see that if the element $w_1^{d_1} * \cdots * w_m^{d_m}$ is nonzero, then it is the leading term of $v_1^{d_1} * \cdots * v_m^{d_m}$, which is in K because $Kv_1 + \cdots + Kv_m$ is Kummer. Otherwise $w_1^{d_1} * \cdots * w_m^{d_m}$ is zero, which is also in K. Consequently $Kw_1 + \cdots + Kw_m$ is a Kummer space. □

Corollary 4.2. *If $p = 3$, then the upper bound for the dimension of a Kummer subspace of A is $3n + 1$.*

Proof. Follows immediately from the previous theorem and Section 3. □

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