SPLITTING NUMBERS OF LINKS AND THE FOUR-GENUS

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ABSTRACT. The splitting number of a link is the minimum number of crossing changes between distinct components that is required to convert the link into a split link. We provide a bound on the splitting number in terms of the four-genus of related knots.

1. INTRODUCTION

A link $L \subset S^3$ has *splitting number* $\mathrm{sp}(L) = n$ if n is the least nonnegative integer for which some choice of n crossing changes between distinct components results in a totally split link. The study of splitting numbers and closely related invariants includes [1–5, 12, 14]. (In [1, 12, 14], the term *splitting number* permits self-crossing changes.) Here we will investigate the splitting number from the perspective of the four-genus of knots, an approach that is closely related to the use of concordance to study the splitting number in [3, 4] and earlier work considering concordances to split links [11]. Recent work by Jeong [10] develops a new infinite family of invariants that bound the splitting number, based on Khovanov homology. We will be working in the category of smooth oriented links, but notice that the splitting number is independent of the choice of orientation.

To state our results, we remind the reader of the notion of a band connected sum of a link L. A *band* b is an embedding $b\colon [0,1] \times [0,1] \to S^3$ such that $\mathrm{Image}(b) \cap L = b([0,1] \times \{0,1\})$. The orientation of the band must be consistent with the orientation of the link. To make this precise, first choose an orientation on $[0,1] \times [0,1]$. Then $b([0,1] \times \{0,1\})$ has an orientation arising as a subspace of the boundary of an embedded oriented disk; it is also oriented as a subspace of L. Those two orientations must agree. For an illustration, see Figure 1. The link L_b is defined to be $L \setminus (b([0,1] \times \{0,1\})) \cup (b(\{0,1\} \times [0,1]))$. Similarly, for a link L of k components, we can consider a set of $k-1$ disjoint bands $\beta = \{b_1, \ldots, b_{k-1}\}$ and use these to construct a link L_β; we will always work in the setting that β has the property that L_β is connected. We will call such a set of bands a *minimal connecting set of bands*.

For a knot K, we denote the mirror image of K with string orientation reversed by $\overline{K}$. The minimum genus of a smooth properly embedded oriented surface in the 4–ball with boundary K is called the four-genus of K, denoted $g_4(K)$.

Theorem 1.1. *Let $L = L_1 \cup \cdots \cup L_k$ be an oriented k–component link with linking numbers $lk(L_i, L_j) = l_{i,j}$ for $i \neq j$, and let β be a minimal connecting set of bands.*

Let N be the total linking number: $N = \left| \sum_{i<j} l_{i,j} \right|$. Then

$$sp(L) \geq 2g_4(L_\beta \#_{i=1}^k \overline{L}_i) - N.$$

As a simple corollary, we have:

Corollary 1.2. *If L is a k–component link with unknotted components and with all linking numbers 0, then for any minimal set of connecting bands, $sp(L) \geq 2g_4(L_\beta)$.*

Example 1.3. The simplest nontrivial link, the Hopf link, illustrates the role of the choice of β. One band connected sum yields the unknot, with four-genus 0, and another yields the trefoil, with four-genus 1; from this, Theorem 1.1 implies the obvious, that the splitting number is 1.

Example 1.4. Two basic examples of nonsplit links for which all linking numbers are 0 are the Whitehead link and the Borromean link. Most tools for studying splittings handle these examples, as does Corollary 1.2. For both links, band moves yield the trefoil knot, of four-genus 1, showing the splitting number is at least 2. Splittings with exactly two crossing changes are easily constructed.

Here is a generalization of an example in [4], studied in more depth in [8]. Consider the two-bridge link illustrated in Figure 1, with m, n and l nonnegative. The numbers in the boxes represent full twists. Without loss of generality, we can assume $m \geq n$. The linking number is $m - n$. The illustrated band leads to a knot L_b whose signature is easily computed to be $-2m$, so $g_4(L_b) \geq m$. (In fact, L_b is the connected sum of the torus knot $T_{2,2m+1}$ and a genus 1 knot of signature 0.) Thus, by Theorem 1.1, $sp(L) \geq 2m - (m - n) = n + m$. The link can evidently be split with $n + m$ crossings changes, so $sp(L) = n + m$.
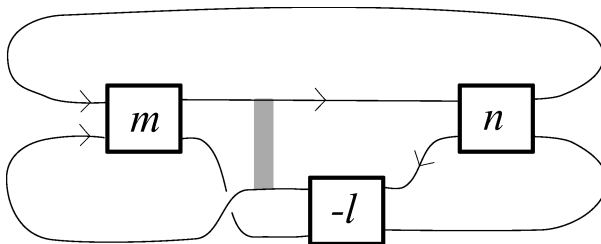

FIGURE 1. A family of two-bridge links.

The cases of $(m, n, l) = (1, 2, 1)$ and $(m, n, l) = (2, 3, 1)$ are the links L_{9a30} and L_{11a372}. The splitting numbers of these were determined in [5], with L_{9a30} serving as a basic example and L_{11a372} as an example of a case which could not be resolved in [2].

Theorem 1.1 provides a surprisingly easy and effective tool in determining splitting numbers, but it is not difficult to find examples for which it is weaker than previously developed methods. One reason is that the bound given in Theorem 1.1 is in fact a bound on the *concordance splitting number*, $csp(L)$, which is implicitly studied in [4]. This invariant is discussed in Section 3. The next family of examples presents the distinction between the two invariants.

Example 1.5. Figure 2 illustrates a link L_K, the Bing double of a knot K. The presence of an incompressible torus in its complement shows that if K is nontrivial, then $sp(L_K) = 2$. If K is slice, then L_K is concordant to the unlink, so $csp(L) = 0$ and the splitting number cannot be detected by Theorem 1.1.

The indicated band move on the Bing double produces the untwisted Whitehead double, $\mathrm{Wh}(K)$. Thus, by Corollary 1.2, if K is such that $\mathrm{Wh}(K)$ is not slice, then L_K is not concordant to a split link. As an example, letting $K = \mathrm{Wh}(T_{2,3})$ yields an example of a link L_K which is topologically but not smoothly concordant to a split link. Presumably, algebraic invariants would not detect the splitting number in this case.

Alternative approaches to showing the concordance splitting number of L_K is 2 (for specific choices of K) can be based on showing that the Bing double is not strongly slice, which was done, for instance, in [6, 7].
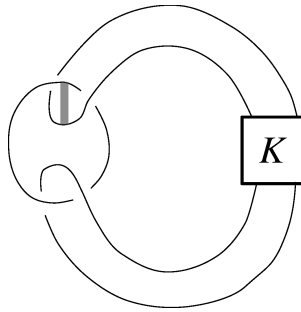

FIGURE 2. The Bing double, L_K.

2. Proof of Theorem 1.1

2.1. **The trace of the isotopy.** A set of crossing changes of a link $L^0 = L_1 \cup \cdots \cup L_k$ into a split link, which we will denote L^1, corresponds to an isotopy of L^0 with double points, which we call the *splitting isotopy*. We will write $L^1 = L_1 \sqcup \cdots \sqcup L_k$ to distinguish it from L^0 and to emphasize that the individual components are identical as knots. In general we will use the symbol $\sqcup$ to indicate split links.

The trace of the isotopy from L^0 to L^1 in $S^3 \times [0, 1]$ is an *immersed concordance*. To be specific, an immersed concordance between k–component links L^0 and L^1 is a smooth immersion

$$F \colon S^1 \times [0, 1] \times \{1, \ldots, k\} \to S^3 \times [0, 1]$$

such that

$$F(S^1 \times i \times j) = L_j^i \subset S^3 \times i$$

for $i = 0, 1$ and $j = 1, \ldots, k$. Singular points are required to be isolated transverse double points.

In the setting of Theorem 1.1, $L^0 = L = L_1 \cup \cdots \cup L_k$ and $L^1 = L_1 \sqcup \cdots \sqcup L_k$. The immersed concordance consists of a set of k *embedded* concordances intersecting transversely in double points. These embedded concordances are called *the components* of the immersed concordance, although they need not be disjoint.

Projection of $S^3 \times [0, 1]$ onto $[0, 1]$ defines a height function. In the current situation, there are no critical points for the height function on the concordance; each component is an embedded product concordance.

The splitting isotopy of L^0 to L^1 in S^3 can be extended to the bands of β by isotoping the bands so that they do not interfere with the crossings. Using this, we can construct an immersed concordance from L_β^0 to $L_{\beta'}^1$ for some set of bands β'.

2.2. Forming the connected sum with the $\overline{L}_i$.
We now form the connected sum of L_β^0 with $\#_i \overline{L}_i$. We do this by forming the connected sum of each L_i with the corresponding $\overline{L}_i$, so that the $\overline{L}_i$ is in a small ball far from the basepoints of any $b \in \beta$. It is now clear that we can modify the immersed concordance to form an immersed concordance from $L_\beta \#_i \overline{L}_i$ to $(\bigsqcup_i (L_i \# \overline{L}_i))_{\beta'}$.

2.3. Forming an immersed slice disk.
Observe that the knot $(\bigsqcup_i (L_i \# \overline{L}_i))_{\beta'}$ is slice. A set of $k - 1$ band moves (dual to the bands of β') yields the link $(L_1 \# \overline{L}_1) \sqcup \cdots \sqcup (L_k \# \overline{L}_k)$. Since the components are split and each is slice, we see that the original knot is slice.

Since the knot $L_\beta^0 \#_i \overline{L}_i$ bounds a singular concordance to a slice knot, it bounds a singular slice disk in B^4 with corresponding singular points.

2.4. Counting and resolving the double points.
For each pair (i, j), let $p_{i,j}$ and $n_{i,j}$ denote the number of positive and negative crossing changes between L_i and L_j in the splitting sequence for L, respectively. The linking numbers are given by $l_{i,j} = p_{i,j} - n_{i,j}$.

Since $p_{i,j} \geq 0$ and $n_{i,j} \geq 0$, it follows that $(p_{i,j} + n_{i,j}) - |p_{i,j} - n_{i,j}| \geq 0$. The difference is clearly even, so we write

$$(p_{i,j} + n_{i,j}) - |p_{i,j} - n_{i,j}| = 2m_{i,j},$$

where $m_{i,j} \geq 0$.

Let $\mathcal{P}$ be the set of pairs $(i, j), i < j$, such that the linking number $l_{i,j} \geq 0$ and let $\mathcal{N}$ be the set of pairs $(i, j), i < j$, such that the linking number $l_{i,j} < 0$. For $(i, j) \in \mathcal{P}$, one sees that $m_{i,j} = n_{i,j}$; for $(i, j) \in \mathcal{N}$, $m_{i,j} = p_{i,j}$.

For each pair $(i, j) \in \mathcal{P}$, we have $|l_{i,j}| = p_{i,j} - n_{i,j}$, so that $p_{i,j} = |l_{i,j}| + n_{i,j} = |l_{i,j}| + m_{i,j}$. The number of negative crossing changes is $n_{i,j} = m_{i,j}$.

Similarly, for each pair $(i, j) \in \mathcal{N}$, the number of negative crossing changes between the i and j components during the splitting is $|l_{i,j}| + m_{i,j}$, and the number of positive crossing changes is $p_{i,j} = m_{i,j}$.

It follows from this count that the total number of positive double points in the immersed concordance is

$$A = \sum_{(i,j) \in \mathcal{P}} |l_{i,j}| + \sum_{(i,j) \in \mathcal{P} \cup \mathcal{N}} m_{i,j}.$$

The number of negative double points is

$$B = \sum_{(i,j) \in \mathcal{N}} |l_{i,j}| + \sum_{(i,j) \in \mathcal{P} \cup \mathcal{N}} m_{i,j}.$$

2.5. Building an embedded surface in the 4–ball bounded by $L_\beta \#_i \overline{L}_i$.

Assume now that the initial sequence of crossing changes was a minimal splitting sequence. Our goal is to build an embedded surface bounded by $L_\beta \#_i \overline{L}_i$ in the 4–ball by tubing together pairs of canceling double points in the immersed slice disk (which we temporarily denote $\overline{B}$) and then resolving the remaining double points individually. Here is a summary of the construction.

Let x_0 and x_1 be a positive and negative double point on the immersed surface. There is an embedded path γ on $\overline{B}$ from x_0 to x_1 which misses all other double points. A tubular neighborhood $N(\gamma)$ of γ is diffeomorphic to $B^3 \times I$. Up to diffeomorphism, we have

$$N(\gamma) \cap \overline{B} \cong \Big(\{(0, x, y)\} \times \{0\}\Big) \cup \Big(\{(z, 0, 0)\} \times I\Big) \cup \Big(\{(0, x, y)\} \times \{1\}\Big),$$

where the set of pairs $\{(x, y)\}$ ranges over $B^2 \setminus (0, 0)$, and z ranges over B^1. The immersed surface $\overline{B}$ is now modified by removing the sets $\Big(\{(0, x, y)\} \times \{0\}\Big)$ and $\Big(\{(0, x, y)\} \times \{1\}\Big)$ and replacing them with $\{(0, s, t)\} \times I$, where the set of pairs $\{(s, t)\}$ is restricted to range over the unit circle. This construction, called ambient surgery, has replaced a pair of disks on the immersed surface with an annulus, and the new immersed surface has two fewer double points. The resulting surface is orientable because the intersection points are of opposite sign. This operation has increased the genus by one.

If any double points remain after removing these pairs of double points, they can be eliminated one at a time by removing a pair of transversely intersecting disks and replacing them with an annulus; this also increases the genus by one.

On $\overline{B}$ we had A positive double points and B negative double points. Thus, we can find a set of $\min(A, B)$ of pairs of double points having opposite sign. After performing the surgery to remove these pairs of points, the number of remaining double points is $A + B - 2\min(A, B) = \max(A, B) - \min(A, B)$. Thus, after removing these double points, the genus of the resulting surface is $\min(A, B) + (\max(A, B) - \min(A, B)) = \max(A, B)$.

We have now constructed an embedded bounding surface of genus

$$g = \max(A, B) = (|A + B| + |A - B|)/2.$$

Using the formulas for A and B, this becomes

$$2g = \left(\sum_{(i,j)\in\mathcal{P}\cup\mathcal{N}} |l_{i,j}| + 2\sum_{(i,j)\in\mathcal{P}\cup\mathcal{N}} m_{i,j}\right) + \left|\Big(\sum_{(i,j)\in\mathcal{P}} |l_{i,j}| - \sum_{(i,j)\in\mathcal{N}} |l_{i,j}|\Big)\right|.$$

It follows from the definition of $m_{i,j}$ that $|l_{i,j}| + 2m_{i,j} = p_{i,j} + n_{i,j}$. From this we see that the expression in the first set of parentheses equals the splitting number; the second term (the absolute value of the difference of sums) is simply the absolute value of the sum of the linking numbers, called N in the statement of Theorem 1.1. Thus,

$$2g = sp(L) + N,$$

and so, as desired,

$$sp(L) = 2g - N \geq 2g_4(L_\beta \#_{i=1}^k \overline{L}_i) - N.$$

3. Concordance splitting

The lower bound on the splitting number given in Theorem 1.1 is in fact a bound on the *concordance splitting number*.

Definition 3.1. A link L has concordance splitting number $csp(L) = n$ if n is the least nonnegative integer such that there is an immersed concordance form L to a split link having n double points and each component is embedded.

Example 3.2. Example 1.5 demonstrates that for some links $csp(L) < sp(L)$.

Notice that in the definition, the concordance need not be to the link $L_1 \sqcup \cdots \sqcup L_k$. However, we have the following.

Lemma 3.3. *If $csp(L) = n$, then there is an immersed concordance, with n double points and each component embedded, from L to $L_1 \sqcup \cdots \sqcup L_k$.*

Proof. The end of the immersed concordance is a link $L'_1 \sqcup \cdots \sqcup L'_k$. Since the components of the immersed concordance are embedded, each L'_i is concordant to L_i. Thus, the immersed concordance can be extended using these individual concordances so that the ending link is $L_1 \sqcup \cdots \sqcup L_k$. □

We have the following analog of Theorem 1.1.

Theorem 3.4. *Let $L = L_1 \cup \cdots \cup L_k$ be an oriented k–component link with linking numbers $lk(L_i, L_j) = l_{i,j}$ for $i \neq j$, and let β be a set of $k - 1$ bands for which L_β is connected. Let N be the total linking number: $N = \left| \sum_{i<j} l_{i,j} \right|$. Then*

$$csp(L) \geq 2g_4(L_\beta \#_{i=1}^k \overline{L}_i) - N.$$

Proof. Much of the proof proceeds as before, but there is one significant difficulty. The presence of possible maximum points in the concordance prevents one from converting the concordance of the link into a concordance of its band connected sum. The bands might interfere with capping off unknotted components that arise from index two critical points. Here is how the proof is adjusted.

The concordance can be modified so that all critical points of index 2 occur at height 3/4 and all other critical points and double points occur below the height of 1/4. At level 1/2 we have the link $L' \sqcup U_1 \sqcup \cdots \sqcup U_r$, where the U_r form an unlink split from L' (each component of which is capped off at level 3/4).

The index 0 and index 1 critical points do not interfere with the constructions used earlier, and from this one finds that there is a genus 0 immersed cobordism from $L_\beta \#_i \overline{L}_i$ to $(\bigsqcup_i (L_i \# \overline{L}_i))_{\beta'} \sqcup U_1 \sqcup \cdots \sqcup U_r$. Notice that the bands in β' might link the U_i, and this is the point of difficulty. However, we can use this cobordism to construct an immersed slice disk: perform the band moves dual to the β' to build a split link with all components slice knots (some are the $L_i \# \overline{L}_i$ and some are the U_i); these can be capped off to form the immersed slice disk.

The rest of the proof is identical to that of Theorem 1.1. □

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