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SYMBOLIC POWERS OF COVER IDEAL OF VERY WELL-COVERED AND BIPARTITE GRAPHS

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Abstract. Let G be a graph with n vertices and $S = \mathbb{K}[x_1, \dots, x_n]$ be the polynomial ring in n variables over a field $\mathbb{K}$. Assume that $J(G)$ is the cover ideal of G and $J(G)^{(k)}$ is its k-th symbolic power. We show that if G is a very well-covered graph such that $J(G)$ has a linear resolution, then for every integer $k \geq 1$, the ideal $J(G)^{(k)}$ has a linear resolution and moreover, the modules $J(G)^{(k)}$ and $S/J(G)^{(k)}$ satisfy Stanley's inequality, i.e., their Stanley depth is an upper bound for their depth. Finally, we determine a linear upper bound for the Castelnuovo–Mumford regularity of powers of cover ideals of bipartite graphs.

1. Introduction

Over the last 25 years the study of algebraic, homological and combinatorial properties of powers of ideals has been one of the major topics in Commutative Algebra. In this paper we study the minimal free resolution of the powers of cover ideals of graphs. The cover ideal of a graph is the Alexander dual of its edge ideal and has been studied by several authors (see, e.g., [4], [6], [13], [14], [27]).

In Section 3, we study the minimal free resolution of symbolic powers of cover ideals of very well-covered graphs. A graph G is said to be very well-covered if the cardinality of every maximal independent set of G is half the number of vertices of G. The family of very well-covered graphs includes all the unmixed bipartite graphs, which have no isolated vertex. Further interest comes from a complete combinatorial characterization of Cohen–Macaulayness in this case. This class of graphs is studied from the algebraic point of view in [2], [8], [25] [26]. Let G be a graph with n vertices and $S = \mathbb{K}[x_1, \dots, x_n]$ be the polynomial ring in n variables over a field $\mathbb{K}$. Assume that $J(G)$ is the cover ideal of G. The first main result of Section 3 determines a class of monomial ideals, such that all symbolic powers of an ideal in this class have a linear resolution. In fact, there are many attempts to characterize the monomial ideals with a linear resolution. One of the most important results in this direction is due to Fröberg [15, Theorem 1], who characterized all squarefree monomial ideals generated by quadratic monomials, which have a linear resolution. It is also known [23] that polymatroidal ideals have a linear resolution and that powers of polymatroidal ideals are again polymatroidal (see [20]). In particular they have again a linear resolution. In general, however, powers of ideals with a linear resolution need not have

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a linear resolution. The first example of such an ideal was given by Terai. He showed that over a base field of characteristic $\neq 2$ the Stanley Reisner ideal $I = (x_1x_2x_3, x_1x_2x_5, x_1x_3x_6, x_1x_4x_5, x_1x_4x_6, x_2x_3x_4, x_2x_4x_6, x_2x_5x_6, x_3x_4x_5, x_3x_5x_6)$ of the minimal triangulation of the projective plane has a linear resolution, while I^2 does not have a linear resolution. This example depends on the characteristic of the base field. If the base field has characteristic 2, then I itself does not have a linear resolution. Another example, namely $I = (x_4x_5x_6, x_3x_5x_6, x_3x_4x_6, x_3x_4x_5, x_2x_5x_6, x_2x_3x_4, x_1x_3x_6, x_1x_4x_5)$ is given by Sturmfels [35]. Again I has a linear resolution, while I^2 does not have a linear resolution. However, Herzog, Hibi and Zheng [22] prove that a monomial ideal I generated in degree 2 has a linear resolution if and only if every power of I has a linear resolution. Also, it follows from [28, Theorem 2.2] that if G is a bipartite graph such that $J(G)$ has a linear resolution, then every power of $J(G)$ has a linear resolution too. Our Theorem 3.6 is a generalization of this result and asserts that if G is a very well-covered graph, such that $J(G)$ has a linear resolution, then for every integer $k \geq 1$, the k-th symbolic power of $J(G)$, denoted by $J(G)^{(k)}$, has a linear resolution and even more, it has linear quotients. In order to prove this result, in Proposition 3.1, we introduce a construction to obtain a Cohen–Macaulay very well-covered graph from a given Cohen–Macaulay very well-covered graph. We will see that the cover ideal of the resulting graph is related to the symbolic powers of the cover ideal of the primary graph, via polarization. In Corollary 3.7, we prove that the converse of Theorem 3.6 is true for bipartite graphs. In other words, for a bipartite graph G, the cover ideal $J(G)$ has a linear resolution if and only if $J(G)^{(k)}$ has a linear resolution for some integer $k \geq 1$. Next, in Corollary 3.8, we prove that if G is a very well-covered graph such that $J(G)$ has a linear resolution, then for every integer $k \geq 1$, the modules $J(G)^{(k)}$ and $S/J(G)^{(k)}$ satisfy Stanley's inequality, i.e., their Stanley depth is an upper bound for their depth. In the proof of Corollary 3.8, we use the result obtained in [26], which states that for very well-covered graphs the notions of Cohen–Macaulayness and vertex decomposability are the same.

Computing and finding bounds for the regularity of powers of a monomial ideal have been studied by a number of researchers (see for example [1], [3], [5], [18]). It follows from [9, Theorem 1.1] that $\mathrm{reg}(I^s)$ is asymptotically a linear function for $s \gg 0$. However, it is usually difficult to compute this linear function or estimate it. In Section 4, we study the regularity of (ordinary) powers of cover ideals of a bipartite graph (note that by [16, Corollary 2.6], for the cover ideal of bipartite graphs the ordinary and symbolic powers coincide). In Theorem 4.3, we determine a linear upper bound for the regularity of these ideals. More explicit, we prove that for a bipartite graph G and every integer $k \geq 1$, the regularity of $S/J(G)^k$ is at most $k\mathrm{deg}(J(G)) + \mathrm{reg}(S/J(G)) - 1$, where for a monomial ideal I, we denote the maximum degree of minimal monomial generators of I by $\mathrm{deg}(I)$.

2. Preliminaries

In this section, we provide the definitions and basic facts which will be used in the next sections. We refer the reader to [20] for undefined terminology.

Let G be a simple graph with vertex set $V(G) = \{x_1, \dots, x_n\}$ and edge set $E(G)$ (by abusing the notation, we identify the vertices of G with the variables of S). For a vertex x_i, the *neighbor set* of x_i is $N_G(x_i) = \{x_j \mid \{x_i, x_j\} \in E(G)\}$ and we set $N_G[x_i] = N_G(x_i) \cup \{x_i\}$ and call it the *closed neighborhood* of x_i. For a subset $F \subseteq V(G)$, we set $N_G[F] = \bigcup_{x_i \in F} N_G[x_i]$. For every subset $A \subset V(G)$, the graph $G \setminus A$ is the graph with vertex set $V(G \setminus A) = V(G) \setminus A$ and edge set $E(G \setminus A) = \{e \in E(G) \mid e \cap A = \emptyset\}$. A subgraph H of G is called induced provided that two vertices of H are adjacent if and only if they are adjacent in G. A *matching* in a graph G is a subgraph consisting of pairwise disjoint edges. If the subgraph is an induced subgraph, the matching is an *induced matching*. The cardinality of the maximum induced matching of G is denoted by $\mathrm{indmatch}(G)$. A *perfect matching* in G is a matching whose vertex set is the same as $V(G)$. A subset W of $V(G)$ is called an *independent subset* of G if there are no edges among the vertices of W. The graph G is said to be *very well-covered* if n is an even number and every maximal independent subset of G has cardinality $n/2$.

A *simplicial complex* Δ on the set of vertices $V(\Delta) = \{v_1, \ldots, v_n\}$ is a collection of subsets of $V(\Delta)$ which is closed under taking subsets; that is, if $F \in \Delta$ and $F' \subseteq F$, then also $F' \in \Delta$. Every element $F \in \Delta$ is called a *face* of Δ, and its *dimension* is defined to be $|F| - 1$. The *dimension* of Δ which is denoted by $\dim \Delta$, is $d - 1$, where $d = \max\{|F| \mid F \in \Delta\}$. A *facet* of Δ is a maximal face of Δ with respect to inclusion. Let $\mathcal{F}(\Delta)$ denote the set of facets of Δ. It is clear that $\mathcal{F}(\Delta)$ determines Δ. When $\mathcal{F}(\Delta) = \{F_1, \ldots, F_m\}$, we write $\Delta = \langle F_1, \ldots, F_m \rangle$. We say that Δ is *pure* if all facets of Δ have the same cardinality. For every subset $F \subseteq V(\Delta)$, we set $x_F = \prod_{v_i \in F} x_i$. The *Stanley–Reisner ideal of* Δ *over* $\mathbb{K}$ is the ideal I_Δ of S which is generated by those squarefree monomials x_F with $F \notin \Delta$. The *Stanley–Reisner ring of* Δ *over* $\mathbb{K}$, denoted by $\mathbb{K}[\Delta]$, is defined to be $\mathbb{K}[\Delta] = S/I_\Delta$.

Let G be a graph. *Independence simplicial complex* of G is defined by

$$\Delta(G) = \{A \subseteq V(G) \mid A \text{ is an independent set in } G\}.$$

Note that the Stanley–Reisner ideal of $\Delta(G)$ is the *edge ideal* of G which is defined to be

$$I(G) = (x_i x_j \mid \{x_1, x_j\} \in E(G)) \subset S.$$

A subset C of $V(G)$ is called a *vertex cover* of the graph G if every edge of G is incident to at least one vertex of C. A vertex cover C is called a *minimal vertex cover* of G if no proper subset of C is a vertex cover of G. Note that C is a minimal vertex cover if and only if $V(G) \setminus C$ is a maximal independent set, that is, a facet of $\Delta(G)$. A graph G is called *unmixed* if all minimal vertex covers of G have the same number of elements. Hence G is an unmixed graph if and only if $\Delta(G)$ is a pure simplicial complex. The size of the smallest vertex cover of G will be denoted by $\tau(G)$. The Alexander dual of the edge ideal of G in S, i.e., the ideal

$$J(G) = I(G)^\vee = \bigcap_{\{x_i, x_j\} \in E(G)} (x_i, x_j),$$

is called the *cover ideal* of G and is the main object of study in this paper. The reason for this name is due to the well-known fact that the generators of $J(G)$ correspond to minimal vertex covers of G.

Definition 2.1. Let I be an ideal of S and $\mathrm{Min}(I)$ denote the set of minimal prime of I. For every integer $k \geq 1$, the k-th *symbolic power* of I, denoted by $I^{(k)}$, is defined to be

$$I^{(k)} = \bigcap_{P \in \mathrm{Min}(I)} \mathrm{Ker}(R \to (R/I^k)_P).$$

Let I be a squarefree monomial ideal in S and suppose that I has the irredundant primary decomposition

$$I = \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_r,$$

where every $\mathfrak{p}_i$ is an ideal of S generated by a subset of the variables of S. It follows from [20, Proposition 1.4.4] that for every integer $k \geq 1$,

$$I^{(k)} = \mathfrak{p}_1^k \cap \ldots \cap \mathfrak{p}_r^k.$$

For a graded S-module M, we denote the graded Betti numbers of M by $\beta_{i,j}(M)$. The Castelnuovo–Mumford regularity (or simply, regularity) of M, denoted by $\mathrm{reg}(M)$, is defined as follows:

$$\mathrm{reg}(M) = \max\{j - i | \ \beta_{i,j}(M) \neq 0\}.$$

The module M is said to have a *linear resolution*, if for some integer d, $\beta_{i,i+t}(M) = 0$ for all i and every $t \neq d$. It is clear from the definition that if an ideal has a linear resolution, then all the minimal generators of I have the same degree. Next, we recall the definition of monomial ideals with linear quotients. We recall that for a monomial ideal I, the set of minimal monomial generators of I is denoted by $\mathrm{Gens}(I)$.

Definition 2.2 ([23])**.** Let I be a monomial ideal. Assume that $u_1 \prec u_2 \prec \ldots \prec u_m$ is a linear order on $\mathrm{Gens}(I)$. We say that I has *linear quotients with respect to* $\prec$, if for every $2 \leq i \leq m$, the ideal $(u_1, \ldots, u_{i-1}) : u_i$ is generated by a subset of the variables. We say that I has *linear quotients*, if it has linear quotients with respect to a linear order on $\mathrm{Gens}(I)$.

By [20, Proposition 8.2.1], we know that if I is a monomial ideal which is generated in a single degree and has linear quotients, then it admits a linear resolution. Monomial ideals with linear quotients are related to an important class of simplicial complexes, namely shellable simplicial complexes.

Definition 2.3. A simplicial complex Δ is called *shellable* if its facets can be arranged in linear order $F_1, F_2, \ldots, F_t$ in such a way that the subcomplex $\langle F_1, \ldots, F_{k-1}\rangle \cap \langle F_k \rangle$ is pure and has dimension $\dim F_k - 1$ for every k with $2 \leq k \leq t$.

By [20, Theorem 8.2.5], a simplicial complex Δ is shellable if and only if $I_{\Delta^\vee}$ has linear quotients, where $\Delta^\vee$ is the Alexander dual of Δ. A simplicial complex Δ is called Cohen–Macaulay over $\mathbb{K}$ if its Stanley–Reisner ring $\mathbb{K}[\Delta]$ is a Cohen–Macaulay ring. In this paper, we fix a field $\mathbb{K}$. When we say that a simplicial complex Δ is Cohen–Macaulay without qualification, we implicitly mean that Δ is Cohen–Macaulay over $\mathbb{K}$. A fundamental result in combinatorial commutative algebra says that a pure shellabe simplicial complex is Cohen–Macaulay (over any field $\mathbb{K}$). Also, it follows from the Eagon–Reiner theorem [20, Theorem 8.1.9], that a simplicial complex Δ is Cohen–Macaulay if and only if $I_{\Delta^\vee}$ has a linear resolution.

Let Δ be a simplicial complex. The *link* of Δ with respect to a face $F \in \Delta$, denoted by $\mathrm{lk}_\Delta(F)$, is the simplicial complex $\mathrm{lk}_\Delta(F) = \{G \subseteq [n] \setminus F \mid G \cup F \in \Delta\}$ and the *deletion* of F, denoted by $\mathrm{del}_\Delta(F)$, is the complex $\mathrm{del}_\Delta(F) = \{G \subseteq [n] \setminus F \mid G \in \Delta\}$. When $F = \{x\}$ is a single vertex, we abuse notation and write $\mathrm{lk}_\Delta(x)$ and $\mathrm{del}_\Delta(x)$. We are now ready to define vertex decomposable simplicial complexes which will be used in the proof of Corollary 3.8.

Definition 2.4. Let Δ be a simplicial complex. Then we say that Δ is *vertex decomposable* if either

(1) Δ is a simplex, or
(2) Δ has a vertex x such that $\mathrm{del}_\Delta(x)$ and $\mathrm{lk}_\Delta(x)$ are vertex decomposable and every facet of $\mathrm{del}_\Delta(x)$ is a facet of Δ.

A graph G is called Cohen–Macaulay/shellable/vertex decomposable if $\Delta(G)$ has the same property. Thus, the graph G is Cohen–Macaulay if and only if $J(G)$ has a linear resolution and it is shellable if and only if $J(G)$ has linear quotients. We know from [26, Theorem 1.1] (see also [7, Theorem 2.3]) that for very well-covered graphs, the concepts of Cohen–Macaulayness, shellability and vertex decomposability are equivalent.

3. Very well-covered graphs

The aim of this section is to study the minimal free resolution of symbolic powers of cover ideals of very well-covered graphs. Before stating our results, we notice that if G is not a bipartite graph, then there is an integer $k \geq 1$, for which $J(G)^{(k)} \neq J(G)^k$ (see [21, Theorem 5.1]).

As the first result of this section, we prove in Theorem 3.6 that if G is a very well-covered graph such that $J(G)$ has a linear resolution, then every symbolic power $J(G)^{(k)}$ has a linear resolution too. Notice that if G is not a bipartite graph, then there is an integer $k \geq 1$, for which $J(G)^{(k)} \neq J(G)^k$. In order to prove this result, we introduce a construction to obtain a Cohen–Macaulay very well-covered graph from a given Cohen–Macaulay very well-covered graph. In the following construction, for every graph G and every integer $k \geq 1$, we build a new graph G_k whose cover ideal is strongly related to the k-th symbolic power of $J(G)$ (see Lemma 3.4).

Construction. Let G be a graph with vertex set $V(G) = \{x_1, \dots, x_n\}$ and let $k \geq 1$ be an integer. We define the new graph G_k on new vertices

$$V(G_k) = \{x_{i,p} \mid 1 \leq i \leq n \text{ and } 1 \leq p \leq k\},$$

(thus G_k has nk vertices) and the edge set of G_k is

$$E(G_k) = \{\{x_{i,p}, x_{j,q}\} \mid \{x_i, x_j\} \in E(G) \text{ and } p + q \leq k + 1\}.$$

Proposition 3.1. *Let G be a graph without isolated vertices and $k \geq 1$ be an integer.*

(a) *If G is very well-covered, then G_k is very well-covered too.*
(b) *If G is Cohen–Macaulay and very well-covered, then G_k is Cohen–Macaulay too.*

Proof. Since G is very well-covered, $n = |V(G)|$ is an even integer. Set $h = n/2$. By [26, Lemma 4.1] and [8, Proposition 2.3], the vertices of G can be relabeled, say $V(G) = \{w_1, \dots, w_h, z_1, \dots, z_h\}$ such that

(i) $\{w_1, \dots, w_h\}$ is a minimal vertex cover of G and $\{z_1, \dots, z_h\}$ is a maximal independent set of G;
(ii) $\{w_1, z_1\}, \dots, \{w_h, z_h\} \in E(G)$;

(iii) if $\{y_i, w_j\}, \{z_j, w_l\} \in E(G)$, then $\{y_i, w_l\} \in E(G)$ for distinct i, j, l and for $y_i \in \{w_i, z_i\}$;
(iv) if $\{w_i, z_j\} \in E(G)$, then $\{w_i, w_j\} \notin E(G)$.

We rename the vertices of G_k as follows:

$$a_1 := w_{1,1},\ a_2 := w_{2,1}, \ldots, a_h := w_{h,1},\ a_{h+1} := w_{1,2}, \ldots, a_{2h} := w_{h,2}, \ldots, a_{kh} := w_{h,k},$$
$$b_1 := z_{1,k},\ b_2 := z_{2,k}, \ldots, b_h := z_{h,k},\ b_{h+1} := z_{1,k-1}, \ldots, b_{2h} := z_{h,k-1}, \ldots, b_{kh} := z_{h,1}.$$

It is clear from (i) and the construction of G_k that $\{a_1, a_2, \ldots, a_{kh}\}$ is a minimal vertex cover of G_k and $\{b_1, b_2, \ldots, b_{kh}\}$ is a maximal independent set of G_k. This shows that $\tau(G_k) \leq kh$. Also, it follows from (ii) that $\{\{a_1, b_1\}, \ldots, \{a_{kh}, b_{kh}\}\}$ is a perfect matching of G_k. Therefore $\tau(G_k) = kh$. This implies that $\mathrm{ht}(I(G_k)) = kh$.

Assume that i, j, l are distinct integers with $1 \leq i, j, l \leq kh$. Then there exist integers m, p, q, r, s, t such that $a_i = w_{m,p}, b_i = z_{m,k+1-p}, a_j = w_{q,r}, b_j = z_{q,k+1-r}$ and $a_l = w_{s,t}$. We continue the proof in several steps.

Step 1. If $\{a_i, a_j\}, \{b_j, a_l\} \in E(G_k)$, then $\{a_i, a_l\} \in E(G_k)$.

Proof. It follows from the assumptions that $\{w_m, w_q\}, \{w_s, z_q\} \in E(G)$, $p+r \leq k+1$ and $k+1-r+t \leq k+1$. Hence, $p+t = (p+r) + (k+1-r+t) - (k+1) \leq k+1$. Since $\{w_m, w_q\} \in E(G)$, we conclude that $m \neq q$. It also follows from (iv) that $s \neq m$. If $s = q$, then $\{w_m, w_s\} = \{w_m, w_q\} \in E(G)$. Thus, $\{a_i, a_l\}$ is an edge of G_k. If $s \neq q$, then it follows from (iii) that $\{w_m, w_s\}$ is an edge of G. Thus, again $\{a_i, a_l\} \in E(G_k)$. □

Step 2. If $\{b_i, a_j\}, \{b_j, a_l\} \in E(G_k)$, then $\{b_i, a_l\} \in E(G_k)$.

Proof. It follows from the assumptions that $\{z_m, w_q\}, \{w_s, z_q\} \in E(G)$, $k+1-p+r \leq k+1$ and $k+1-r+t \leq k+1$. Hence, $(k+1-p)+t = (k+1-p+r)+(k+1-r+t)-(k+1) \leq k+1$. If $s = q$, then it follows from $\{z_m, w_s\} = \{z_m, w_q\} \in E(G)$ that $\{b_i, a_l\} \in E(G_k)$. Therefore, assume that $s \neq q$. Similarly, we can assume that $m \neq q$. If $s = m$, it follows from (ii) that $\{z_m, w_s\} = \{z_m, w_m\} \in E(G)$, which implies that $\{b_i, a_l\} \in E(G_k)$. Thus, suppose that $s \neq m$. Hence, m, q and s are distinct. Then (iii) implies that $\{z_m, w_s\}$ is an edge of G. Thus, again $\{b_i, a_l\} \in E(G_k)$. □

Step 3. If $\{a_i, b_j\} \in E(G_k)$, then $\{w_m, z_q\} \in E(G)$ and it follows from (iv) that $\{w_m, w_q\} \notin E(G)$. Thus, $\{a_i, a_j\} \notin E(G_k)$. □

Now, Steps 1, 2, 3 and [29, Theorem 2.9] (see also [8, Proposition 2.3]) imply that G is unmixed and since $\mathrm{ht}(I(G_k)) = kh = |V(G)|/2$, we conclude that G_k is a very well-covered graph.

Next assume that G is a Cohen–Macaulay very well-covered graph. By [26, Lemma 3.1] there is a relabeling for the vertices of G which satisfies conditions (i)-(iv), mentioned above and the following condition:

(v) If $\{w_i, z_j\} \in E(G)$, then $i \leq j$.

Step 4. If $\{a_i, b_j\} \in E(G_k)$, then $i \leq j$.

Proof. It follows from the assumption that $\{w_m, z_q\} \in E(G)$ and $p + (k+1-r) \leq k+1$. Thus $p \leq r$ and it follows from (v) that $m \leq q$. Therefore $i \leq j$.

Finally, it follows from Steps 1, 2, 3, 4 and [26, Lemma 3.1] that G_k is Cohen–Macaulay. □

Remark 3.2. Although we proved in Proposition 3.1 that for every very well-covered graph G, the graph G_k is also a very well-covered graph, but it is not in general true that G_k is unmixed if G is. For example, let $G = K_3$ be the complete graph with three vertices. Then G is unmixed (even Cohen–Macaulay) but G_2 is not unmixed.

We next recall the definition of polarization. It is a very useful machinery to convert a monomial ideal to a squarefree one.

Definition 3.3. Let I be a monomial ideal of $S = \mathbb{K}[x_1, \ldots, x_n]$ with minimal generators $u_1, \ldots, u_m$, where $u_j = \prod_{i=1}^n x_i^{a_{i,j}}$, $1 \le j \le m$. For every i with $1 \le i \le n$, let $a_i = \max\{a_{i,j} \mid 1 \le j \le m\}$, and suppose that

$$T = \mathbb{K}[x_{1,1}, x_{1,2}, \ldots, x_{1,a_1}, x_{2,1}, x_{2,2}, \ldots, x_{2,a_2}, \ldots, x_{n,1}, x_{n,2}, \ldots, x_{n,a_n}]$$

is a polynomial ring over the field $\mathbb{K}$. Let I^{pol} be the squarefree monomial ideal of T with minimal generators $u_1^{\mathrm{pol}}, \ldots, u_m^{\mathrm{pol}}$, where $u_j^{\mathrm{pol}} = \prod_{i=1}^n \prod_{k=1}^{a_{i,j}} x_{i,k}$, $1 \le j \le m$. The monomial u_j^{pol} is called the *polarization* of u_j, and the ideal I^{pol} is called the *polarization* of I.

As we mentioned at the beginning of this section, for every graph G and every integer $k \ge 1$, the cover ideal of G_k is related to the k-th symbolic power of the cover ideal of G. This is the content of the following lemma.

Lemma 3.4. *Let G be a graph. For every integer $k \ge 1$, the ideal $(J(G)^{(k)})^{\mathrm{pol}}$ is the cover ideal of G_k.*

Proof. We know that polarization commutes with the intersection (see [12, Proposition 2.3]). Therefore,

$$(J(G)^{(k)})^{\mathrm{pol}} = \bigcap_{\{x_i,x_j\} \in E(G)} ((x_i, x_j)^k)^{\mathrm{pol}}.$$

Moreover, by [12, Proposition 2.5], it holds that

$$((x_i, x_j)^k)^{\mathrm{pol}} = \bigcap_{p+q \le k+1} (x_{i,p}, x_{j,q}).$$

Thus,

$$(J(G)^{(k)})^{\mathrm{pol}} = \bigcap_{\{x_i,x_j\} \in E(G)} \; \bigcap_{p+q \le k+1} (x_{i,p}, x_{j,q}).$$

Therefore, $(J(G)^{(k)})^{\mathrm{pol}}$ is the cover ideal of G_k. □

We know from [20, Corollary 1.6.3] that polarization preserves the graded Betti numbers. Thus a monomial ideal has a linear resolution if and only if its polarization has a linear resolution. In the following lemma, we show that a similar statement is true if one replaces the linear resolution by linear quotients.

Lemma 3.5. *A monomial ideal I has linear quotients if and only if I^{pol} has linear quotients.*

Proof. We use the notation of Definition 3.3.

($\Rightarrow$) By [20, Lemma 8.2.3], the elements of $\mathrm{Gens}(I)$ can be ordered $u_1, \ldots, u_m$ such that for every pair of integers $j < i$ there exist an integer $k < i$ and a variable x_p such that

$$\frac{u_k}{\gcd(u_k, u_i)} = x_p \quad \text{and} \quad x_p \text{ divides } \frac{u_j}{\gcd(u_j, u_i)}.$$

Let t be the largest integer with $x_p^t | u_i$. It follows from the equality

$$\frac{u_k}{\gcd(u_k, u_i)} = x_p$$

that $x_p^{t+1} | u_k$ and $x_p^{t+2} \nmid u_k$, Therefore

$$\frac{u_k^{\rm pol}}{\gcd(u_k^{\rm pol}, u_i^{\rm pol})} = x_{p,t+1}.$$

On the other hand, x_p divides

$$\frac{u_j}{\gcd(u_j, u_i)}$$

by the choice of t; we conclude that $x_p^{t+1} | u_j$. This shows that $x_{p,t+1}$ divides

$$\frac{u_j^{\rm pol}}{\gcd(u_j^{\rm pol}, u_i^{\rm pol})}.$$

It again follows from [20, Lemma 8.2.3] that $I^{\rm pol}$ has linear quotients.

($\Leftarrow$) By [20, Lemma 8.2.3], the elements of $G(I^{\rm pol})$ can be ordered $u_1^{\rm pol}, \ldots, u_m^{\rm pol}$ such that for every pair of integers $j < i$ there exist an integer $k < i$ and a variable $x_{p,q} \in T$ such that

$$\frac{u_k^{\rm pol}}{\gcd(u_k^{\rm pol}, u_i^{\rm pol})} = x_{p,q} \qquad \text{and} \qquad x_{p,q} \ \text{ divides } \ \frac{u_j^{\rm pol}}{\gcd(u_j^{\rm pol}, u_i^{\rm pol})}.$$

This shows that

$$\frac{u_k}{\gcd(u_k, u_i)} = x_p \qquad \text{and} \qquad x_p \ \text{ divides } \ \frac{u_j}{\gcd(u_j, u_i)},$$

and hence, I has linear quotients. □

We are now ready to prove the first main result of this section.

Theorem 3.6. *Let G be a very well-covered graph and suppose that its cover ideal $J(G)$ has a linear resolution. Then*

(i) *$J(G)^{(k)}$ has a linear resolution, for every integer $k \geq 1$.*
(ii) *$J(G)^{(k)}$ has linear quotients, for every integer $k \geq 1$.*

Proof. Since the isolated vertices have no effect on the cover ideal, we assume that G has no isolated vertex. Since $J(G)$ has a linear resolution, it follows from [20, Theorem 8.1.9] that G is a Cohen–Macaulay graph. Thus, by Proposition 3.1, the graph G_k is Cohen–Macaulay.

(i) Notice that [20, Theorem 8.1.9] and Lemma 3.4 imply that $(J(G)^{(k)})^{\rm pol} = J(G_k)$ has a linear resolution. Hence, it follows from [20, Corollary 1.6.3] that $J(G)^{(k)}$ has a linear resolution.

(ii) By [26, Theorem 1.1] (see also [7, Theorem 2.3]), we know that every Cohen–Macaulay very well-covered graph is shellable. Therefore, G_k is a shellable graph and hence, [20, Theorem 8.2.5] and Lemma 3.4 imply that $(J(G)^{(k)})^{\rm pol} = J(G_k)$ has linear quotients. We now conclude from Lemma 3.5 that $J(G)^{(k)}$ has linear quotients. □

We do not know whether the converse of the above theorem is true. However, in the following corollary, we prove that the converse of Theorem 3.6 is true for bipartite graphs. As mentioned in the introduction, the "only if" part of the following corollary is already known by [28, Theorem 2.2]. However, we provide an alternative proof using Theorem 3.6.

Corollary 3.7. *Let G be a bipartite graph and $k \geq 1$ be an integer. Then $J(G)$ has a linear resolution if and only if $J(G)^k$ has a linear resolution.*

Proof. Without loss of generality, we assume that G has no isolated vertex. Note that by [16, Corollary 2.6], the symbolic and the ordinary powers of cover ideals of bipartite graphs coincide. If $J(G)$ has a linear resolution, then it follows from [20, Theorem 8.1.9] that G is Cohen–Macaulay. In particular, G is unmixed and hence it is very well-covered. Therefore, the "only if" part follows from Theorem 3.6. To prove the "if" part assume that $V(G) = X \cup Y$ is a bipartition for the vertex set of G. Suppose that $X = \{x_1, \ldots, x_s\}$ and $Y = \{y_1, \ldots, y_t\}$. Clearly, we can assume that $k \geq 2$. It follows from [20, Corollary 1.6.3] and Lemma 3.4 that $(J(G)^k)^{\rm pol} = J(G_k)$ has a linear resolution. Then [20, Theorem 8.1.9] implies that G_k is a Cohen–Macaulay graph. Notice that the set

$$F = \{x_{i,j} \mid 1 \leq i \leq s \text{ and } 2 \leq j \leq k\}$$

is an independent subset of vertices of G_k. Since G_k has no isolated vertex, one can easily check that

$$N_{G_k}[F] = F \cup \{y_{i,j} \mid 1 \leq i \leq t \text{ and } 1 \leq j \leq k-1\}.$$

Thus $G_k \setminus N_{G_k}[F]$ is isomorphic to G. This means that ${\rm lk}_{\Delta(G_k)} F = \Delta(G)$. Since G_k is Cohen–Macaulay, it follows that G is Cohen–Macaulay too. Hence, [20, Theorem 8.1.9] implies that $J(G)$ has a linear resolution. □

Let M be a finitely generated $\mathbb{Z}^n$-graded S-module. Let $u \in M$ be a homogeneous element and $Z \subseteq \{x_1, \ldots, x_n\}$. The $\mathbb{K}$-subspace $u\mathbb{K}[Z]$ generated by all elements uv with $v \in \mathbb{K}[Z]$ is called a *Stanley space* of dimension $|Z|$, if it is a free $\mathbb{K}[Z]$-module. Here, as usual, $|Z|$ denotes the number of elements of Z. A decomposition $\mathcal{D}$ of M as a finite direct sum of Stanley spaces is called a *Stanley decomposition* of M. The minimum dimension of a Stanley space in $\mathcal{D}$ is called the *Stanley depth* of $\mathcal{D}$ and is denoted by ${\rm sdepth}(\mathcal{D})$. The quantity

$${\rm sdepth}(M) := \max\big\{{\rm sdepth}(\mathcal{D}) \mid \mathcal{D} \text{ is a Stanley decomposition of } M\big\}$$

is called the *Stanley depth* of M. We say that a $\mathbb{Z}^n$-graded S-module M satisfies *Stanley's inequality* if

$${\rm depth}(M) \leq {\rm sdepth}(M).$$

In fact, Stanley [36] conjectured that every $\mathbb{Z}^n$-graded S-module satisfies Stanley's inequality. This conjecture has been recently disproved in [11]. However, it is still interesting to find the classes of $\mathbb{Z}^n$-graded S-modules which satisfy Stanley's inequality. For a reader's friendly introduction to Stanley depth, we refer to [31] and for a nice survey on this topic, we refer to [19]. In [32, Corollary 3.4], the author proves that for a bipartite graph G, the modules $J(G)^k$ and $S/J(G)^k$ satisfy Stanley's inequality for every integer $k \gg 0$. In the following corollary, we prove Stanley's inequality for every symbolic power of the cover ideal of Cohen–Macaulay very well-covered graphs.

Corollary 3.8. *Let G be a very well-covered graph and suppose that its cover ideal $J(G)$ has a linear resolution. Then $J(G)^{(k)}$ and $S/J(G)^k$ satisfy Stanley's inequality, for every integer $k \geq 1$.*

Proof. By Theorem 3.6, we know that $J(G)^{(k)}$ has linear quotients. On the other hand, it is known [34] that Stanley's inequality holds true for every monomial ideal with linear quotients. Thus, $J(G)^{(k)}$ satisfies Stanley's inequality.

To prove that $S/J(G)^k$ satisfies Stanley's inequality, by [24, Corollary 4.5], it is sufficient to prove that $T/(J(G)^{(k)})^{\rm pol}$ satisfies Stanley's inequality (where T is the new polynomial ring). By assumption and [20, Theorem 8.1.9], we conclude that G is a Cohen–Macaulay graph. Again, we can assume that G has no isolated vertex. Thus, Proposition 3.1 implies that G_k is a Cohen–Macaulay very well-covered graph. It then follows from [26, Theorem 1.1] (see also [7, Theorem 2.3]) that G_k is a vertex decomposable graph. Now, It follows from [33] that $T/J(G_k)$ satisfies Stanley's inequality. Finally, the assertion follows from Lemma 3.4. □

4. Bipartite graphs

In this section, we determine a linear upper bound for the regularity of powers of cover ideals of bipartite graphs. For a monomial ideal I, let $\deg(I)$ denote the maximum of the degrees of the elements of $\mathrm{Gens}(I)$. Thus in particular, $\deg(J(G))$ is the cardinality of the largest minimal vertex cover of the graph G. It is clear that for every integer $k \geq 1$ and every graph G, the regularity of $S/J(G)^k$ is at least $k\deg(J(G)) - 1$. In Theorem 4.3, we prove that the regularity of $S/J(G)^k$ cannot be much larger than $k\deg(J(G)) - 1$, when G is a bipartite graph. We first need the following simple lemma.

Lemma 4.1. *Assume that $I \subseteq S$ is a monomial ideal. Let $S' = \mathbb{K}[x_2, \dots, x_n]$ be the polynomial ring obtained from S by deleting the variable x_1 and set $I' = I \cap S'$. Then $\mathrm{reg}(I') \leq \mathrm{reg}(I)$.*

Proof. Using polarization, we can assume that I is a squarefree monomial ideal. Then the assertion follows immediately from [17, Lemma 3.1]. □

The following lemma is a consequence of Lemma 4.1.

Lemma 4.2. *Let I be a monomial ideal of S. Then for every monomial $u \in S$, we have $\mathrm{reg}(S/(I : u)) \leq \mathrm{reg}(S/I)$.*

Proof. Clearly, we can assume that u is a variable, say $u = x_1$. By applying [30, Corollary 18.7] on the exact sequence

$$0 \longrightarrow S/(I : x_1)(-1) \longrightarrow S/I \longrightarrow S/(I, x_1) \longrightarrow 0,$$

we obtain that

$$\mathrm{reg}(S/(I : x_1)) + 1 \leq \max\{\mathrm{reg}(S/I), \mathrm{reg}(S/(I, x_1)) + 1\} \leq \mathrm{reg}(S/I) + 1,$$

where the last inequality follows from Lemma 4.1. □

We are now ready to prove the main result of this section. Let G be a bipartite graph with cover ideal $J(G)$ and assume that $k \geq 1$ is an integer. In the following theorem, we determine an interval of length $\mathrm{reg}(S/J(G))$, which contains the regularity of $S/J(G)^k$.

Theorem 4.3. *Let G be a bipartite graph with n vertices. Then for every integer $k \geq 1$, we have*

$$k\deg(J(G)) - 1 \leq \mathrm{reg}(S/J(G)^k) \leq k\deg(J(G)) + \mathrm{reg}(S/J(G)) - 1.$$

Proof. The first inequality is trivial, because $\deg(J(G)^k) = k\deg(J(G))$. Thus, we prove the second inequality. Assume that $V(G) = U \cup W$ is a bipartition for the vertex set of G. Without loss of generality, we may assume that $U = \{x_1, \ldots, x_t\}$ and $W = \{x_{t+1}, \ldots, x_n\}$, for some integer t with $1 \leq t \leq n$. Let m be the number of edges of G. We prove the assertions by induction on $m + k$. Clearly, we can assume that G has no isolated vertex.

There is nothing to prove for $k = 1$. If $m = 1$, then $J(G) = (x_1, y_1)$. Hence $\deg(J(G)) = 1$ and $\mathrm{reg}(S/J(G)^k) = k - 1$. Thus, the desired inequality is true for $m = 1$. Therefore, assume that $k, m \geq 2$. Let $S_1 = \mathbb{K}[x_2, \ldots, x_n]$ be the polynomial ring obtained from S by deleting the variable x_1 and consider the ideals $J_1 = J(G)^k \cap S_1$ and $J_1' = (J(G)^k : x_1)$. It follows from [10, Lemma 2.10] that

$$\mathrm{reg}(S/J(G)^k) \leq \max\{\mathrm{reg}_{S_1}(S_1/J_1), \mathrm{reg}_S(S/J_1') + 1\}.$$

Notice that $J_1 = (J(G) \cap S_1)^k$. Hence, by Lemma [32, Lemma 2.2], there exists a monomial $u_1 \in S_1$ with $\deg(u_1) = \deg_G(x_1)$ such that

$$J(G) \cap S_1 = u_1 J(G \setminus N_G[x_1]) S_1$$

and thus, $J_1 = u_1^k J(G \setminus N_G[x_1])^k S_1$. Notice that if C is a minimal vertex cover of $G \setminus N_G[x_1]$, then $C \cup N_G(x_1)$ is a minimal vertex cover of G. This shows that $\deg(J(G \setminus N_G[x_1])) + \deg_G(x_1) \leq \deg(J(G))$. On the other hand, Lemma 4.1 implies that $\mathrm{reg}(J(G) \cap S_1) \leq \mathrm{reg}(J(G))$ and therefore,

$$\mathrm{reg}_{S_1}(S_1/J(G \setminus N_G[x_1])S_1) \leq \mathrm{reg}(S/J(G)) - \deg(u_1).$$

Hence, by the induction hypothesis we conclude that

$$\begin{aligned}
&\mathrm{reg}_{S_1}(S_1/J_1) = \mathrm{reg}_{S_1}(S_1/J(G \setminus N_G[x_1])^k S_1) + k\deg(u_1)\\
&\leq k\deg(J(G \setminus N_G[x_1])) + \mathrm{reg}(S/J(G \setminus N_G[x_1])) - 1 + k\deg_G(x_1)\\
&\leq k(\deg(J(G)) - \deg_G(x_1)) + \mathrm{reg}(S/J(G)) - \deg_G(x_1) - 1 + k\deg_G(x_1)\\
&\leq k\deg(J(G)) + \mathrm{reg}(S/J(G)) - 1.
\end{aligned}$$

Thus, using the inequality (4), it is enough to prove that

$$\mathrm{reg}_S(S/J_1') \leq k\deg(J(G)) + \mathrm{reg}(S/J(G)) - 2.$$

For every integer i with $2 \leq i \leq t$, let $S_i = \mathbb{K}[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]$ be the polynomial ring obtained from S by deleting the variable x_i and consider the ideals $J_i' = (J_{i-1}' : x_i)$ and $J_i = J_{i-1}' \cap S_i$. □

Claim. For every integer i with $1 \leq i \leq t - 1$ we have

$$\mathrm{reg}(S/J_i') \leq \max\{k\deg(J(G)) + \mathrm{reg}(S/J(G)) - 2, \mathrm{reg}_S(S/J_{i+1}') + 1\}.$$

Proof of the claim. For every integer i with $1 \leq i \leq t-1$, we know from [10, Lemma 2.10] that

$$(*) \qquad \mathrm{reg}(S/J_i') \leq \max\{\mathrm{reg}_{S_{i+1}}(S_{i+1}/J_{i+1}), \mathrm{reg}_S(S/J_{i+1}') + 1\}.$$

Notice that for every integer i with $1 \leq i \leq t-1$, we have $J'_i = (J(G)^k : x_1x_2\dots x_i)$. Thus $J_{i+1} = J'_i \cap S_{i+1} = ((J(G)^k \cap S_{i+1}) :_{S_{i+1}} x_1x_2\dots x_i)$. Hence, it follows from Lemma 4.2 that

$$(**) \qquad \mathrm{reg}_{S_{i+1}}(S_{i+1}/J_{i+1}) \leq \mathrm{reg}_{S_{i+1}}(S_{i+1}/(J(G)^k \cap S_{i+1})).$$

By lemma [32, Lemma 2.2], we conclude that there exists a monomial $u_{i+1} \in S_{i+1}$, with $\deg(u_{i+1}) = \deg_G(x_{i+1})$ such that $J(G) \cap S_{i+1} = u_{i+1}J(G \backslash N_G[x_{i+1}])S_{i+1}$. Therefore

$$J(G)^k \cap S_{i+1} = u_{i+1}^k J(G \setminus N_G[x_{i+1}])^k S_{i+1}.$$

Notice that if C is a minimal vertex cover of $G \setminus N_G[x_{i+1}]$, then $C \cup N_G(x_{i+1})$ is a minimal vertex cover of G. This shows that $\deg(J(G \setminus N_G[x_{i+1}])) + \deg_G(x_{i+1}) \leq \deg(J(G))$. On the other hand, Lemma 4.1 implies that $\mathrm{reg}(J(G) \cap S_{i+1}) \leq \mathrm{reg}(J(G))$ and therefore,

$$\mathrm{reg}_{S_{i+1}}(S_{i+1}/J(G \setminus N_G[x_{i+1}])S_{i+1}) \leq \mathrm{reg}(S/J(G)) - \deg(u_{i+1}).$$

Hence, by the induction hypothesis we conclude that

$$\begin{aligned}
&\mathrm{reg}_{S_{i+1}}(S_{i+1}/(J(G)^k \cap S_{i+1})) = \mathrm{reg}_{S_{i+1}}(S_{i+1}/J(G \setminus N_G[x_{i+1}])^k S_{i+1}) + k\deg(u_{i+1})\\
&\leq k\deg(J(G \setminus N_G[x_{i+1}])) + \mathrm{reg}(S/J(G \setminus N_G[x_{i+1}])) - 1 + k\deg_G(x_{i+1})\\
&\leq k(\deg(J(G)) - \deg_G(x_{i+1})) + \mathrm{reg}(S/J(G)) - \deg_G(x_{i+1}) - 1 + k\deg_G(x_{i+1})\\
&\leq k\deg(J(G)) + \mathrm{reg}(S/J(G)) - 2.
\end{aligned}$$

Now the claim follows by inequalities $(*)$ and $(**)$.

Now, $J'_t = (J(G)^k : x_1x_2\dots x_t)$ and hence, [32, Lemma 3.2] implies that $J'_t = J(G)^{k-1}$ and thus, by the inductive hypothesis we conclude that $\mathrm{reg}(S/J'_t) \leq (k-1)\deg(J(G)) + \mathrm{reg}(S/J(G)) - 1$. Therefore, using the claim repeatedly, implies that

$$\begin{aligned}
&\mathrm{reg}(S/J'_1) \leq \max\{k\deg(J(G)) + \mathrm{reg}(S/J(G)) - 2, \mathrm{reg}_S(S/J'_t) + t - 1\}\\
&\leq \max\{k\deg(J(G)) + \mathrm{reg}(S/J(G)) - 2, (k-1)\deg(J(G)) + \mathrm{reg}(S/J(G)) + t - 2\}.
\end{aligned}$$

Note that $t = \mid U \mid \leq \deg(J(G))$. Thus, the above inequalities imply that $\mathrm{reg}(S/J'_1) \leq k\deg(J(G)) + \mathrm{reg}(S/J(G)) - 2$. This completes the proof of the theorem. □

Remark 4.4. It follows from Corollary 3.7 that if G is a bipartite graph and $J(G)$ has a linear resolution, then for every integer $k \geq 1$, we have

$$\mathrm{reg}(S/J(G)^k) = k\deg(J(G)) - 1.$$

This shows that the first inequality of Theorem 4.3 is sharp. Although the difference of the lower and the upper bound given by Theorem 4.3 is small, but we have no example of a bipartite graph G, for which there exists an integer $k \geq 2$ with

$$\mathrm{reg}(S/J(G)^k) = k\deg(J(G)) + \mathrm{reg}(S/J(G)) - 1.$$

In fact, we believe that the second inequality of Theorem 4.3 can still be improved.

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