

QUANTITATIVE QUASISYMMETRIC UNIFORMIZATION OF COMPACT SURFACES

LUKAS GEYER AND KEVIN WILDRICK

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ABSTRACT. Bonk and Kleiner showed that any metric sphere which is Ahlfors 2-regular and linearly locally contractible is quasimetrically equivalent to the standard sphere in a quantitative way. We extend this result to arbitrary metric compact orientable surfaces.

1. INTRODUCTION AND STATEMENT OF RESULTS

Through isothermal coordinates, every Riemannian metric on a compact orientable surface S determines a Riemann surface structure on S . By the classical uniformization theorem, S carries a conformally equivalent Riemannian metric of constant curvature 1 (in the case of a sphere), 0 (for a torus), or -1 (for higher genus surfaces). Hence, the original Riemannian metric on S can be *conformally deformed* to a metric of constant curvature.

The purpose of this note is to extend the above discussion to certain classes of possibly non-smooth distances on compact orientable surfaces. In this setting we have a metric, but no smooth structure, and the appropriate category replacing the class of conformal mappings is the class of *quasisymmetric mappings*.

In a metric space X we will denote the distance between a and b by $|a - b|_X$, or $|a - b|$ if the metric space X is clear from the context. Let X and Y be metric spaces. An embedding $\phi: X \rightarrow Y$, i.e., a homeomorphism from X onto its image $\phi(X)$, is a *quasisymmetric embedding* if there is a homeomorphism $\eta: [0, \infty) \rightarrow [0, \infty)$ such that for all triples of distinct points a, b , and c in X ,

$$\frac{|\phi(a) - \phi(b)|_Y}{|\phi(a) - \phi(c)|_Y} \leq \eta \left(\frac{|a - b|_X}{|a - c|_X} \right).$$

The homeomorphism η is called a *distortion function* for the mapping ϕ . If a quasisymmetric embedding ϕ has a distortion function η , it will be called an η -quasisymmetric embedding. A *quasisymmetric map* is a quasisymmetric homeomorphism, i.e., a quasisymmetric embedding $\phi: X \rightarrow Y$ with $\phi(X) = Y$. The inverse of an η -quasisymmetric map is η' -quasisymmetric with $\eta'(t) = 1/\eta^{-1}(1/t)$. A proof of this fact can be found in the book of Heinonen [Hei01, Proposition 10.6], which also serves as an excellent introduction to the theory of quasisymmetric mappings on metric spaces. A quasisymmetric mapping distorts *relative* distances by a controlled amount. While it may distort distances, it must do so more-or-less isotropically.

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A motivating example arises from Cannon's Conjecture in geometric group theory. The boundary at infinity of a Gromov hyperbolic group carries a natural family of visual metrics, any two of which are quasimetrically equivalent. For this reason, one might say that the metric on the boundary of a Gromov hyperbolic group is defined only up to quasimetricity. Cannon's Conjecture can be phrased as follows: If the boundary $\partial_\infty G$ of a Gromov hyperbolic group G is homeomorphic to the sphere \mathbb{S}^2 , then each visual metric on $\partial_\infty G$ is quasimetrically equivalent to the standard metric on \mathbb{S}^2 . If Cannon's conjecture is true, then a Gromov hyperbolic group whose boundary is homeomorphic to \mathbb{S}^2 acts discretely and cocompactly by isometries on 3-dimensional hyperbolic space. See [Bon06] and the references therein for a more detailed discussion of this problem.

The following quasimetric uniformization theorem of Bonk and Kleiner [BK02] led to significant progress on Cannon's Conjecture:

Theorem 1 (Bonk-Kleiner). *Let (X, d) be an Ahlfors 2-regular metric space that is homeomorphic to \mathbb{S}^2 . Then (X, d) is quasimetrically equivalent to \mathbb{S}^2 equipped with the smooth metric of constant curvature 1 if and only if (X, d) is linearly locally contractible.*

A metric space X is *linearly locally contractible* if there is a constant $\Lambda \geq 1$ such that for each point $x \in X$ and radius $0 < r < (\text{diam } X)/\Lambda$, the ball $B(x, r)$ is contractible inside the dilated ball $B(x, \Lambda r)$. The metric space X is *Ahlfors 2-regular* if there is a constant $K \geq 1$ such that for each $x \in X$ and radius $0 < r < 2 \text{ diam } X$, the two-dimensional Hausdorff measure \mathcal{H}^2 satisfies

$$(1) \quad \frac{r^2}{K} \leq \mathcal{H}^2(B(x, r)) \leq Kr^2.$$

Given a Gromov-hyperbolic group G with $\partial_\infty G$ homeomorphic to \mathbb{S}^2 , each visual metric on $\partial_\infty G$ is linearly locally contractible. However, it is not known whether there always is an Ahlfors 2-regular visual metric on $\partial_\infty G$. Simple examples show that Theorem 1 fails without the assumption of Ahlfors 2-regularity, although Ahlfors 2-regularity is not preserved by quasimetric mappings in general.

The linear local contractibility condition is a quantitative quasimetric invariant; i.e., if $f: X \rightarrow Y$ is η -quasimetric and X is linearly locally contractible with constant Λ , then Y is linearly locally contractible with a constant that depends only on Λ and η .

Theorem 1 is quantitative in the sense that the quasimetric map can be chosen to have a distortion depending only on the *data* of X , i.e., the constants Λ and K appearing in the linear local contractibility and Ahlfors 2-regularity conditions.

Problems outside geometric group theory, such as the search for an intrinsic characterization of \mathbb{R}^2 up to bi-Lipschitz equivalence (note that Ahlfors 2-regularity is a bi-Lipschitz invariant), led to the development of versions of Theorem 1 for noncompact simply connected surfaces [Wil08] and for a large class of planar domains [MW13], as well as a local version [Wil10]. In this work, we provide a version of Theorem 1 that applies to all orientable compact surfaces, including those of higher genus. This is part of an ongoing program to complete the classification of Ahlfors 2-regular and linearly locally contractible metric surfaces up to quasimetricity.

Theorem 2. *Let (X, d) be a metric compact orientable surface. Assume that (X, d) is Ahlfors 2-regular and linearly locally contractible. Then there exists a Riemannian metric \hat{d} of constant curvature 1, 0, or -1 on X such that the identity map $\text{id} : (X, d) \rightarrow (X, \hat{d})$ is quasimetric with a distortion function depending only on the data of X .*

Here the *data* of X are the constants in the Ahlfors 2-regularity and linear local contractibility conditions.

Except for the statement of dependence of the distortion function only on the data of (X, d) , Theorem 2 follows immediately from the local uniformization theorem of [Wil10] and an elementary local-to-global result for quasimetric mappings due to Tukia and Väisälä [TV80, Theorem 2.23]. A quantitative proof is significantly more involved.

We note that the data of (X, d) do *not* measure the “quasimetric distance” of (X, d) to a Riemannian metric of constant curvature. For each genus $g \geq 1$ and any $\Lambda \geq 1$, there is a compact Riemannian surface of genus g and constant curvature that fails to be Λ -linearly locally contractible; this occurs when the injectivity radius is sufficiently small compared to the diameter. Note also that as the genus grows, the ratio of the injectivity radius to the diameter necessarily tends to 0.

This indicates a very interesting, though vague, question: given (X, d) as in Theorem 2, find a Riemannian metric \hat{d} on X and a quasimetric mapping $f : (X, d) \rightarrow (X, \hat{d})$ with “minimal” quasimetric distortion over all such metrics and mappings.

One might hope to improve Theorem 2 by separating the data into parts which control the *scale* and *severity* of the conditions separately. Suppose that (X, d) is a metric compact orientable surface with constants $C_1, C_2 \geq 1$ so that for all $x \in X$ and $0 < r < (\text{diam } X)/C_1$, the ball $B(x, r)$ is contractible inside $B(x, C_2r)$ and satisfies (1) with K replaced by C_2 . Unfortunately, Theorem 2 does not hold if the data of X is defined to be the single number C_2 ; a counter-example is provided by a sequence of smooth dumbbell-shaped spheres with narrowing “bars”; by allowing C_1 to grow, we may choose C_2 to be constant for the entire sequence. Despite this, the quasimetric distortion of mappings to a sphere of constant curvature must grow. A similar construction can be made in any genus.

Our proof of Theorem 2 proceeds as follows. Let (X, d) be a metric surface as given in the theorem. By the local uniformization result of [Wil10], (X, d) has an atlas of uniformly quasimetric mappings. Our first step is to create a compatible conformal atlas, giving (X, d) the structure of a Riemann surface. This step can be thought of as creating “quasi-isothermal” coordinates; while the chart transitions are conformal, the chart mappings themselves are only quasimetrics. The classical uniformization theorem then provides a globally defined conformal homeomorphism $F : X \rightarrow Y$ to a Riemann surface that is the quotient of the sphere, plane, or disk by an appropriate group of Möbius transformations. The Riemann surface Y inherits a Riemannian metric d_Y of constant curvature 1, 0, or -1 , respectively. Although conformal maps are locally quasimetric, this does not immediately give us global information about the metric properties of the uniformizing map $F : (X, d) \rightarrow (Y, d_Y)$. We show that the uniformizing mapping F is in fact globally quasimetric with a distortion function that depends only on the data of X . The key idea in this step is a type of Harnack inequality. The claim

in the theorem then follows by taking \hat{d} to be the pull-back of the metric d_Y under F .

2. BACKGROUND AND PRELIMINARY RESULTS

2.1. **Notation.** In a metric space X , we will denote the distance between points x and y of X by $|x - y|_X$ or $|x - y|$ when the space X is understood. We denote by

$$B_X(x, r) = B(x, r) = \{y : |x - y| < r\}$$

the open ball of radius r centered at x and by $\overline{B}(x, r)$ the corresponding closed ball. For an open or closed ball B of radius r and a number $\lambda > 0$, the notation λB denotes the same type of ball with the same center and radius λr .

We denote the complex plane with the standard Euclidean metric by \mathbb{C} and the disk model of hyperbolic space equipped with the standard hyperbolic metric of curvature -1 by \mathbb{D} . The open ball of radius $\alpha > 0$ centered at 0 is denoted by \mathbb{C}_α in \mathbb{C} and by \mathbb{D}_α in \mathbb{D} , i.e.,

$$\mathbb{C}_\alpha = B_{\mathbb{C}}(0, \alpha) \text{ and } \mathbb{D}_\alpha = B_{\mathbb{D}}(0, \alpha).$$

Note that α is the Euclidean radius for \mathbb{C}_α , whereas it is the hyperbolic radius for \mathbb{D}_α .

2.2. **Quasisymmetric mappings.** Let $\phi: X \rightarrow Y$ be an embedding of metric spaces. For $x \in X$ and $0 < r < \frac{1}{2} \text{diam } X$, define

$$L_\phi(x, r) := \sup\{|\phi(x) - \phi(y)|_Y : |x - y|_X \leq r\} \text{ and}$$

$$l_\phi(x, r) := \inf\{|\phi(x) - \phi(y)|_Y : |x - y|_X \geq r\}.$$

If ϕ is an η -quasisymmetric embedding, then for $x \in X$ and $0 < r_1, r_2 < \frac{1}{2} \text{diam } X$,

$$\frac{L_\phi(x, r_1)}{l_\phi(x, r_2)} \leq \eta \left(\frac{r_1}{r_2} \right).$$

We will need a statement about the equicontinuity of quasisymmetric maps, which is a slightly more general version of [Hei01, Corollary 10.27].

Lemma 3. *Let $\phi: U \rightarrow V$ be an η -quasisymmetric embedding of metric spaces and let D_U and D_V be positive real numbers. If $\text{diam } U \geq D_U$ and $\text{diam } V \leq D_V$, then*

$$\omega(t) = \max \left\{ \frac{3D_V t}{D_U}, D_V \eta \left(\frac{3t}{D_U} \right) \right\}$$

is a modulus of continuity for ϕ .

Proof. Given $x_1, x_2 \in U$ with $0 < |x_1 - x_2| < D_U/3$, there exists $x_3 \in U$ such that $|x_1 - x_3| \geq D_U/3$. Then

$$|\phi(x_1) - \phi(x_2)| \leq |\phi(x_1) - \phi(x_3)| \eta \left(\frac{|x_1 - x_2|}{|x_1 - x_3|} \right) \leq D_V \eta \left(\frac{3|x_1 - x_2|}{D_U} \right).$$

If $|x_1 - x_2| \geq D_U/3$, the trivial estimate $|\phi(x_1) - \phi(x_2)| \leq D_V$ yields the desired estimate. □

Note that the modulus of continuity provided by Lemma 3 is not scale-invariant. If the metrics on U and V are scaled by the same quantity, the resulting modulus of continuity may still change; see subsection 2.4.

The following result, which is a slight variation of [TV80, Theorem 2.23], gives a local-to-global result for quasisymmetric homeomorphisms between compact spaces

in terms of Lebesgue numbers. We include a proof for the reader's convenience. Recall that $L > 0$ is a *Lebesgue number* for a covering $\{X_j\}$ if for every set E with $\text{diam } E \leq L$ there exists j such that $E \subseteq X_j$.

Theorem 4 (Tukia-Väisälä). *Let $F: X \rightarrow Y$ be a homeomorphism between compact connected metric spaces X and Y . Suppose that*

- $\{X_j\}_{j=1}^n$ is a finite open covering of X with Lebesgue number $L_X < \text{diam } X$.
- $\delta > 0$ satisfies the implication

$$|x - x'| = L_X/2 \implies |F(x) - F(x')| \geq \delta,$$

- there is a quasisymmetric distortion function η such that for each $j = 1, \dots, n$, the restricted mapping $F|_{X_j}$ is η -quasisymmetric.

Then F is quasisymmetric with distortion function depending only on η and the ratios $\text{diam}(X)/L_X$ and $\text{diam}(Y)/\delta$.

Proof. Let a, b , and c be distinct points of X . Set

$$\rho = \frac{|a - b|}{|a - c|} \text{ and } \rho' = \frac{|F(a) - F(b)|}{|F(a) - F(c)|}.$$

We consider four cases.

- (1) Suppose that $\max\{|a - b|, |a - c|\} \leq L_X/2$. Then $\{a, b, c\}$ is contained in some X_j , and so $\rho' \leq \eta(\rho)$.
- (2) Suppose that $|a - b| \leq L_X/2$ but $|a - c| > L_X/2$. Since X is connected with $\text{diam } X > L_X$, there exists $x \in X$ with $|a - x| = L_X/2$. Then there exists j such that $\{a, b, x\} \subseteq X_j$ and

$$|F(a) - F(x)| \geq \delta \text{ and } \frac{|a - b|}{|a - x|} \leq \frac{2\rho \text{ diam } X}{L_X}.$$

This implies that

$$\rho' = \frac{|F(a) - F(b)|}{|F(a) - F(x)|} \frac{|F(a) - F(x)|}{|F(a) - F(c)|} \leq \eta \left(\frac{2\rho \text{ diam } X}{L_X} \right) \frac{\text{diam } Y}{\delta}.$$

- (3) Suppose that $|a - b| > L_X/2$ but $|a - c| \leq L_X/2$. By the same argument as in the previous case, there exists $x \in X$ with $|a - x| = L_X/2$, as well as an index j such that $\{a, c, x\} \subseteq X_j$. This gives

$$|F(a) - F(x)| \geq \delta \text{ and } \frac{|a - x|}{|a - c|} \leq \frac{2\rho \text{ diam } X}{L_X},$$

which implies that

$$\rho' = \frac{|F(a) - F(b)|}{|F(a) - F(x)|} \frac{|F(a) - F(x)|}{|F(a) - F(c)|} \leq \frac{\text{diam } Y}{\delta} \eta \left(\frac{2\rho \text{ diam } X}{L_X} \right).$$

- (4) Suppose that $\min\{|a - b|, |a - c|\} > L_X/2$. Then $\rho \geq \frac{L_X}{2 \text{ diam } X}$ and $\rho' \leq \frac{\text{diam } Y}{\delta}$, so

$$\rho' \leq 2 \left(\frac{\text{diam } X}{L_X} \right) \left(\frac{\text{diam } Y}{\delta} \right) \rho.$$

Combining these estimates on ρ' in terms of ρ yields the desired result. □

2.3. Conformality and quasisymmetry. A conformal mapping $f: U \rightarrow V$ between domains in \mathbb{C} is quasisymmetric when restricted to a relatively compact subdomain $U' \subset U$, with quasisymmetric distortion depending only on U and U' . This fact, which should be compared to Koebe’s distortion theorem, will play a key role in the proof of Theorem 12. A more general statement is even true: one may consider quasiconformal mappings in higher dimensional Euclidean spaces, although the quasisymmetric distortion then also depends on the maximal quasiconformal dilatation. See [Väi81, Theorem 2.4] for a proof of this fact, which we will use in the following form:

Proposition 5. *Let $G: \mathbb{C}_1 \rightarrow \mathbb{C}$ be a conformal embedding. For each $\beta \in (0, 1)$, the restriction $G: \mathbb{C}_\beta \rightarrow \mathbb{C}$ is a quasisymmetric embedding with distortion function that depends only on β .*

An easy consequence of Proposition 5 is the following.

Corollary 6. *There is a universal constant $0 < \beta_1 < 1/2$ such that for any conformal embedding $G: \mathbb{C}_1 \rightarrow \mathbb{C}$,*

$$G(\mathbb{C}_{\beta_1}) \subseteq B_{\mathbb{C}} \left(G(0), \frac{l_G(0, 1/2)}{6} \right).$$

Proof. By Proposition 5, there is a universal quasisymmetric distortion function η for $G: \mathbb{C}_{1/2} \rightarrow \mathbb{C}$. Hence, if $\beta \in (0, 1/2)$, then

$$L_G(0, \beta) \leq \eta(2\beta) l_G(0, 1/2).$$

Thus, choosing $0 < \beta_1 < 1/2$ so small that $\eta(2\beta_1) < 1/6$ fulfills the requirements of the statement. □

We will also need a hyperbolic version of this result. Recall that the hyperbolic disk \mathbb{D} is equipped not just with a conformal structure but also with the standard hyperbolic metric. Our proof employs Koebe’s distortion theorem for specificity, but could also be carried out using Proposition 5 alone.

Proposition 7. *Let $G: \mathbb{C}_1 \rightarrow \mathbb{D}$ be a conformal embedding of the Euclidean unit disk into the hyperbolic plane. Then the restriction $G: \mathbb{C}_{1/10} \rightarrow \mathbb{D}$ is a quasisymmetric embedding with a universal distortion function.*

Proof. The map $f(z) = \frac{G(z)-G(0)}{G'(0)} = z + O(z^2)$ is conformal in \mathbb{C}_1 , so by Koebe’s distortion theorem [Dur83], the image $f(\mathbb{C}_1)$ contains the Euclidean disk $\mathbb{C}_{1/4}$, and the image of the Euclidean disk $\mathbb{C}_{1/10}$ is contained in the Euclidean disk $\mathbb{C}_{10/81} \subseteq \mathbb{C}_{1/8}$. This implies that

$$B_{\mathbb{C}} \left(G(0), \frac{|G'(0)|}{4} \right) \subseteq G(\mathbb{C}_1) \subseteq \mathbb{D}$$

and

$$G(\mathbb{C}_{1/10}) \subseteq B_{\mathbb{C}} \left(G(0), \frac{|G'(0)|}{8} \right) \subseteq B_{\mathbb{C}} \left(G(0), \frac{1 - |G(0)|}{2} \right).$$

It follows from the definition of the hyperbolic metric that the identity mapping

$$\text{id}: B_{\mathbb{C}} \left(G(0), \frac{1 - |G(0)|}{2} \right) \rightarrow \mathbb{D}$$

is, up to scaling, a bi-Lipschitz embedding with universal constants, so it is quasisymmetric with universal distortion. Since the composition of quasisymmetric maps is quasisymmetric, quantitatively, the result follows from Proposition 5. \square

The following statement is an easy corollary of Proposition 7.

Corollary 8. *There is a universal constant $0 < \beta_2 < 1/10$ such that for any conformal embedding $G: \mathbb{C}_1 \rightarrow \mathbb{D}$,*

$$G(\mathbb{C}_{\beta_2}) \subseteq B_{\mathbb{D}} \left(G(0), \frac{l_G(0, 1/2)}{6} \right).$$

Proof. By Proposition 7, there is a universal quasisymmetric distortion function η for $G: \mathbb{C}(1/10) \rightarrow \mathbb{D}$. Hence, if $\beta \in (0, 1/10)$, then

$$L_G(0, \beta) \leq \eta(10\beta) l_G(0, 1/10) \leq \eta(10\beta) l_G(0, 1/2).$$

Choosing β_2 so small that $\eta(10\beta_2) < 1/6$, we get the claim of the corollary. \square

For the remainder of the paper, for convenience we define $\beta = \min\{\beta_1, \beta_2\}$.

2.4. The data of X , scalings, and normalization. Let X be a metric space that is Ahlfors 2-regular with constant K , linearly locally contractible with constant Λ , and homeomorphic to a compact orientable surface. We will refer to Λ and K as the *data* of X .

For $\lambda > 0$, we may form a new metric space X_λ by multiplying the original metric on X by λ . Then X_λ is again Ahlfors 2-regular and linearly locally contractible, and X_λ has the same data as X . Moreover, the identity mapping from X to X_λ is η -quasisymmetric with $\eta(t) = t$. Hence, in the proof of Theorem 2, we may scale the domain as we see fit.

Convention. For the remainder of this article, let X be a metric space that is Ahlfors 2-regular with constant K , linearly locally contractible with constant Λ , homeomorphic to a compact orientable surface, and with $\text{diam } X = 1$. When we state that a quantity depends only the data of X , we are implicitly assuming this normalization.

The main reason for this convention (aside from notational convenience) is that Lemma 3 is not scale-invariant. Without this normalization, we would not be able to say that certain moduli of continuity depend only on the data of X .

3. FINDING A CONFORMAL STRUCTURE

By [Wil10, Theorem 4.1], X possesses a quasisymmetric atlas:

Theorem 9. *There is a quantity $A_0 \geq 1$ and a quasisymmetric distortion function η , each depending only on the data of X , such that for each $0 < R \leq 1/A_0$ there is a neighborhood U of x such that*

- (1) $B(x_0, R/A_0) \subseteq U \subseteq B(x_0, A_0R)$,
- (2) *there exists an η -quasisymmetric map $\phi: U \rightarrow \mathbb{C}_1$ with $\phi(x_0) = 0$.*

The original theorem in [Wil10] does not include the normalization $\phi(x_0) = 0$. However, since $\phi: U \rightarrow \mathbb{C}_1$ is η -quasisymmetric, the basic distortion estimates [Hei01, Proposition 10.8] imply that $\phi(x_0) \in \mathbb{C}_\alpha$ where $\alpha < 1$ depends only on η . As the Möbius transformation $T(z) = \frac{z - \phi(x_0)}{1 - \overline{\phi(x_0)}z}$ has quasisymmetric distortion depending only on $|\phi(x_0)|$, we may assume that $\phi(x_0) = 0$. Accordingly, given a

pair (U, ϕ) where $\phi: U \rightarrow \mathbb{C}_1$ is a homeomorphism, we will call $\phi^{-1}(0)$ the *center* of (U, ϕ) .

We use the atlas provided by Theorem 9 to produce a conformal atlas on X that is adapted to its metric. We have separated the construction into two lemmas. The first is purely metric; the second modifies the output of the first.

Lemma 10. *Let $\rho \in (0, 1)$ be given. Then there exists a quasisymmetric distortion function η , radii $\alpha, r_0 > 0$, and a positive integer $n \in \mathbb{N}$, such that the following statements hold:*

- (1) *There exists an atlas $\mathcal{A}_\rho = \{(U_j, \mathbb{C}_1, \phi_j) \mid j = 1, \dots, n\}$ of X , where each mapping $\phi_j: U_j \rightarrow \mathbb{C}_1$ is an η -quasisymmetric homeomorphism with center denoted by x_j .*
- (2) *The collection $\{B(x_j, r_0)\}_{j=1}^n$ is pairwise disjoint.*
- (3) *The collection $\{B(x_j, 2r_0)\}_{j=1}^n$ covers X .*
- (4) *For each $j = 1, \dots, n$, it holds that $B(x_j, 10r_0) \subseteq U_j$ and*

$$\mathbb{C}_\alpha \subseteq \phi_j(B(x_j, r_0)) \subseteq \phi_j(B(x_j, 10r_0)) \subseteq \mathbb{C}_\rho.$$

Moreover, η depends only on the data of X , while α, r_0 , and n depend only on the data of X and ρ .

Note that the collection $\{B(x_j, 10r_0)\}_{j=1}^n$ forms an open cover of X for which $8r_0$ is a Lebesgue number; cf. Theorem 4. Moreover, Lemma 3 implies that for each $j = 1, \dots, n$, the restriction $\phi_j|_{B(x_j, 10r_0)}$ and its inverse have moduli of continuity that depend only on ρ and the data of X .

Proof. For each $x \in X$, let $\phi_x: U_x \rightarrow \mathbb{C}_1$ be the η -quasisymmetric mapping provided by Theorem 9 with $R = 1/A_0$, so that

$$B(x, 1/A_0^2) \subseteq U_x \subseteq B(x, 1).$$

Applying Lemma 3 to ϕ_x and its inverse, we see that there is a common modulus of continuity ω for all of the mappings $\{\phi_x, \phi_x^{-1}\}_{x \in X}$, depending only on the data of X .

Hence, there is a radius $0 < r_0 < 1/10A_0^2$ and a number $0 < \alpha < \rho$, each depending only on ρ and the data of X , such that for each $x \in X$,

$$(2) \quad \phi_x(B(x, 10r_0)) \subseteq \mathbb{C}_\rho \text{ and } \phi_x(B(x, r_0)) \supseteq \mathbb{C}_\alpha.$$

Let $\{x_1, \dots, x_n\}$ be a maximal $2r_0$ -separated set in X . Then the open balls $\{B(x_j, r_0)\}_{j=1}^n$ are pairwise disjoint, while the open balls $\{B(x_j, 2r_0)\}_{j=1}^n$ cover X . Since X is Ahlfors 2-regular and we have assumed that $\text{diam } X = 1$, this implies that n is comparable to r_0^{-2} and therefore depends only on ρ and the data. Moreover, as $10r_0 < 1/(A_0)^2$, for each $j = 1, \dots, n$, it holds that $B(x_j, 10r_0) \subseteq U_{x_j}$. In particular $\{U_{x_j}\}_{j=1}^n$ is also a cover of X . This shows that $\{(U_{x_j}, \mathbb{C}_1, \phi_{x_j})\}_{j=1}^n$ is the desired atlas. □

We now adapt the atlas given in Lemma 10 so that the transition mappings are conformal. This step is similar to the proof that a quasiconformal structure on a surface has a compatible conformal structure; see [Kuu67] and [Can69].

We recall that $\beta \in (0, 1)$ is a universal constant given by Corollaries 6 and 8.

Lemma 11. *There exists a quasisymmetric distortion function η , radii $\alpha, r_0 > 0$, and a positive integer $n \in \mathbb{N}$, all depending only on the data of X , such that the following statements hold:*

- (1) *There exists an atlas $\mathcal{B} = \{(U_j, \mathbb{C}_1, \psi_j)\}_{j=1}^n$ of X , where each mapping $\psi_j: U_j \rightarrow \mathbb{C}_1$ is an η -quasisymmetric homeomorphism with center denoted by x_j .*
- (2) *The collection $\{B(x_j, r_0)\}_{j=1}^n$ is pairwise disjoint.*
- (3) *The collection $\{B(x_j, 2r_0)\}_{j=1}^n$ covers X .*
- (4) *For each $j = 1, \dots, n$, it holds that $B(x_j, 10r_0) \subseteq U_j$ and*

$$\mathbb{C}_\alpha \subseteq \psi_j(B(x_j, r_0)) \subseteq \psi_j(B(x_j, 10r_0)) \subseteq \mathbb{C}_\beta.$$

- (5) *The transition maps $\psi_j \circ \psi_k^{-1}$ are conformal wherever defined.*

Proof. We begin by letting $0 < \rho < 1$ be a number which will be determined below and will depend only on the data of X . Consider the atlas \mathcal{A}_ρ provided by Lemma 10. Let us say that this atlas is given by η' -quasisymmetric charts $\{\phi_j: U_j \rightarrow \mathbb{C}_1\}_{j=1}^n$ where η' depends only on the data of X . The associated radii α', r'_0 and the number of charts n depend only on the data of X and ρ .

Since X is connected, the charts can be relabeled to satisfy

$$U_{j+1} \cap (U_1 \cup \dots \cup U_j) \neq \emptyset$$

for $j = 1, \dots, n - 1$.

Define $\psi_1: U_1 \rightarrow \mathbb{C}_1$ by setting $\psi_1 = \phi_1$. We will iteratively construct $\psi_j: U_j \rightarrow \mathbb{C}_1$ for $j = 2, \dots, n$ as follows. For $k < j$, we assume that the already constructed map ψ_k is quasisymmetric on U_k , has center x_k , and that if $k, k' < j$, then the transition functions $\psi_k \circ \psi_{k'}^{-1}$ are conformal where defined.

In the following, we will write $D_{a,b} := \phi_a(U_a \cap U_b)$ for indices $a > b$ in $\{1, \dots, n\}$.

For $k < j$, define $T_{j,k} = \psi_k \circ \phi_j^{-1}$. Then $T_{j,k}$ is quasisymmetric and hence quasiconformal on $D_{j,k}$. Therefore, the complex dilatation $\mu_{j,k}$ of $T_{j,k}$ is well-defined (up to a.e. equivalence) on $D_{j,k}$. Given another index $k' < j$, it holds that $T_{j,k} = (\psi_k \circ \psi_{k'}^{-1}) \circ T_{j,k'}$ on $D_{j,k} \cap D_{j,k'}$. It therefore follows from the conformality of $\psi_k \circ \psi_{k'}^{-1}$ that $\mu_{j,k} = \mu_{j,k'}$ a.e. on the intersection $D_{j,k} \cap D_{j,k'}$. This shows that there exists a measurable Beltrami coefficient $\mu_j: \mathbb{C}_1 \rightarrow \mathbb{C}_1$ with $\mu_j = \mu_{j,k}$ a.e. on $D_{j,k}$ for each index $1 \leq k < j$, and $\mu_j = 0$ on $\mathbb{C}_1 \setminus \bigcup_{k=1}^{j-1} D_{j,k}$. We extend μ_j to the whole plane by $\mu_j(z) = (z/\bar{z})^2 \overline{\mu_j(1/\bar{z})}$ for $|z| > 1$. By the Measurable Riemann Mapping Theorem there exists a unique quasiconformal map $h_j: \mathbb{C} \rightarrow \mathbb{C}$, normalized by $h_j(0) = 0, h_j(1) = 1$, with complex dilatation $\mu_{h_j} = \mu_j$ a.e. By the symmetry of μ_j and the chosen normalization, $h_j(1/\bar{z}) = 1/\overline{h_j(z)}$, and so $h_j(\mathbb{C}_1) = \mathbb{C}_1$. We define $\psi_j = h_j \circ \phi_j$. The transformation formula for Beltrami coefficients (see, e.g., [LV73, IV.5.1]) shows that if $k < j$, then $\psi_k \circ \psi_j^{-1} = T_{j,k} \circ h_j^{-1}$ is conformal. Moreover, $\psi_j(x_j) = 0$.

We claim that for each $j = 1, \dots, n$, the mapping ψ_j has a quasisymmetric distortion function that depends only on the data of X . As this is true of ϕ_j , it suffices to prove the same of h_j , and we may also assume that $j > 1$. As a normalized quasiconformal self-map of the unit disk, h_j is quasisymmetric with distortion controlled by the maximal dilatation $\|\mu_j\|_\infty$; see e.g. [Väi81, Theorem 2.4]. By an inductive argument, it is easy to show that this dilatation is bounded by a constant depending only on η' (which depends only on the data of X) and

the number of charts n (which depends on the data and ρ). However, the bound is actually independent of ρ , by the following argument.

Fix $j > 1$. By the uniform quasismetry of $\{\phi_k\}_{k=1}^n$, there is a quantity $0 \leq \kappa < 1$, depending only on the data of X , such that for all indices $k < j$ with $D_{j,k} \neq \emptyset$, the complex dilatation $\mu_{\phi_k \circ \phi_j^{-1}}$ of $\phi_k \circ \phi_j^{-1}: D_{j,k} \rightarrow \phi_k(D_{j,k})$ satisfies $\|\mu_{\phi_k \circ \phi_j^{-1}}\|_\infty \leq \kappa$. For $k \in \{1, \dots, j-1\}$, define

$$F_{j,k} = D_{j,k} \setminus \bigcup_{l=1}^{k-1} D_{j,l}, \quad F_k = \mathbb{C}_1 \setminus \bigcup_{l=1}^{k-1} D_{k,l}, \quad \text{and} \quad F_j = \mathbb{C}_1 \setminus \bigcup_{k=1}^{j-1} D_{j,k}.$$

If $z \in F_j$, then $\mu_j(z) = 0$. If $z \in F_{j,k}$, then $\phi_k \circ \phi_j^{-1}(z) \in F_k$, and hence

$$\mu_k|_{\phi_k \circ \phi_j^{-1}(F_{j,k})} = 0.$$

The transformation formula for Beltrami coefficients now shows that for almost every point $z \in F_{j,k}$,

$$\mu_j(z) = \mu_{j,k}(z) = \mu_{\psi_k \circ \phi_j^{-1}}(z) = \mu_{h_k \circ \phi_k \circ \phi_j^{-1}}(z) = \mu_{\phi_k \circ \phi_j^{-1}}(z).$$

Since $\mathbb{C}_1 = \bigcup_{k=1}^{j-1} F_{j,k} \cup F_j$, we see that $\|\mu_j\|_\infty \leq \kappa$. As discussed above, we may now conclude that each of the mappings $\{\psi_j\}_{j=1}^n$ has a quasismetric distortion function that depends only on the data of X .

We have now seen that the atlas $\mathcal{B} := \{(U_j, \mathbb{C}_1, \psi_j) | j = 1, \dots, n\}$ of X satisfies conditions (1) and (5) of the statement. Moreover, setting $r_0 := r'_0$, the conditions (2) and (3) follow directly from the corresponding statements for the atlas $\mathcal{A}_{X,\rho}$. Note that only condition (4) involves the constant β . We now show how to choose ρ so that condition (4) is satisfied.

Let us make the convention that $h_1: \mathbb{C}_1 \rightarrow \mathbb{C}_1$ is the identity. As discussed above, the mappings $\{h_j\}_{j=1}^n$ are uniformly quasismetric with a distortion function depending only on the data of X . Hence, by Lemma 3, there is a common modulus of continuity for all of the mappings $\{h_j, h_j^{-1}\}_{j=1}^n$ that depends only on the data of X . Since β is a universal constant, we may choose $\rho > 0$ depending only on the data of X such that for each $j = 1, \dots, n$, it holds that $h_j(\mathbb{C}_\rho) \subseteq \mathbb{C}_\beta$. Having so chosen ρ , the radius α' depends only on the data of X , and so we may also choose $\alpha > 0$ depending only on the data of X such that for each $j = 1, \dots, n$, $h_j(\mathbb{C}_{\alpha'}) \supseteq \mathbb{C}_\alpha$. This establishes condition (4). \square

3.1. Uniformizing to a standard metric. The atlas \mathcal{B} given by Lemma 11 determines a conformal structure on the compact orientable surface X ; i.e., the pair (X, \mathcal{B}) determines a Riemann surface. By the classical uniformization theorem, (X, \mathcal{B}) is conformally equivalent to a standard Riemann surface $Y = U/\Gamma$, where U denotes the standard Riemann surface structure on the sphere, the plane, or the unit disk, and Γ is a discrete group of Möbius transformations acting freely and properly discontinuously on U . The standard spherical, plane, or hyperbolic Riemannian metric on U then descends to a Riemannian metric of constant curvature $+1, 0$, or -1 on Y , compatible with the conformal structure. We fix a uniformizing conformal homeomorphism $F: (X, \mathcal{B}) \rightarrow Y$ and equip Y with the distance function d_Y arising from the Riemannian metric of constant curvature. Denote by π the quotient mapping from U to Y . Recall that d_Y may be expressed as $d_Y(p, q) = \inf_\gamma \text{length}(\gamma)$, where the infimum is taken over all smooth paths γ in U such that the projected path $\pi \circ \gamma$ connects p and q . A priori, it is not clear how

the properties of the distance d_Y or the map F depend on the original metric space (X, d) . The following statement is the main result of this paper and completes the proof of Theorem 2.

Theorem 12. *The uniformizing map $F: (X, d) \rightarrow (Y, d_Y)$ is η -quasisymmetric, with distortion η depending only on the data of X .*

Proof of Theorem 12. In the case that $U = \hat{\mathbb{C}}$, X itself must be homeomorphic to $\hat{\mathbb{C}}$, and so Theorem 1 implies Theorem 12. We consider the remaining planar and hyperbolic cases together.

Define $g_j: \mathbb{C}_1 \rightarrow F(U_j)$ by $g_j = F \circ \psi_j^{-1}$. Then g_j is a conformal homeomorphism that lifts to a conformal embedding $G_j: \mathbb{C}_1 \rightarrow U$.

We use a sequence of claims to complete the proof.

Claim 1. Let $j \in \{1, \dots, n\}$. The projection $\pi: U \rightarrow Y$ maps $G_j(\mathbb{C}_\beta)$ isometrically onto $g_j(\mathbb{C}_\beta)$. In other words, for each $z, z' \in \mathbb{C}_\beta$, $|g_j(z) - g_j(z')|_Y = |G_j(z) - G_j(z')|_U$.

Proof of Claim 1. For $u \in U$, define

$$r(u) = \min\{|u - \gamma(u)|_U : \gamma \in \Gamma \setminus \{\text{id}\}\}.$$

If $U = \mathbb{C}$, then $r(u)$ is independent of $u \in U$ and depends only on the group Γ . If $U = \mathbb{D}$, then $r(u)$ is bounded below by the minimal translation distance of a non-identity element of $\Gamma \setminus \{\text{id}\}$ and is a 2-Lipschitz function of u ; see e.g. [Bea83, section 7.35]. These facts imply that for any $u \in U$ and $r > 0$,

- if $r \leq r(u)/6$, then $\pi|_{B_U(u,r)}$ is an isometry;
- if $\pi|_{B_U(u,r)}$ is injective, then $r \leq r(u)$.

Since $\pi: G_j(\mathbb{C}_1) \rightarrow g_j(\mathbb{C}_1)$ is injective, it follows that $l_{G_j}(0, 1) \leq r(G_j(0))$. Corollaries 6 and 8 now complete the proof of this claim. \square

Claim 2. For each $j = 1, \dots, n$, the mapping $F|_{B(x_j, 10r_0)}$ is quasisymmetric with distortion that depends only on the data of X .

Proof of Claim 2. By Propositions 5 and 7, the mapping G_j restricted to \mathbb{C}_β is quasisymmetric with a universal distortion function. By Claim 1, this is also true of g_j . Since we may write $F = g_j \circ \psi_j$, and ψ_j is η -quasisymmetric where η depends only on the data, it follows that for each $j = 1, \dots, n$, the mapping $F|_{\psi_j^{-1}(\mathbb{C}_\beta)}$ is quasisymmetric with distortion that depends only on the data of X . According to Lemma 11, $\psi_j^{-1}(\mathbb{C}_\beta) \supseteq B(x_j, 10r_0)$, implying the claim. \square

Claim 3. There is a constant $C \geq 1$ depending only on the data of X such that for each $j = 1, \dots, n$,

$$(3) \quad l_{g_j}(0, \alpha) \geq \frac{\text{diam } Y}{C}.$$

Proof of Claim 3. As $\{B_X(x_j, 2r_0)\}_{j=1}^n$ covers X and $\psi_j(B_X(x_j, 10r_0)) \subseteq \mathbb{C}_\beta$, it holds that $\{\psi_j^{-1}(\mathbb{C}_\beta)\}_{j=1}^n$ also covers X . Since F is a homeomorphism, this implies that $Y = \bigcup_{j=1}^n g_j(\mathbb{C}_\beta)$. As Y is connected, it follows that

$$(4) \quad \max\{\text{diam } g_j(\mathbb{C}_\beta) : j = 1, \dots, n\} \geq \frac{\text{diam } Y}{n}.$$

Recall that the number of charts n depends only on the data of X .

Consider indices j and k in $\{1, \dots, n\}$, and suppose that $|x_j - x_k|_X < 4r_0$. Then $\psi_j(x_k) \in \mathbb{C}_\beta$, and so $d_Y(F(x_j), F(x_k)) \leq L_{g_j}(0, \beta)$. On the other hand, $|x_j - x_k| \geq r_0$, and so $F(x_k) \notin g_j(\mathbb{C}_\alpha)$, implying $d_Y(F(x_j), F(x_k)) \geq l_{g_j}(0, \alpha)$. Moreover, since $g_j: \mathbb{C}_\beta \rightarrow Y$ is a quasisymmetric embedding with universal distortion function, there is a quantity $C_0 \geq 1$ depending only on the ratio of β to α , and hence only on the data of X , such that

$$l_{g_j}(0, \alpha) \geq \frac{L_{g_j}(0, \beta)}{C_0} \geq \frac{\text{diam } g_j(\mathbb{C}_\beta)}{2C_0}.$$

Since, in addition,

$$\text{diam } g_j(\mathbb{C}_\beta) \geq L_{g_j}(0, \beta) \geq l_{g_j}(0, \alpha),$$

we have now shown that the quantities

$$l_{g_j}(0, \alpha), L_{g_j}(0, \beta), \text{diam } g_j(\mathbb{C}_\beta), \text{ and } d_Y(F(x_j), F(x_k))$$

are all comparable with constants that depend only on the data of X . The same is true with the roles of j and k reversed.

Since the open sets $\{B(x_j, 2r_0)\}_{j=1}^n$ cover the connected space X , for any pair of indices $j, j' \in \{1, \dots, n\}$, there is a sequence of distinct indices $j = j_1, \dots, j_k = j'$ of length at most n so that $|x_{j_i} - x_{j_{i+1}}| < 4r_0$ for each $i = 1, \dots, k-1$. Since n depends only on the data of X , (4) proves the claim. \square

We complete the proof of Theorem 12 by employing Theorem 4. We consider the covering of X given by $\{B(x_j, 10r_0) : j = 1, \dots, n\}$. By Claim 2, the mapping F is quasisymmetric on each element of this cover with a distortion function η that depends only on the data of X . Moreover, Lemma 11 implies that this cover has Lebesgue number $8r_0$. Suppose that $x, x' \in X$ satisfy $|x - x'| = 4r_0$. We may find indices j and k in $\{1, \dots, n\}$ such that $|x - x_j| < 2r_0$ and $|x' - x_k| < 2r_0$. Then $2r_0 \leq |x' - x_j| \leq 6r_0$, so $x, x', x_j \in B(x_j, 10r_0)$ and

$$|F(x') - F(x)| \geq \frac{|F(x') - F(x_j)|}{\eta\left(\frac{|x' - x_j|}{|x' - x|}\right)} \geq \frac{|F(x') - F(x_j)|}{\eta\left(\frac{3}{2}\right)}.$$

Since $x' \notin B(x_j, r_0)$, we see that $F(x') \notin g_j(\mathbb{C}_\alpha)$, so by Claim 3,

$$|F(x') - F(x_j)| = |F(x') - g_j(0)| \geq l_{g_j}(0, \alpha) \geq \frac{\text{diam } Y}{C}.$$

Now Theorem 4 implies that $F: X \rightarrow Y$ is quasisymmetric with distortion function depending only on the data of X . \square

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REFERENCES

- [Bea83] Alan F. Beardon, *The geometry of discrete groups*, Graduate Texts in Mathematics, vol. 91, Springer-Verlag, New York, 1983. MR698777
- [BK02] Mario Bonk and Bruce Kleiner, *Quasisymmetric parametrizations of two-dimensional metric spheres*, Invent. Math. **150** (2002), no. 1, 127–183, DOI 10.1007/s00222-002-0233-z. MR1930885
- [Bon06] Mario Bonk, *Quasiconformal geometry of fractals*, International Congress of Mathematicians. Vol. II, Eur. Math. Soc., Zürich, 2006, pp. 1349–1373. MR2275649

- [Can69] Raymond J. Cannon Jr., *Quasiconformal structures and the metrization of 2-manifolds*, Trans. Amer. Math. Soc. **135** (1969), 95–103, DOI 10.2307/1995004. MR0232932
- [Dur83] Peter L. Duren, *Univalent functions*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 259, Springer-Verlag, New York, 1983. MR708494
- [Hei01] Juha Heinonen, *Lectures on analysis on metric spaces*, Universitext, Springer-Verlag, New York, 2001. MR1800917
- [Kuu67] Tapani Kuusalo, *Verallgemeinerter Riemannscher Abbildungssatz und quasikonforme Mannigfaltigkeiten* (German), Ann. Acad. Sci. Fenn. Ser. A I No. **409** (1967), 24 pp. MR0218560
- [LV73] O. Lehto and K. I. Virtanen, *Quasiconformal mappings in the plane*, 2nd ed., translated from the German by K. W. Lucas; Die Grundlehren der mathematischen Wissenschaften, Band 126, Springer-Verlag, New York-Heidelberg, 1973. MR0344463
- [MW13] Sergei Merenkov and Kevin Wildrick, *Quasisymmetric Koebe uniformization*, Rev. Mat. Iberoam. **29** (2013), no. 3, 859–909, DOI 10.4171/RMI/743. MR3090140
- [TV80] P. Tukia and J. Väisälä, *Quasisymmetric embeddings of metric spaces*, Ann. Acad. Sci. Fenn. Ser. A I Math. **5** (1980), no. 1, 97–114, DOI 10.5186/aasfm.1980.0531. MR595180
- [Väi81] Jussi Väisälä, *Quasisymmetric embeddings in Euclidean spaces*, Trans. Amer. Math. Soc. **264** (1981), no. 1, 191–204, DOI 10.2307/1998419. MR597876
- [Wil08] K. Wildrick, *Quasisymmetric parametrizations of two-dimensional metric planes*, Proc. Lond. Math. Soc. (3) **97** (2008), no. 3, 783–812, DOI 10.1112/plms/pdn023. MR2448247
- [Wil10] Kevin Wildrick, *Quasisymmetric structures on surfaces*, Trans. Amer. Math. Soc. **362** (2010), no. 2, 623–659, DOI 10.1090/S0002-9947-09-04861-2. MR2551500

DEPARTMENT OF MATHEMATICAL SCIENCES, MONTANA STATE UNIVERSITY, BOZEMAN, MONTANA 59717

E-mail address: `geyer@montana.edu`

DEPARTMENT OF MATHEMATICAL SCIENCES, MONTANA STATE UNIVERSITY, BOZEMAN, MONTANA 59717

E-mail address: `kevin.wildrick@montana.edu`