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ON A GENERAL MACLAURIN'S INEQUALITY

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ABSTRACT. Maclaurin's inequality provides a sequence of inequalities that interpolate between the arithmetic mean at the high end and the geometric mean at the low end. We introduce a similar interpolating sequence of inequalities between the weighted arithmetic and geometric mean with arbitrary weights. Maclaurin's inequality arises for uniform weights. As a by-product we obtain inequalities that may be of interest in the theory of Jacobi polynomials.

1. INTRODUCTION

The inequality between the arithmetic mean and the geometric mean, or briefly the AM-GM inequality, is one of the most well-known inequalities in mathematical analysis; see, e.g., Bullen [3] and Hardy et al. [8]. In particular, if $\mathbf{x} = (x_1, \ldots, x_n)$ is a collection of positive real numbers, for any $n \geq 1$, then the AM-GM inequality states that the arithmetic mean of $\mathbf{x}$ is greater than or equal to the geometric mean of $\mathbf{x}$, i.e.,

$$\frac{1}{n}\sum_{i=1}^{n} x_i \geq \prod_{i=1}^{n} x_i^{\frac{1}{n}}. \tag{1.1}$$

The equality in (1.1) follows if and only if the x_i's are all equal. Maclaurin's inequality, first stated in Maclaurin [10], is a natural refinement of the AM-GM inequality. Let

$$E_k(\mathbf{x}) = \left[\frac{\sum_{1\leq i_1<i_2<\cdots<i_k\leq n} x_{i_1}x_{i_2}\cdots x_{i_k}}{\binom{n}{k}}\right]^{\frac{1}{k}} \tag{1.2}$$

for any $k = 1, \ldots, n$, where the numerator of (1.2) is the k-th elementary symmetric polynomial in $\mathbf{x}$, and the binomial coefficient in the denominator of (1.2) is the number of terms in the numerator. Maclaurin's inequality is the following chain of inequalities,

$$E_1(\mathbf{x}) \geq E_2(\mathbf{x}) \geq \cdots \geq E_{n-1}(\mathbf{x}) \geq E_n(\mathbf{x}), \tag{1.3}$$

with the extreme terms $E_1(\mathbf{x})$ and $E_n(\mathbf{x})$ being the arithmetic mean and the geometric mean, respectively. The inequality (1.3) thus interpolates terms between the left-hand side and the right-hand side of (1.1). Note that, as for the AM-GM inequality, the equality in (1.3) follows if and only if the x_i's are all equal. See, e.g., Bullen [3], Steel [11], Cvetkovski [4] for a detailed account on the Maclaurin's inequality.

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Among the numerous generalizations of the AM-GM inequality, the so-called weighted AM-GM inequality certainly stands out. Again, we refer to the monographs by Bullen [3] and Hardy et al. [8] for details. Specifically, for any $n \geq 1$ let $\mathbf{x} = (x_1, \dots, x_n)$ be a collection of positive real numbers and let $(w_1, \dots, w_n)$ be a collection of positive weights such that $\sum_{1 \leq i \leq n} w_i = 1$. Then, the weighted AM-GM inequality is

$$\sum_{i=1}^{n} w_i x_i \geq \prod_{i=1}^{n} x_i^{w_i}. \tag{1.4}$$

The aim of this paper is to introduce an interpolative sequence of inequalities between the two sides of (1.4), in the same way as (1.3) is an interpolative sequence of inequalities between the two sides of (1.1). Let $m > 0$ such that $r_i = m\, w_i$ is an integer for all $i = 1, \dots, n$, and set $\mathbf{r} = (r_1, \dots, r_n)$. Then, for any positive integer l define

$$T_l(\mathbf{x}, \mathbf{r}) = \left[\sum_{(l_1, \dots, l_n) \in \mathcal{P}_{n,l}} \frac{\prod_{i=1}^{n} \binom{r_i}{l_i} x_i^{l_i}}{\binom{m}{l}} \right]^{1/l}, \tag{1.5}$$

where $\mathcal{P}_{n,l} = \{(l_1, \dots, l_n) : l_i \geq 0 \text{ and } \sum_{1 \leq i \leq n} l_i = l\}$. In particular, for $l = 1$ and $l = m$,

$$T_1(\mathbf{x}, \mathbf{r}) = \sum_{i=1}^{n} w_i x_i$$

and

$$T_m(\mathbf{x}, \mathbf{r}) = \prod_{i=1}^{n} x_i^{w_i},$$

respectively, so that $T_1(\mathbf{x}, \mathbf{r}) \geq T_m(\mathbf{x}, \mathbf{r})$ is the weighted AM-GM inequality, and if in addition $w_i = 1/n$ for any $i = 1, \dots, n$, then it reduces to the AM-GM inequality. On the other hand, Maclaurin's inequality arises when $w_i = 1/n$, for any $i = 1, \dots, n$, and $m = n$. The next theorem states our general Maclaurin's inequality.

Theorem 1.1. *For $n \geq 1$ let $\mathbf{x} = (x_1, \dots, x_n)$ such that $x_i > 0$ and $(w_1, \dots, w_n)$ such that $w_i > 0$ and $\sum_{1 \leq i \leq n} w_i = 1$. If $\mathbf{r} = (r_1, \dots, r_n)$, with $r_i = m w_i$ being an integer for $m > 0$, then*

$$T_1(\mathbf{x}, \mathbf{r}) \geq T_2(\mathbf{x}, \mathbf{r}) \geq \cdots \geq T_{m-1}(\mathbf{x}, \mathbf{r}) \geq T_m(\mathbf{x}, \mathbf{r}). \tag{1.6}$$

The proof of Theorem 1.1 is given in Section 2. In particular, we start by showing that the key step consists in proving Theorem 1.1 for $n = 2$. The general case then follows by an inductive argument. Hence most of Section 2 focuses on Theorem 1.1 for $n = 2$. Under the assumption $n = 2$ we have x_1 and x_2 and we assume without loss of generality that $x_2 < x_1$. We then put $t = x_1/x_2$, so we aim to show that, for $t \geq 1$,

$$T_l((t, x_2), (mw, m(1 - w))) \geq T_{l+1}((t, x_2), (mw, m(1 - w))),$$

i.e.,

$$(1.7)\qquad T_l^*(t,mw) = \left[\sum_{i=\max\{0,l-m(1-w)\}}^{\min\{l,mw\}} \frac{\binom{mw}{i}\binom{m(1-w)}{l-i}}{\binom{m}{l}} t^i\right]^{\frac{1}{l}}$$

$$\geq \left[\sum_{i=\max\{0,l+1-m(1-w)\}}^{\min\{l+1,mw\}} \frac{\binom{mw}{i}\binom{m(1-w)}{l+1-i}}{\binom{m}{l+1}} t^i\right]^{\frac{1}{l+1}} = T_{l+1}^*(t,mw)$$

for any $w \in (0,1)$ and $m > 0$ such that $mw = 1,\dots,m$, and any $l = 1,\dots,m$. It is worth pointing out that $T_l^*(t,mw)$ can be represented in terms of Jacobi polynomials; see, e.g., Erdélyi [5] and Szegő [12]. Indeed, for any real x, any reals α and β, and any integer n, a Jacobi polynomial $P_n^{\alpha,\beta}(x)$ admits the finite-sum representation

$$(1.8)\qquad P_n^{\alpha,\beta}(x) = \frac{1}{2^n}\sum_{i=0}^{n}\binom{n+\alpha}{i}\binom{n+\beta}{n-i}(x-1)^{n-i}(x+1)^i.$$

The proof of the inequalities (1.7) thus naturally leads to introducing a collection of inequalities for Jacobi polynomials that may be of separate interest in the theory of orthogonal polynomials. In particular, if $l \leq mw$ and $0 \geq l-m(1-w)$, we obtain an inequality which is somehow reminiscent of the celebrated Turán inequality for Jacobi polynomials introduced in Gasper [6] and [7]. See also Baricz [1], Baricz [2] and references therein for recent developments on the Turán inequality for special functions.

2. Proof of Theorem 1.1

As anticipated in the Introduction, we start by showing that it is sufficient to prove Theorem 1.1 for $n = 2$. The general case then follows by an inductive argument which relies on the fact that $[T_l(\mathbf{x},\mathbf{t})]^l$ corresponds to the probability generating function of the multivariate hypergeometric distribution with parameter $(n,\mathbf{r},l)$; see, e.g., Johnson et al. [9]. For the inductive argument to extend to all n, we assume that Theorem 1.1 is true for $n = 2$, and hence, for x_1 and x_2 positive, we have that

$$T_l((x_1,x_2),(r_1,r_2)) = \left[\sum_{(l_1,l_2)\in\mathcal{P}_{2,l}} \frac{\binom{r_1}{l_1}\binom{r_2}{l_2}}{\binom{m}{l}} x_1^{l_1}x_2^{l_2}\right]^{\frac{1}{l}}$$

is decreasing in l. Let

$$(2.1)\qquad T_l((x_1,x_2,x_3),(r_1,r_2,r_3)) = \left[\sum_{(l_1,l_2,l_3)\in\mathcal{P}_{3,l}} \frac{\binom{r_1}{l_1}\binom{r_2}{l_2}\binom{r_3}{l_3}}{\binom{m}{l}} x_1^{l_1}x_2^{l_2}x_3^{l_3}\right]^{\frac{1}{l}}.$$

In particular, if now we define $t_1 = x_1/x_3$ and $t_2 = x_2/x_3$, then we can write (2.1) as follows,

$$T_l((t_1,t_2,x_3),(r_1,r_2,r_3)) = x_3\left[\sum_{(l_1,l_2,l_3)\in\mathcal{P}_{3,l}} \frac{\binom{r_1}{l_1}\binom{r_2}{l_2}\binom{r_3}{l_3}}{\binom{m}{l}} t_1^{l_1}t_2^{l_2}\right]^{\frac{1}{l}},$$

i.e.,

$$T_l((t_1,t_2,x_3),(r_1,r_2,r_3)) = x_3 \left\{ \mathbb{E}\left[t_1^{L_1} t_2^{L_2} \right] \right\}^{\frac{1}{l}},$$

where $\mathbb{E}$ denotes the expected value with respect to the random variable (L_1, L_2) distributed according to a multivariate hypergeometric distribution with parameter $(3,(r_1,r_2,r_3),l)$ and with $m = r_1 + r_2 + r_3$. The marginal distribution of (L_1, L_2) is also a multivariate hypergeometric distribution with parameter $(2, r_1, m - r_1, l)$ and hence

$$T_l((t_1,t_2,x_3),(r_1,r_2,r_3)) = x_3 \left[\sum_{(l_1,l_2)\in\mathcal{P}_{2,l}} \frac{\binom{r_1}{l_1}\binom{m-r_1}{l_2}}{\binom{m}{l}} t_1^{l_1} t_2^{l_2} \right]^{\frac{1}{l}},$$

which is decreasing in l by assumption. These arguments extend to all n using marginal properties of the multivariate hypergeometric distribution. Hence the key is to prove Theorem 1.1 for $n = 2$, i.e., (1.7). According to the values of w, m and l, in the next subsections we prove (1.7) for: C1) $m(1-w) \geq l$ and $mw \geq l$; C2) $m(1-w) \geq l$ and $mw \leq l$; C3) $m(1-w) \leq l$ and $mw \geq l$; C4) $m(1-w) \leq l$ and $mw \leq l$.

2.1. **Case C1).** Note that $l \leq m(1-w)$ and $l \leq mw$ implies $l \leq m - l$. We assume $w \leq 1/2$, and the case $w > 1/2$ follows by similar arguments. Under this set of conditions for m, l and w, we write $T_l^*(t, mw)$ in terms of a Jacobi polynomial. For any $y < 0$,

$$T_l^*(1-y, mw) = \frac{\binom{mw}{l}}{\binom{m}{l}} (-y)^l \frac{P_l^{mw-l,m(1-w)-l}\left(\frac{y-2}{y}\right)}{\binom{mw}{l}}, \tag{2.2}$$

with

$$\begin{aligned} (-y)^l \frac{P_l^{mw-l,m(1-w)-l}\left(\frac{y-2}{y}\right)}{\binom{mw}{l}} &= \sum_{i=0}^{l} \frac{\binom{mw}{i}\binom{m(1-w)}{l-i}}{\binom{mw}{l}} (1-y)^i \\ &\leq \sum_{i=0}^{l+1} \frac{\binom{mw}{i}\binom{m(1-w)}{l+1-i}}{\binom{mw}{l+1}} (1-y)^i \\ &= (-y)^{l+1} \frac{P_{l+1}^{mw-l-1,m(1-w)-l-1}\left(\frac{y-2}{y}\right)}{\binom{mw}{l+1}} \end{aligned} \tag{2.3}$$

because

$$\frac{\frac{\binom{m(1-w)}{l-i}}{\binom{mw}{l}}}{\frac{\binom{m(1-w)}{l+1-i}}{\binom{mw}{l+1}}} = \frac{(1-i+l)(l-mw)}{(l+1)(-i+l-m(1-w))} \leq 1$$

for any $i = 0, \ldots, l$, and $(1-y)^{l+1} \geq 0$. Indeed, $0 \leq l+1-i \leq l+1$ and $0 \leq mw - l \leq m(1-w) - l + i$ from the set of conditions C1), and because $0 \leq i \leq l$ and $w \leq 1/2$. Note that

i) $P_l^{mw-l,m(1-w)-l}((y-2)/y)$, as $y \in (-\infty, 0)$, is positive, monotone decreasing and

$$\lim_{y\to-\infty} \frac{P_l^{mw-l,m(1-w)-l}\left(\frac{y-2}{y}\right)}{\binom{mw}{l}} = 1;$$

ii) $(-y)^l P_l^{mw-l,m(1-w)-l}((y-2)/y)$, as $y \in (-\infty, 0)$, is positive, monotone decreasing and

$$\lim_{y\to 0^-} (-y)^l \frac{P_l^{mw-l,m(1-w)-l}\left(\frac{y-2}{y}\right)}{\binom{mw}{l}} = \frac{\binom{m}{l}}{\binom{mw}{l}} \geq 1.$$

This follows from the definition of Jacobi polynomial in (1.8), and by the Vandermonde identity. Also,

$$(2.4) \qquad (-y)^{l+1} \frac{P_{l+1}^{mw-l-1,m(1-w)-l-1}\left(\frac{y-2}{y}\right)}{\binom{mw}{l+1}} - (-y)^l \frac{P_l^{mw-l,m(1-w)-l}\left(\frac{y-2}{y}\right)}{\binom{mw}{l}}$$
$$= \sum_{i=0}^{l} \binom{mw}{i} \left[\frac{\binom{m(1-w)}{l+1-i}}{\binom{mw}{l+1}} - \frac{\binom{m(1-w)}{l-i}}{\binom{mw}{l}} \right] (1-y)^i + (1-y)^{l+1},$$

is positive and increasing in l. In other terms, the gap between the first term and the second term appearing on the left-hand side of (2.4) is positive and it increases as l increases. Accordingly, by combining this property with the inequality (2.3), we have that, if

$$(2.5) \quad (-y) \frac{P_1^{mw-1,m(1-w)-1}\left(\frac{y-2}{y}\right)}{\binom{mw}{1}} \leq \left[(-y)^2 \frac{P_{1+1}^{mw-1-1,m(1-w)-1-1}\left(\frac{y-2}{y}\right)}{\binom{mw}{1+1}} \right]^{\frac{1}{2}},$$

then

(2.6)

$$\left[(-y)^l \frac{P_l^{mw-l,m(1-w)-l}\left(\frac{y-2}{y}\right)}{\binom{mw}{l}} \right]^{\frac{1}{l}} \leq \left[(-y)^{l+1} \frac{P_{l+1}^{mw-l-1,m(1-w)-l-1}\left(\frac{y-2}{y}\right)}{\binom{mw}{l+1}} \right]^{\frac{1}{l+1}}.$$

The inequality (2.5) can be easily proved by means of (1.8). Indeed, (2.5) reduces to the inequality

$$\left(1 - \frac{1}{wy}\right)^2 \leq \frac{-1 - wy(y-2) + m(-1+wy)^2}{wy^2(mw-1)},$$

i.e., $(wy-1)^2(-w^{-1}) \leq -1 - wy(y-2)$, which holds because $0 \leq w \leq 1/2$. This proves (2.6). Therefore, with respect to the probability generating function (2.2), one has the inequalities:

i)

(2.7)

$$\left[(-y)^l \frac{P_l^{mw-l,m(1-w)-l}\left(\frac{y-2}{y}\right)}{\binom{mw}{l}} \right]^{\frac{1}{l}} \leq \left[(-y)^{l+1} \frac{P_{l+1}^{mw-l-1,m(1-w)-l-1}\left(\frac{y-2}{y}\right)}{\binom{mw}{l+1}} \right]^{\frac{1}{l+1}},$$

ii)

$$(2.8) \qquad \left[\frac{\binom{mw}{l}}{\binom{m}{l}} \right]^{\frac{1}{l}} \geq \left[\frac{\binom{mw}{l+1}}{\binom{m}{l+1}} \right]^{\frac{1}{l+1}},$$

i.e.,

$$\frac{mw!(m-l)!}{m!(mw-l)!} \geq \left(\frac{l-mw}{l-m}\right)^{l}. \tag{2.9}$$

In particular, (2.9) follows by a straightforward induction on l. The left-hand side of (2.7) increases as l increases, whereas the left-hand side of (2.8) decreases as l increases. In order that the product of these two functions decreases as l increases, we have to prove

$$\frac{\left[(-y)^{l}\frac{P_{l}^{mw-l,m(1-w)-l}\left(\frac{y-2}{y}\right)}{\binom{mw}{l}}\right]^{\frac{1}{l}}}{\left[(-y)^{l+1}\frac{P_{l+1}^{mw-l-1,m(1-w)-l-1}\left(\frac{y-2}{y}\right)}{\binom{mw}{l+1}}\right]^{\frac{1}{l+1}}} \geq \frac{\left[\frac{\binom{mw}{l+1}}{\binom{m}{l+1}}\right]^{\frac{1}{l+1}}}{\left[\frac{\binom{mw}{l}}{\binom{m}{l}}\right]^{\frac{1}{l}}}. \tag{2.10}$$

The left-hand side of (2.10) is bounded in $(-\infty, 0)$ and monotone increasing. In particular,

$$\begin{aligned}
\frac{\left[\frac{\binom{m}{l}}{\binom{mw}{l}}\right]^{\frac{1}{l}}}{\left[\frac{\binom{m}{l+1}}{\binom{mw}{l+1}}\right]^{\frac{1}{l+1}}} &= \lim_{y\to 0^-}\frac{\left[(-y)^{l}\frac{P_{l}^{mw-l,m(1-w)-l}\left(\frac{y-2}{y}\right)}{\binom{mw}{l}}\right]^{\frac{1}{l}}}{\left[(-y)^{l+1}\frac{P_{l+1}^{mw-l-1,m(1-w)-l-1}\left(\frac{y-2}{y}\right)}{\binom{mw}{l+1}}\right]^{\frac{1}{l+1}}} \\
&\leq \frac{\left[(-y)^{l}\frac{P_{l}^{mw-l,m(1-w)-l}\left(\frac{y-2}{y}\right)}{\binom{mw}{l}}\right]^{\frac{1}{l}}}{\left[(-y)^{l+1}\frac{P_{l+1}^{mw-l-1,m(1-w)-l-1}\left(\frac{y-2}{y}\right)}{\binom{mw}{l+1}}\right]^{\frac{1}{l+1}}},
\end{aligned}$$

so that $[T_l^*(1-y,mw)]^{1/l} \geq [T_{l+1}^*(1-y,mw)]^{1/(l+1)}$ for any $y<0$. This completes the proof under the set of conditions C1) and $w \leq 1/2$. We now consider the case $w \geq 1/2$. For any $y<0$,

$$T_l^*\left(\frac{y-1}{y}, mw\right) = \frac{\binom{m(1-w)}{l}}{\binom{m}{l}}\frac{1}{(y)^l}\frac{P_l^{m(1-w)-l,mw-l}(-1+2y)}{\binom{m(1-w)}{l}}$$

and, along lines similar to the proof of the case $w \leq 1/2$, we obtain the following inequalities:

i)

$$\begin{aligned}
&\left[\frac{1}{(y)^l}\frac{P_l^{m(1-w)-l,mw-l}(-1+2y)}{\binom{m(1-w)}{l}}\right]^{\frac{1}{l}} \\
&\qquad \leq \left[\frac{1}{(y)^{(l+1)}}\frac{P_{l+1}^{m(1-w)-l-1,mw-l-1}(-1+2y)}{\binom{m(1-w)}{l+1}}\right]^{\frac{1}{l+1}},
\end{aligned} \tag{2.11}$$

ii)

$$\left[\frac{\binom{m(1-w)}{l}}{\binom{m}{l}}\right]^{\frac{1}{l}} \geq \left[\frac{\binom{m(1-w)}{l+1}}{\binom{m}{l+1}}\right]^{\frac{1}{l+1}}. \tag{2.12}$$

By combining (2.11) and (2.12) in the same way presented under the assumption $w \le 1/2$, we obtain $[T_l^*((y-1)/y, mw)]^{1/l} \ge [T_{l+1}^*((y-1)/y, mw)]^{1/(l+1)}$ for any $y < 0$. This completes the proof under the set of conditions C1), that is, $m(1-w) \ge l$ and $mw \ge l$.

2.2. **Case C2).** Note that $mw \le m - l$ and $mw \le l$ implies $w \le 1/2$. Under this set of conditions on m and l, we write $T_l^*(t, mw)$ as a Jacobi polynomial. For any $y < 0$,

$$T_l^*(1-y, mw) = \frac{\binom{l}{mw}}{\binom{m}{mw}}(-y)^{mw} \frac{P_{mw}^{l-mw, m(1-w)-l}\left(\frac{y-2}{y}\right)}{\binom{l}{mw}}, \tag{2.13}$$

with

$$\begin{aligned}
&\frac{P_{mw}^{l-mw, m(1-w)-l}\left(\frac{y-2}{y}\right)}{\binom{l}{mw}} \\
&\quad \ge \frac{y}{y-1} \frac{P_{mw}^{l-mw, m(1-w)-l-1}\left(\frac{y-2}{y}\right)}{\binom{l}{mw}} + \frac{1}{1-y} \frac{P_{mw}^{l+1-mw, m(1-w)-l-1}\left(\frac{y-2}{y}\right)}{\binom{l+1}{mw}} \\
&\quad = \frac{y}{y-1}\left(\frac{P_{mw}^{l-mw, m(1-w)-l-1}\left(\frac{y-2}{y}\right)}{\binom{l}{mw}} - \frac{P_{mw}^{l+1-mw, m(1-w)-l-1}\left(\frac{y-2}{y}\right)}{\binom{l+1}{mw}}\right) \\
&\qquad + \frac{P_{mw}^{l+1-mw, m(1-w)-l-1}\left(\frac{y-2}{y}\right)}{\binom{l+1}{mw}} \\
&\quad \ge \frac{P_{mw}^{l+1-mw, m(1-w)-l-1}\left(\frac{y-2}{y}\right)}{\binom{l+1}{mw}},
\end{aligned} \tag{2.14}$$

where the first inequality in (2.14) arises from equation (34) in Erdélyi [5], and the second inequality in (2.14) arises from the definition of Jacobi polynomial in (1.8). Indeed,

$$\frac{P_{mw}^{l-mw, m(1-w)-l-1}\left(\frac{y-2}{y}\right)}{\binom{l}{mw}} = \left(-\frac{1}{y}\right)^{mw} \sum_{i=0}^{mw} \frac{\binom{l}{i}\binom{m-l-1}{mw-i}}{\binom{l}{mw}}(1-y)^i$$

and

$$\frac{P_{mw}^{l+1-mw, m(1-w)-l-1}\left(\frac{y-2}{y}\right)}{\binom{l+1}{mw}} = \left(-\frac{1}{y}\right)^{mw} \sum_{i=0}^{mw} \frac{\binom{l+1}{i}\binom{m-l-1}{mw-i}}{\binom{l+1}{mw}}(1-y)^i,$$

where

$$\frac{\frac{\binom{l}{i}\binom{m-l-1}{mw-i}}{\binom{l}{mw}}}{\frac{\binom{l+1}{i}\binom{m-l-1}{mw-i}}{\binom{l+1}{mw}}} = \frac{1-i+l}{1+l-mw} \ge 1 \tag{2.15}$$

for any index $i = 0, \ldots, mw$. Indeed, one has $1 + l - i \ge 1 + l - mw \ge 0$ from the set of conditions C2), and because $0 \le i \le mw$. Hence the inequality displayed in

(2.14) leads to

$$(-y)^{mw}\frac{P_{mw}^{l-mw,m(1-w)-l}\left(\frac{y-2}{y}\right)}{\binom{l}{mw}} \geq (-y)^{mw}\frac{P_{mw}^{l+1-mw,m(1-w)-l-1}\left(\frac{y-2}{y}\right)}{\binom{l+1}{mw}}$$

and

$$\left[(-y)^{mw}\frac{P_{mw}^{l-mw,m(1-w)-l}\left(\frac{y-2}{y}\right)}{\binom{l}{mw}}\right]^{\frac{1}{l}} \geq \left[(-y)^{mw}\frac{P_{mw}^{l+1-mw,m(1-w)-l-1}\left(\frac{y-2}{y}\right)}{\binom{l+1}{mw}}\right]^{\frac{1}{l+1}},$$

where

i) $P_{l}^{mw-l,m(1-w)-l}((y-2)/y)$, as $y \in (-\infty,0)$, is positive, monotone decreasing and

$$\lim_{y\to-\infty}\frac{P_{mw}^{l-mw,m(1-w)-l}\left(\frac{y-2}{y}\right)}{\binom{l}{mw}} = 1,$$

ii) $(-y)^{mw}P_{mw}^{l-mw,m(1-w)-l}((y-2)/y)$, as $y \in (-\infty,0)$ is positive, monotone decreasing and

$$\lim_{y\to 0^-}(-y)^{mw}\frac{P_{mw}^{l-mw,m(1-w)-l}\left(\frac{y-2}{y}\right)}{\binom{l}{mw}} = \frac{\binom{m}{mw}}{\binom{l}{mw}} \geq 1.$$

This follows from (1.8) and by a direct application of the Vandermonde identity. Therefore, with respect to the probability generating function (2.13), one has the following inequalities:

i)

$$\left[(-y)^{mw}\frac{P_{mw}^{l-mw,m(1-w)-l}\left(\frac{y-2}{y}\right)}{\binom{l}{mw}}\right]^{\frac{1}{l}} \geq \left[(-y)^{mw}\frac{P_{mw}^{l+1-mw,m(1-w)-l-1}\left(\frac{y-2}{y}\right)}{\binom{l+1}{mw}}\right]^{\frac{1}{l+1}}, \tag{2.16}$$

ii)

$$\left[\frac{\binom{l}{mw}}{\binom{m}{mw}}\right]^{\frac{1}{l}} \leq \left[\frac{\binom{l+1}{mw}}{\binom{m}{mw}}\right]^{\frac{1}{l+1}}, \tag{2.17}$$

i.e.,

$$\frac{l!(m(1-w))!}{m!(l-mw)!} \leq \left(\frac{1+l}{1+l-mw}\right)^{l}. \tag{2.18}$$

In particular, (2.18) follows by a straightforward induction on l. The left-hand side of (2.16) decreases as l increases, whereas the left-hand side of (2.17) increases as l increases. In order that the product of these two functions decreases as l increases, we have to prove the following

$$\frac{\left[\frac{\binom{l}{mw}}{\binom{m}{mw}}\right]^{\frac{1}{l}}}{\left[\frac{\binom{l+1}{mw}}{\binom{m}{mw}}\right]^{\frac{1}{l+1}}} \geq \frac{\left[(-y)^{mw}\frac{P_{mw}^{l+1-mw,m(1-w)-l-1}\left(\frac{y-2}{y}\right)}{\binom{l+1}{mw}}\right]^{\frac{1}{l+1}}}{\left[(-y)^{mw}\frac{P_{mw}^{l-mw,m(1-w)-l}\left(\frac{y-2}{y}\right)}{\binom{l}{mw}}\right]^{\frac{1}{l}}}. \tag{2.19}$$

The right-hand side of (2.19) is bounded in $(-\infty, 0)$ and monotone decreasing. In particular,

$$\frac{\left[\frac{\binom{l}{mw}}{\binom{m}{mw}}\right]^{\frac{1}{l}}}{\left[\frac{\binom{l+1}{mw}}{\binom{m}{mw}}\right]^{\frac{1}{l+1}}} = \lim_{y\to 0^-} \frac{\left[(-y)^{mw}\frac{P_{mw}^{l+1-mw,m(1-w)-l-1}\left(\frac{y-2}{y}\right)}{\binom{l+1}{mw}}\right]^{\frac{1}{l+1}}}{\left[(-y)^{mw}\frac{P_{mw}^{l-mw,m(1-w)-l}\left(\frac{y-2}{y}\right)}{\binom{l}{mw}}\right]^{\frac{1}{l}}}$$
$$\geq \frac{\left[(-y)^{mw}\frac{P_{mw}^{l+1-mw,m(1-w)-l-1}\left(\frac{y-2}{y}\right)}{\binom{l+1}{mw}}\right]^{\frac{1}{l+1}}}{\left[(-y)^{mw}\frac{P_{mw}^{l-mw,m(1-w)-l}\left(\frac{y-2}{y}\right)}{\binom{l}{mw}}\right]^{\frac{1}{l}}},$$

so that we have the desired inequality $[T_l^*(1-y, mw)]^{1/l} \geq [T_{l+1}^*(1-y, mw)]^{1/(l+1)}$ for any $y < 0$. This completes the proof under the set of conditions C2), that is, $m(1-w) \geq l$ and $mw \leq l$.

2.3. **Case C3).** Note that $m(1-w) \leq l$ and $m(1-w) \leq m-l$ implies $w \geq 1/2$. Under this set of conditions on m and l, we write $T_l^*(t, mw)$ as a Jacobi polynomial. For any $y < 0$,

$$T_l^*(1-y, mw) = \left(\frac{-y}{1-y}\right)^{m(1-w)} (1-y)^l \frac{P_{m(1-w)}^{mw-l,l-m(1-w)}\left(\frac{y-2}{y}\right)}{\binom{m}{mw}}, \tag{2.20}$$

with

(2.21)
$$\begin{aligned}
&P_{m(1-w)}^{mw-l,l-m(1-w)}\left(\frac{y-2}{y}\right)\\
&\quad= yP_{m(1-w)}^{mw-l-1,l-m(1-w)}\left(\frac{y-2}{y}\right) + (1-y)P_{m(1-w)}^{mw-l-1,l+1-m(1-w)}\left(\frac{y-2}{y}\right)\\
&\quad= y\left(P_{m(1-w)}^{mw-l-1,l-m(1-w)}\left(\frac{y-2}{y}\right) - P_{m(1-w)}^{mw-l-1,l+1-m(1-w)}\left(\frac{y-2}{y}\right)\right)\\
&\qquad+ P_{m(1-w)}^{mw-l-1,l+1-m(1-w)}\left(\frac{y-2}{y}\right)\\
&\quad\geq P_{m(1-w)}^{mw-l-1,l+1-m(1-w)}\left(\frac{y-2}{y}\right),
\end{aligned}$$

where the first equality in (2.21) arises from equation (34) in Erdélyi [5], and the inequality in (2.21) arises from the definition of Jacobi polynomial in (1.8). Indeed,

$$\begin{aligned}
&P_{m(1-w)}^{mw-l-1,l-m(1-w)}\left(\frac{y-2}{y}\right)\\
&\quad= \left(-\frac{1}{y}\right)^{m(1-w)} \sum_{i=0}^{m(1-w)} \binom{m-l-1}{i}\binom{l}{m(1-w)-i}(1-y)^i
\end{aligned}$$

and

$$P_{m(1-w)}^{mw-l-1,l+1-m(1-w)}\left(\frac{y-2}{y}\right)$$
$$=\left(-\frac{1}{y}\right)^{m(1-w)}\sum_{i=0}^{m(1-w)}\binom{m-l-1}{i}\binom{l+1}{m(1-w)-i}(1-y)^i,$$

where

$$\frac{\binom{m-l-1}{i}\binom{l+1}{m(1-w)-i}}{\binom{m-l-1}{i}\binom{l}{m(1-w)-i}}=\frac{l+1}{1+i+l-m(1-w)}\geq 1 \tag{2.22}$$

for any index $i=0,\ldots,m(1-w)$. Indeed, one has $l+1\geq 1+i+l-m(1-w)\geq 0$ from the set of conditions C3), and because $0\leq i\leq m(1-w)$. Hence, the inequality (2.21) leads to

$$\left(\frac{-y}{1-y}\right)^{m(1-w)}P_{m(1-w)}^{mw-l,l-m(1-w)}\left(\frac{y-2}{y}\right)$$
$$\geq\left(\frac{-y}{1-y}\right)^{m(1-w)}P_{m(1-w)}^{mw-l-1,l+1-m(1-w)}\left(\frac{y-2}{y}\right)$$

and

$$\left[\left(\frac{-y}{1-y}\right)^{m(1-w)}(1-y)^lP_{m(1-w)}^{mw-l,l-m(1-w)}\left(\frac{y-2}{y}\right)\right]^{\frac{1}{l}}$$
$$\geq\left[\left(\frac{-y}{1-y}\right)^{m(1-w)}(1-y)^{l+1}P_{m(1-w)}^{mw-l-1,l+1-m(1-w)}\left(\frac{y-2}{y}\right)\right]^{\frac{1}{l+1}},$$

where

i) $P_{m(1-w)}^{mw-l,l-m(1-w)}((y-2)/y)$, as $y\in(-\infty,0)$, is positive, monotone decreasing and

$$\lim_{y\to-\infty}P_{m(1-w)}^{mw-l,l-m(1-w)}\left(\frac{y-2}{y}\right)=\binom{m-l}{m(1-w)}\geq 1,$$

ii) $(-y)^{m(1-w)}(1-y)^{l-m(1-w)}P_{m(1-w)}^{mw-l,l-m(1-w)}((y-2)/y)$, as $y\in(-\infty,0)$, is positive, monotone decreasing and

$$\lim_{y\to 0^-}\left(\frac{-y}{1-y}\right)^{m(1-w)}(1-y)^lP_{m(1-w)}^{mw-l,l-m(1-w)}\left(\frac{y-2}{y}\right)=\binom{m}{m(1-w)}\geq 1.$$

This follows from (1.8) and by a direct application of the Vandermonde identity. Therefore, with respect to the probability generating function (2.20), one has the following inequality

i)

$$\left[\left(\frac{-y}{1-y}\right)^{m(1-w)}(1-y)^lP_{m(1-w)}^{mw-l,l-m(1-w)}\left(\frac{y-2}{y}\right)\right]^{\frac{1}{l}} \tag{2.23}$$
$$\geq\left[\left(\frac{-y}{1-y}\right)^{m(1-w)}(1-y)^{l+1}P_{m(1-w)}^{mw-l-1,l+1-m(1-w)}\left(\frac{y-2}{y}\right)\right]^{\frac{1}{l+1}},$$

ii)

$$\left[\frac{1}{\binom{m}{mw}}\right]^{\frac{1}{l}} \leq \left[\frac{1}{\binom{m}{mw}}\right]^{\frac{1}{l+1}}, \tag{2.24}$$

i.e.,

$$\frac{1}{\binom{m}{mw}} \leq 1. \tag{2.25}$$

Note that the left-hand side of (2.23) decreases as l increases, whereas the left-hand side of (2.24) increases as l increases. This is similar to what we found under the set of conditions C2). Hence, in order that the product of these two functions decreases as l increases, we have to prove

$$\frac{\left[\frac{1}{\binom{m}{mw}}\right]^{\frac{1}{l}}}{\left[\frac{1}{\binom{m}{mw}}\right]^{\frac{1}{l+1}}} \geq \frac{\left[\left(\frac{-y}{1-y}\right)^{m(1-w)}(1-y)^{l+1}P_{m(1-w)}^{mw-l-1,l+1-m(1-w)}\left(\frac{y-2}{y}\right)\right]^{\frac{1}{l+1}}}{\left[\left(\frac{-y}{1-y}\right)^{m(1-w)}(1-y)^{l}P_{m(1-w)}^{mw-l,l-m(1-w)}\left(\frac{y-2}{y}\right)\right]^{\frac{1}{l}}}. \tag{2.26}$$

The right-hand side of (2.26) is bounded in $(-\infty, 0)$ and monotone decreasing. In particular,

$$\begin{aligned}\frac{\left[\frac{1}{\binom{m}{mw}}\right]^{\frac{1}{l}}}{\left[\frac{1}{\binom{m}{mw}}\right]^{\frac{1}{l+1}}} &= \lim_{y\to 0^-}\frac{\left[\left(\frac{-y}{1-y}\right)^{m(1-w)}(1-y)^{l+1}P_{m(1-w)}^{mw-l-1,l+1-m(1-w)}\left(\frac{y-2}{y}\right)\right]^{\frac{1}{l+1}}}{\left[\left(\frac{-y}{1-y}\right)^{m(1-w)}(1-y)^{l}P_{m(1-w)}^{mw-l,l-m(1-w)}\left(\frac{y-2}{y}\right)\right]^{\frac{1}{l}}}\\ &\geq \frac{\left[\left(\frac{-y}{1-y}\right)^{m(1-w)}(1-y)^{l+1}P_{m(1-w)}^{mw-l-1,l+1-m(1-w)}\left(\frac{y-2}{y}\right)\right]^{\frac{1}{l+1}}}{\left[\left(\frac{-y}{1-y}\right)^{m(1-w)}(1-y)^{l}P_{m(1-w)}^{mw-l,l-m(1-w)}\left(\frac{y-2}{y}\right)\right]^{\frac{1}{l}}},\end{aligned}$$

so that we have the desired inequality $[T_l^*(1-y,mw)]^{1/l} \geq [T_{l+1}^*(1-y,mw)]^{1/(l+1)}$ for any $y<0$. This completes the proof under the set of conditions C3), that is, $m(1-w) \leq l$ and $mw \geq l$.

2.4. **Case C4).** Note that $m-l \leq mw$ and $m-l \leq m(1-w)$ implies $m-l \leq l$. Under this set of conditions on m and l, we write $T_l^*(t,mw)$ as a Jacobi polynomial. For any $y<0$,

$$T_l^*(1-y,mw) = \frac{\binom{m(1-w)}{m-l}}{\binom{m}{l}}(1-y)^{mw}\left(\frac{-y}{1-y}\right)^{m-l}\frac{P_{m-l}^{l-mw,l-m(1-w)}\left(\frac{y-2}{y}\right)}{\binom{m(1-w)}{m-l}} \tag{2.27}$$

with

$$
\begin{aligned}
\left(\frac{-y}{1-y}\right)^{m-l} \frac{P_{m-l}^{l-mw,l-m(1-w)}\left(\frac{y-2}{y}\right)}{\binom{m(1-w)}{m-l}} &= \sum_{i=0}^{m-l} \frac{\binom{m(1-w)}{i}\binom{mw}{m-l-i}}{\binom{m(1-w)}{m-l}}(1-y)^{i-(m-l)} \\
&\geq \sum_{i=0}^{m-l-1} \frac{\binom{m(1-w)}{i}\binom{mw}{m-l-1-i}}{\binom{m(1-w)}{m-l-1}}(1-y)^{i-(m-l-1)} \qquad (2.28)\\
&= \sum_{i=1}^{m-l} \frac{\binom{m(1-w)}{i-1}\binom{mw}{m-l-i}}{\binom{m(1-w)}{m-l-1}}(1-y)^{i-(m-l)} \\
&= \left(\frac{-y}{1-y}\right)^{m-l-1} \frac{P_{m-l-1}^{l+1-mw,l+1-m(1-w)}\left(\frac{y-2}{y}\right)}{\binom{m(1-w)}{m-l-1}}
\end{aligned}
$$

because

$$
\frac{\frac{\binom{m(1-w)}{i}\binom{mw}{m-l-i}}{\binom{m(1-w)}{m-l}}}{\frac{\binom{m(1-w)}{i-1}\binom{mw}{m-l-i}}{\binom{m(1-w)}{m-l-1}}} = \frac{(m-l)(1-i+m(1-w))}{i(1+l-mw)} \geq 1
$$

for any $i = 1, \ldots, m-l$ and $(1-y)^{-(m-l)}\binom{mw}{m-l}/\binom{m(1-w)}{m-l} \geq 0$. Indeed, $m-l \geq i \geq 0$ and $1-i+m(1-w) \geq 1+l-mw \geq 0$ from the set of conditions C4), and because $0 \leq i \leq m-l$. Accordingly, the inequality displayed in (2.28) leads to the following inequality:

$$
\begin{aligned}
&\left[(1-y)^{mw}\left(\frac{-y}{1-y}\right)^{m-l} P_{m-l}^{l-mw,l-m(1-w)}\left(\frac{y-2}{y}\right)\right]^{\frac{1}{l}} \\
&\qquad \geq \left[(1-y)^{mw}\left(\frac{-y}{1-y}\right)^{m-l-1} P_{m-l-1}^{l+1-mw,l+1-m(1-w)}\left(\frac{y-2}{y}\right)\right]^{\frac{1}{l+1}},
\end{aligned}
$$

where

i) $P_{m-l}^{l-mw,l-m(1-w)}((y-2)/y)$, as $y \in (-\infty, 0)$ is positive, monotone decreasing and

$$
\lim_{y\to-\infty} P_{m-l}^{l-mw,l-m(1-w)}\left(\frac{y-2}{y}\right) = \binom{m(1-w)}{m-l} \geq 1,
$$

ii) $(1-y)^{mw}(-y/(1-y))^{m-l}P_{m-l}^{l-mw,l-m(1-w)}((y-2)/y)$, as $y \in (-\infty, 0)$, is positive, monotone decreasing and

$$
\lim_{y\to0^-} (1-y)^{mw}\left(\frac{-y}{1-y}\right)^{m-l} P_{m-l}^{l-mw,l-m(1-w)}\left(\frac{y-2}{y}\right) = \binom{m}{m-l} \geq 1.
$$

This follows from the definition of Jacobi polynomial in (1.8) and the Vandermonde identity. Therefore, with respect to the probability generating function (2.27) one has the inequalities:

i)

$$\left[(1-y)^{mw}\left(\frac{-y}{1-y}\right)^{m-l}P_{m-l}^{l-mw,l-m(1-w)}\left(\frac{y-2}{y}\right)\right]^{\frac{1}{l}} \tag{2.29}$$
$$\geq\left[(1-y)^{mw}\left(\frac{-y}{1-y}\right)^{m-l-1}P_{m-l-1}^{l+1-mw,l+1-m(1-w)}\left(\frac{y-2}{y}\right)\right]^{\frac{1}{l+1}},$$

ii)

$$\left[\frac{\binom{m(1-w)}{m-l}}{\binom{m}{l}}\right]^{\frac{1}{l}}\leq\left[\frac{\binom{m(1-w)}{m-l-1}}{\binom{m}{l+1}}\right]^{\frac{1}{l+1}}, \tag{2.30}$$

i.e.,

$$\frac{l!m(1-w)!}{m!(l-mw)!}\leq\left(\frac{l+1}{1+l-mw}\right)^{l}. \tag{2.31}$$

In particular, (2.31) follows by a straightforward induction on l. In other terms the left-hand side of (2.29) decreases as l increases, whereas the left-hand side of (2.31) increases as l increases. This is similar to what we found under C2) and C3). Hence, in order that the product of these two functions decreases as l increases, we have to prove

$$\frac{\left[\frac{\binom{m(1-w)}{m-l}}{\binom{m}{l}}\right]^{\frac{1}{l}}}{\left[\frac{\binom{m(1-w)}{m-l-1}}{\binom{m}{l+1}}\right]^{\frac{1}{l+1}}}\geq\frac{\left[(1-y)^{mw}\left(\frac{-y}{1-y}\right)^{m-l-1}\frac{P_{m-l-1}^{l+1-mw,l+1-m(1-w)}\left(\frac{y-2}{y}\right)}{\binom{m(1-w)}{m-l-1}}\right]^{\frac{1}{l+1}}}{\left[(1-y)^{mw}\left(\frac{-y}{1-y}\right)^{m-l}\frac{P_{m-l}^{l-mw,l-m(1-w)}\left(\frac{y-2}{y}\right)}{\binom{m(1-w)}{m-l}}\right]^{\frac{1}{l}}}. \tag{2.32}$$

The right-hand side of (2.32) is bounded in $(-\infty,0)$ and monotone decreasing. In particular,

$$\frac{\left[\frac{\binom{m(1-w)}{m-l}}{\binom{m}{l}}\right]^{\frac{1}{l}}}{\left[\frac{\binom{m(1-w)}{m-l-1}}{\binom{m}{l+1}}\right]^{\frac{1}{l+1}}}=\lim_{y\to 0^-}\frac{\left[(1-y)^{mw}\left(\frac{-y}{1-y}\right)^{m-l-1}\frac{P_{m-l-1}^{l+1-mw,l+1-m(1-w)}\left(\frac{y-2}{y}\right)}{\binom{m(1-w)}{m-l-1}}\right]^{\frac{1}{l+1}}}{\left[(1-y)^{mw}\left(\frac{-y}{1-y}\right)^{m-l}\frac{P_{m-l}^{l-mw,l-m(1-w)}\left(\frac{y-2}{y}\right)}{\binom{m(1-w)}{m-l}}\right]^{\frac{1}{l}}}$$
$$\geq\frac{\left[(1-y)^{mw}\left(\frac{-y}{1-y}\right)^{m-l-1}\frac{P_{m-l-1}^{l+1-mw,l+1-m(1-w)}\left(\frac{y-2}{y}\right)}{\binom{m(1-w)}{m-l-1}}\right]^{\frac{1}{l+1}}}{\left[(1-y)^{mw}\left(\frac{-y}{1-y}\right)^{m-l}\frac{P_{m-l}^{l-mw,l-m(1-w)}\left(\frac{y-2}{y}\right)}{\binom{m(1-w)}{m-l}}\right]^{\frac{1}{l}}},$$

so that we have the desired inequality $[T_l^*(1-y,mw)]^{1/l}\geq[T_{l+1}^*(1-y,mw)]^{1/(l+1)}$ for any $y<0$. This completes the proof under the set of conditions C4), that is, $m(1-w)\leq l$ and $mw\leq l$.

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