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ZEROS OF BESSEL FUNCTION DERIVATIVES

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ABSTRACT. We prove that for $\nu > n-1$ all zeros of the nth derivative of the Bessel function of the first kind J_ν are real. Moreover, we show that the positive zeros of the nth and $(n+1)$th derivative of the Bessel function of the first kind J_ν are interlacing when $\nu \geq n$ and n is a natural number or zero. Our methods include the Weierstrassian representation of the nth derivative, properties of the Laguerre-Pólya class of entire functions, and the Laguerre inequalities. Some similar results for the zeros of the first and second derivatives of the Struve function of the first kind $\mathbf{H}_\nu$ are also proved. The main results obtained in this paper generalize and complement some classical results on the zeros of Bessel and Struve functions of the first kind. Some open problems related to Hurwitz's theorem on the zeros of Bessel functions are also proposed.

1. Introduction and Main Results

The zeros of Bessel functions of the first kind have numerous applications in applied mathematics, mathematical physics and engineering sciences. There is an extensive literature on various properties of the zeros of Bessel functions of the first kind. They were investigated by such famous researchers as Bessel, Euler, Fourier, Gegenbauer, Hurwitz, Lommel, Rayleigh and Stokes. We refer to the survey paper [Ke] and the references therein for detailed information on various properties of the zeros of Bessel functions of the first kind. In the past three decades the zeros of the nth derivative of Bessel functions of the first kind for $n \in \{1,2,3\}$ have also been studied by researchers such as Elbert, Ifantis, Ismail, Kokologiannaki, Laforgia, Landau, Lorch, Mercer, Muldoon, Petropoulou, Siafarikas and Szego; for more details see the papers [IM, KP] and the references therein. However, there are no results in the literature about the zeros of the nth derivative of Bessel functions when n is a natural number greater than 3. Some interesting results and open problems about $J_\nu^{(n)}$ when $\nu \in (n-3, n-2)$ were stated by Lorch and Muldoon in [LM]. However the paper [LM] is so far the only study in which the zeros of higher order derivatives of J_ν were considered. In this paper our aim is to partially fill this gap and to present some results for the derivatives of Bessel functions by using the Laguerre-Pólya class of entire functions and the so-called Laguerre inequalities. By using a technique similar to that used for the Bessel functions of the first kind,

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we also present some new results for the zeros of the first and second derivatives of the Struve functions of the first kind. Moreover, by using these results we find explicit expressions for some Rayleigh sums for the zeros of Struve functions and their first and second derivatives. In addition, lower bounds for the first positive zero of these functions are given. At the end of this section we propose some open problems which may be of interest for further research. Throughout this paper $n, s \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$.

1.1. **Zeros of the nth derivative of Bessel functions.** In view of the results on the zeros of the nth derivative of Bessel functions of the first kind, when $n \in \{0, 1, 2, 3\}$, the statements of the following theorem are very natural and somehow expected. Theorem 1 provides extensions of some classical results on the zeros of Bessel function of the first kind and its derivative of the first order.

Theorem 1.

(a) *If $\nu > n - 1$, then $J_\nu^{(n)}(x)$ has infinitely many zeros, which are all real and simple, except the origin.*
(b) *If $\nu \geq n$, then the positive zeros of the nth and $(n+1)$th derivative of J_ν are interlacing.*
(c) *If $\nu > n - 1$, then all zeros of $(n - \nu)J_\nu^{(n)}(x) + xJ_\nu^{(n+1)}(x)$ are real and interlace with the zeros of $J_\nu^{(n)}(x)$.*

We mention that part (b) in particular reduces to the chains of inequalities

$$j'''_{\nu,1} < j''_{\nu,1} < j'''_{\nu,2} < j''_{\nu,2} < j'''_{\nu,3} < j''_{\nu,3} < \dots, \quad \nu \geq 2,$$

and

$$j''_{\nu,1} < j'_{\nu,1} < j''_{\nu,2} < j'_{\nu,2} < j''_{\nu,3} < j'_{\nu,3} < \dots, \quad \nu \geq 1,$$

where $j''_{\nu,n}$ and $j'''_{\nu,n}$ denote the nth positive zero of J''_ν and J'''_ν, respectively. These inequalities complement the well-known ones [OLBC, p. 235]

$$j'_{\nu,1} < j_{\nu,1} < j'_{\nu,2} < j_{\nu,2} < j'_{\nu,3} < j_{\nu,3} < \dots, \quad \nu \geq 0.$$

We also note that part (c) is a generalization of the well-known result that for $\nu > -1$ the zeros of the Bessel functions J_ν and $J_{\nu+1}$ are interlacing (see [OLBC, 10.21.3]). Namely, by choosing $n = 0$ in part (c) we obtain that the zeros of J_ν and $xJ'_\nu(x) - \nu J_\nu(x) = xJ_{\nu+1}(x)$ are interlacing.

By using the main idea from [DC, p. 705], an immediate consequence of part (a) of Theorem 1 in terms of generalized hypergeometric polynomials is given in Theorem 2. The connection between these two results is a special class of real entire functions, called the Laguerre-Pólya class. Recall that a real entire function ψ belongs to the Laguerre-Pólya class $\mathcal{LP}$ if it can be represented in the form

$$\psi(x) = cx^m e^{-ax^2+bx} \prod_{n\geq 1} \left(1 + \frac{x}{x_n}\right) e^{-\frac{x}{x_n}},$$

where c, b, $x_n \in \mathbb{R}$, $a \geq 0$, $m \in \mathbb{N}_0$ and $\sum_{n\geq 1} x_n^{-2} < \infty$. The class $\mathcal{LP}$ consists of entire functions which are uniform limits on compact sets of the complex plane of polynomials with only real zeros. For more details on the class $\mathcal{LP}$ we refer to [DC, p. 703] and the references therein.

Theorem 2. *If $\nu > n-1$, then all the zeros of the Laguerre-type polynomial*

$$ {}_3F_3\left(-s, \frac{\nu+1}{2}, \frac{\nu}{2}+1; \nu+1, \frac{\nu-n+1}{2}, \frac{\nu-n}{2}+1; x\right) $$

are real and simple.

We note that the name Laguerre-type polynomial for the Jensen polynomial appearing in Theorem 2 is justified by the facts that the case $n = 0$ reduces to the well-known generalized Laguerre polynomial ${}_1F_1(-s; \nu+1; x)$ (see [DC, p. 705]), the case $n = 1$ corresponds to the generalized hypergeometric polynomial ${}_2F_2\left(-s, \frac{\nu}{2}+1; \nu+1, \frac{\nu}{2}; x\right)$ and Koornwinder's generalized Laguerre polynomial [Ko, p. 26], while the case $n = 2$ is related to the generalized Laguerre polynomial ${}_3F_3\left(-s, \frac{\nu+1}{2}, \frac{\nu}{2}+1; \nu+1, \frac{\nu-1}{2}, \frac{\nu}{2}; x\right)$ considered by Álvarez-Nodarse and Marcellán [AM].

1.2. **Zeros of the first and second derivatives of Struve functions.** In 1970 Steinig [St] studied the real zeros of the Struve functions and proved that for $|\nu| < \frac{1}{2}$ the zeros of the Struve function $\mathbf{H}_\nu$ are all real and simple, and the positive zeros of $\mathbf{H}_\nu$ interlace with the positive zeros of J_ν and lie in the intervals $(m\pi, (m+1)\pi)$, $m \in \mathbb{N}$. Motivated by these results and to find the radius of convexity of some normalized Struve functions, Baricz and Yağmur [BY] have proved recently that the zeros of the function $\mathbf{H}'_\nu$ are all real and simple for $|\nu| < \frac{1}{2}$, and the positive zeros of the function $\mathbf{H}'_\nu$ interlace with the positive zeros of $\mathbf{H}_\nu$. In this subsection we prove some analogous results for the second derivative of $\mathbf{H}_\nu$.

Theorem 3.

(a) *If $\nu \in \left(0, \frac{1}{2}\right]$, then $\mathbf{H}''_\nu(x)$ has infinitely many zeros, which are all real and simple.*
(b) *$\nu \in \left(0, \frac{1}{2}\right]$, then the positive zeros of the first and second derivatives of $\mathbf{H}_\nu(x)$ are interlacing.*
(c) *If $\nu \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, then all zeros of $-\nu\mathbf{H}'_\nu(x) + x\mathbf{H}''_\nu(x)$ are real and interlace with the zeros of $\mathbf{H}'_\nu(x)$. Moreover, all zeros of $-(\nu+1)\mathbf{H}_\nu(x) + x\mathbf{H}'_\nu(x)$ are real and interlace with the zeros of $\mathbf{H}_\nu(x)$.*

The next result is the counterpart of Theorem 2 for Struve functions.

Theorem 4. *If $\nu \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ and $n \in \{0, 1\}$, or $\nu \in \left(0, \frac{1}{2}\right]$ and $n = 2$, then all zeros of the hypergeometric polynomial*

$$ {}_4F_4\left(-s, 1, \frac{\nu}{2}+1, \frac{\nu}{2}+\frac{3}{2}; \frac{3}{2}, \nu+\frac{3}{2}, \frac{\nu-n}{2}+1, \frac{\nu-n}{2}+\frac{3}{2}; x\right) $$

are real and simple.

1.3. **Rayleigh sums of the zeros of the first and second derivatives of Struve functions.** The Struve function $\mathbf{H}_\nu$ and its derivatives of the first and second order can be represented by infinite series and as infinite products using Hadamard's factorization. So, equating these representations, as was done in [KL] for the zeros of J_ν and J'_ν, it is possible to obtain the Rayleigh sums for the zeros of the Struve function $\mathbf{H}_\nu$ and of its first and second derivatives. For more information on the Rayleigh sums of the zeros of Bessel functions of the first kind, we refer to [OLBC, p. 240], [Wa, p. 502] and the references therein.

Theorem 5. *If $\nu \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, then the first two Rayleigh sums for the nth positive zeros $h_{\nu,n}$, $h'_{\nu,n}$ and $h''_{\nu,n}$ of the Struve functions $\mathbf{H}_\nu$, $\mathbf{H}'_\nu$ and $\mathbf{H}''_\nu$ are*

$$\sum_{n\geq 1} \frac{1}{h_{\nu,n}^2} = \frac{1}{3(2\nu+3)}, \quad \sum_{n\geq 1} \frac{1}{h_{\nu,n}^4} = \frac{7-2\nu}{45(2\nu+3)^2(2\nu+5)},$$

$$\sum_{n\geq 1} \frac{1}{(h'_{\nu,n})^2} = \frac{\nu+3}{3(\nu+1)(2\nu+3)}, \quad \sum_{n\geq 1} \frac{1}{(h'_{\nu,n})^4} = \frac{-2\nu^3-5\nu^2+72\nu+135}{45(\nu+1)^2(2\nu+3)^2(2\nu+5)},$$

$$\sum_{n\geq 1} \frac{1}{(h''_{\nu,n})^2} = \frac{(\nu+2)(\nu+3)}{3\nu(\nu+1)(2\nu+3)},$$

$$\sum_{n\geq 1} \frac{1}{(h''_{\nu,n})^4} = \frac{-2\nu^5-13\nu^4+92\nu^3+763\nu^2+1500\nu+900}{45\nu^2(\nu+1)^2(2\nu+3)^2(2\nu+5)},$$

provided that $\nu > 0$ in the last two relations.

An immediate consequence of the above theorem is Corollary 1. We note that if we keep repeating the procedure in the proof of Theorem 5, then we can derive the sums $\sum_{n\geq 1} h_{\nu,n}^{-2k}$, $k \in \{3, 4, \dots\}$. Using these Rayleigh sums it is possible to derive sharper lower bounds for $h_{\nu,1}$, $h'_{\nu,1}$ and $h''_{\nu,1}$.

Corollary 1. *For $|\nu| < \frac{1}{2}$ we have the following inequalities:*

$$h_{\nu,1} > \sqrt{3(2\nu+3)},$$

$$h_{\nu,1}^2 > 3(2\nu+3)\sqrt{\frac{5(2\nu+5)}{7-2\nu}},$$

$$h'_{\nu,1} > \sqrt{\frac{3(\nu+1)(2\nu+3)}{\nu+3}},$$

$$(h'_{\nu,1})^2 > 3(\nu+1)(2\nu+3)\sqrt{\frac{5(2\nu+5)}{-2\nu^3-5\nu^2+72\nu+135}},$$

$$h''_{\nu,1} > \sqrt{\frac{3\nu(\nu+1)(2\nu+3)}{(\nu+2)(\nu+3)}},$$

$$(h''_{\nu,1})^2 > 3\nu(\nu+1)(2\nu+3)\sqrt{\frac{5(2\nu+5)}{-2\nu^5-13\nu^4+92\nu^3+763\nu^2+1500\nu+900}},$$

provided that $\nu > 0$ in the last two inequalities.

Since $h'_{\nu,1} < h_{\nu,1}$, we can see that the third and fourth inequalities in the above corollary are also bounds for $h_{\nu,1}$, but they are sharper than the ones given in the first and second inequalities. Moreover, for $\nu \in \left(0, \frac{1}{2}\right]$ we have that $h''_{\nu,1} < h'_{\nu,1}$, and thus the fifth and sixth inequalities in the above corollary are also bounds for $h'_{\nu,1}$ and $h_{\nu,1}$.

1.4. **Open problems concerning the zeros of derivatives of Bessel and Struve functions.** Part (a) of Theorem 1 is an extension of the celebrated result of von Lommel (see [Wa, p. 482]), which states that if $\nu > -1$, then all zeros of J_ν are real. Now, recall that by means of Hurwitz's theorem (see [Hu], [Wa, p. 483]) we know that if $\nu > -1$, then all zeros of J_ν are real; when $-2s-2 < \nu < -2s-1$, $s \in \mathbb{N}_0$, then J_ν has $4s+2$ complex zeros, of which two are purely imaginary; and when $-2s-1 < \nu < -2s$, $s \in \mathbb{N}$, then the Bessel function J_ν has $4s$ complex zeros, of which none are purely imaginary. See also [Hu, Ke, KK] for more details. Thus, it is an interesting problem to provide a complete picture of the behavior of the zeros of the derivatives of Bessel functions and to give a generalization of Hurwitz's theorem [Hu].

Open Problem 1. Let $n \in \mathbb{N}_0$.

(a) Is it true that if s is a nonnegative integer and $n-2s-2 < \nu < n-2s-1$, then the function $J_\nu^{(n)}$ has $4s+2$ complex zeros, of which two are purely imaginary?
(b) Is it true that if s is a positive integer and $n-2s-1 < \nu < n-2s$, then the function $J_\nu^{(n)}$ has $4s$ complex zeros, of which none are purely imaginary?

Note that Hurwitz's proof of his famous theorem on the distribution of the zeros of Bessel functions of the first kind was based on the three term recurrence relation for Bessel functions, on Lommel polynomials and on the relation between Bessel functions and Lommel polynomials; see [Hu, Ke] and [Wa, p. 483] for more details. Ki and Kim [KK] presented a nice alternative proof of Hurwitz's theorem by using a Fourier critical point approach. To define the notion of the Fourier critical point let f be a real entire function defined in an open interval $(a,b) \subset \mathbb{R}$. Let $l \in \mathbb{N}$ and suppose that $c \in (a,b)$ is a zero of $f^{(l)}(x)$ of multiplicity $m \in \mathbb{N}$, that is, $f^{(l)}(c) = \cdots = f^{(l+m-1)}(c) = 0$ and $f^{(l+m)}(c) \neq 0$. Now, let $k = 0$ if $f^{(l-1)}(c) = 0$; otherwise let

$$k = \begin{cases} m/2, & \text{if } m \text{ is even,} \\ (m+1)/2, & \text{if } m \text{ is odd and } f^{(l-1)}(c)f^{(l+m)}(c) > 0, \\ (m-1)/2, & \text{if } m \text{ is odd and } f^{(l-1)}(c)f^{(l+m)}(c) < 0. \end{cases}$$

We say that $f^{(l)}(x)$ has k critical zeros and $m-k$ noncritical zeros at $x = c$. A point in (a,b) is said to be a Fourier critical point of f if some derivative of f has a critical zero at the point. For more details on Fourier critical points we refer to [KK].

Motivated by the proof of Hurwitz's theorem given in [KK], we consider the auxiliary function

$$f_{\nu,n}(x) = \sum_{m\geq 0} \frac{\Gamma(\nu+2m+1)}{\Gamma(\nu+2m-n+1)\Gamma(\nu+m+1)} \frac{x^m}{m!}.$$

This function is a real entire function of growth order $\frac{1}{2}$ and consequently of genus 0. Due to [KK, Theorem 4.1] it has just as many Fourier critical points as couples of nonreal zeros (complex conjugate pairs of zeros with nonzero imaginary part). Now, since $2^n x^{n/2} J_\nu^{(n)}(2\sqrt{x}) = x^{\nu/2} f_{\nu,n}(-x)$, part (a) of Theorem 1 implies that the function $f_{\nu,n}$ has no Fourier critical points when $\nu > n-1$. The following problem is motivated by [KK, p. 68] and in case it is true, it would imply that the answers to the questions stated in Open Problem 1 are affirmative.

Open Problem 2. Let $n \in \mathbb{N}_0$.

(a) Is it true that if s is a nonnegative integer and $n-2s-2 < \nu < n-2s-1$, then the function $f_{\nu,n}$ has exactly s Fourier critical points and one positive real zero?

(b) Is it true that if s is a positive integer and $n-2s-1 < \nu < n-2s$, then the function $f_{\nu,n}$ has exactly s Fourier critical points and no positive real zeros?

From Steinig [St, p. 367] we know that when $\nu \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ all zeros of the Struve function $\mathbf{H}_\nu$ are real. We also know that for $\nu > \frac{1}{2}$ we have $\mathbf{H}_\nu(x) > 0$ for $x > 0$, and thus there are no positive real zeros in this case. However, we do not know what happens to the zeros when $\nu < -\frac{1}{2}$. By using the connection between Bessel and Struve functions, that is, the relation $\mathbf{H}_{-m-\frac{1}{2}}(x) = (-1)^m J_{m+\frac{1}{2}}(x)$, where $m \in \mathbb{N}_0$, it is clear that all zeros of the Struve function $\mathbf{H}_\nu$ are real when $\nu = -m-\frac{1}{2}$ and $m \in \mathbb{N}_0$. Moreover, the same is true for $\mathbf{H}'_\nu$, and in general we can state that if $\nu = -m-\frac{1}{2}$, $m+\frac{1}{2} > n-1$ and $m, n \in \mathbb{N}_0$, then all zeros of the function $\mathbf{H}_\nu^{(n)}$ are real. The next two open problems are motivated by this result.

Open Problem 3. Find the number of complex zeros of the function $\mathbf{H}_\nu^{(n)}$ when $\nu \in \mathbb{R}$ and $n \in \mathbb{N}_0$.

Open Problem 4. Is it true that if $\nu \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ and $n \in \mathbb{N}_0$, then all zeros of $\mathbf{H}_\nu^{(n)}$ are real? Find the maximal range for ν for which all zeros of $\mathbf{H}_\nu^{(n)}$ are real.

Recall that Steinig [St, p. 367] showed that the zeros of the functions J_ν and $\mathbf{H}_\nu$ are interlacing when $|\nu| < \frac{1}{2}$. The following open problem is motivated by this result.

Open Problem 5. Is it true that if $\nu \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ and $n \in \mathbb{N}_0$, then the zeros of $\mathbf{H}_\nu^{(n)}$ and $J_\nu^{(n)}$ interlace? Find the maximal range for ν for which all zeros of $\mathbf{H}_\nu^{(n)}$ and $J_\nu^{(n)}$ interlace.

Recently, Baricz and Szász [BS] and Baricz et al. [BCD] have found necessary and sufficient conditions on the parameter ν such that for $n \in \{0, 1, 2, 3\}$ the function $2^\nu \Gamma(\nu - n + 1) z^{\frac{n+2-\nu}{2}} J_\nu^{(n)}(\sqrt{z})$ is starlike (maps the open unit disk of the complex plane into a starlike domain) and all of its derivatives are close-to-convex (and hence univalent). In their proofs the key tool was that for fixed $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$ the mth positive zeros of $J_\nu^{(n)}$, denoted by $j_{\nu,m}^{(n)}$, are increasing with ν on $(n-1, \infty)$, where $n \in \{0, 1, 2, 3\}$ (see [KP, LP, Me, OLBC, WL] for more details). Now, since for $\nu > n-1$ the zeros of $J_\nu^{(n)}$ are real, an affirmative answer to the following problem would enable us to generalize the above-mentioned results of [BS] and [BCD].

Open Problem 6. Is it true that $j_{\nu,m}^{(n)}$ is increasing with ν on $(n-1, \infty)$ for $n \in \mathbb{N}_0$ and $m \in \mathbb{N}$ fixed?

2. Proof of the main results

Proof of Theorem 1. (a) We use mathematical induction to prove that each of the zeros is real. For $n = \{0, 1, 2, 3\}$ and $\nu > n-1$ we know that the zeros of $J_\nu^{(n)}(x)$ are all real; see for example [IM, Ke, KP] and the references therein for more details. We suppose that for fixed $n \in \{4, 5, \dots\}$ and $\nu > n-1$ the function $J_\nu^{(n)}(x)$ has only real zeros and we show that when $\nu > n$ then $J_\nu^{(n+1)}(x)$ also has only real zeros. We denote the mth positive zero of $J_\nu^{(n)}(x)$ by $j_{\nu,m}^{(n)}$, where $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$. From Skelton [Ske, p. 340] we know that the Weierstrassian decomposition

$$J_\nu^{(n)}(x) = \frac{x^{\nu-n}}{2^\nu \Gamma(\nu+1-n)} \prod_{m\geq 1} \left(1 - \frac{x^2}{\left(j_{\nu,m}^{(n)}\right)^2}\right) \tag{2.1}$$

holds, and this infinite product is uniformly convergent on compact subsets of the complex plane. Note that it was not stated in [Ske] for which ν the above infinite product is valid. In fact, the above infinite product (2.1) appears in the proof of [Ske, Theorem 2.1] enounced for $\nu \geq n$. However, (2.1) holds true for all $\nu > n-1$, where n is a natural number or zero. To see this we note that since for fixed n and ν,

$$\lim_{m\to\infty} \frac{m \log m}{\log \frac{2^{2m}}{\Gamma(\nu+1-n)} + \log\Gamma(m+1) + \log\Gamma(\nu+m+1) + \Delta_{n,m}(\nu)} = \frac{1}{2},$$

where $\Delta_{n,m}(\nu) = \log\Gamma(\nu+2m-n+1) - \log\Gamma(\nu+2m+1)$, the real entire function

$$\mathbb{J}_{\nu,n}(x) = 2^\nu\Gamma(\nu+1-n)x^{n-\nu}J_\nu^{(n)}(x) = \sum_{m\geq 0} \frac{(-1)^m\Gamma(\nu+2m+1)\Gamma(\nu+1-n)x^{2m}}{m!2^{2m}\Gamma(\nu+2m-n+1)\Gamma(\nu+m+1)}$$

is of order $\frac{1}{2}$. By the Hadamard factorization theorem [Le, p. 26] it follows that (2.1) is indeed true for $\nu > n-1$. Here we used the limit $\frac{\log\Gamma(am+b)}{m\log m} \to a$, as $m\to\infty$, where $a, b > 0$. To verify this just observe that

$$\begin{aligned}\lim_{x\to\infty} \frac{\log\Gamma(ax+b)}{x\log x} &= a \lim_{x\to\infty} \frac{\psi(ax+b)}{1+\log x} \\ &= a \lim_{x\to\infty} \frac{\log(ax+b) - \frac{1}{2(ax+b)} + \mathcal{O}(x^{-2})}{\log x} = a,\end{aligned}$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the logarithmic derivative of the Euler gamma function. On the other hand, it is known (see [Bo, Theorem 2.9.2]) that if f is an entire function and its growth order is finite and it is not equal to a positive integer, then f has infinitely many zeros or f is a polynomial. Thus, using the fact that the growth order of the real entire function $x^{n-\nu}J_\nu^{(n)}(x)$ is $\frac{1}{2}$ and by the above-mentioned result we obtain that $J_\nu^{(n)}(x)$ has infinitely many zeros when $\nu > n-1$. Moreover, by (2.1) we obtain

$$J_\nu^{(n+1)}(x) = \frac{x^{\nu-n}\prod_{m\geq 1}\left(1 - \frac{x^2}{\left(j_{\nu,m}^{(n)}\right)^2}\right)}{2^\nu\Gamma(\nu+1-n)} \left(\frac{\nu-n}{x} - \sum_{m\geq 1} \frac{2x}{\left(j_{\nu,m}^{(n)}\right)^2 - x^2}\right),$$

which implies that

$$\frac{J_\nu^{(n+1)}(x)}{J_\nu^{(n)}(x)} = \frac{\nu-n}{x} - \sum_{m\geq 1} \frac{2x}{\left(j_{\nu,m}^{(n)}\right)^2 - x^2}, \tag{2.2}$$

where $\nu > n$, x is real (or complex) such that $x \neq 0$, $x \neq j_{\nu,m}^{(n)}$, $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$. Now, we are going to conclude by induction on n that for $\nu > n$ all the zeros of

$J_\nu^{(n+1)}(x)$ are real, provided that for $\nu > n-1$ all the zeros of $J_\nu^{(n)}(x)$ are real. For this, we show first that for $\nu > n$ the zeros of $J_\nu^{(n+1)}(x)$ cannot be purely imaginary. Indeed, letting $J_\nu^{(n+1)}(\mathrm{i}y) = 0$, where $y \in \mathbb{R}$, $y \neq 0$, from (2.2) we deduce that

$$0 = -\mathrm{i}\left(\nu - n + \sum_{m\geq 1} \frac{2y^2}{\left(j_{\nu,m}^{(n)}\right)^2 + y^2}\right),$$

which is a contradiction since $\nu > n$ and the zeros $j_{\nu,m}^{(n)}$ are real. Finally, we show that for $\nu > n$ the zeros of $J_\nu^{(n+1)}(x)$ cannot be complex. Taking $z = x + \mathrm{i}y$, where $xy \neq 0$, a complex zero of $J_\nu^{(n+1)}(z)$ and denoting by $\omega = \left(j_{\nu,m}^{(n)}\right)^2 - x^2 + y^2$, by (2.2) we have

$$z\frac{J_\nu^{(n+1)}(z)}{J_\nu^{(n)}(z)} = \nu - n - 2\sum_{m\geq 1} \frac{(x^2-y^2)\omega - 4x^2y^2}{\omega^2 + 4x^2y^2} - 4\mathrm{i}xy \sum_{m\geq 1} \frac{\omega + x^2 - y^2}{\omega^2 + 4x^2y^2} = 0,$$

that is,

$$\sum_{m\geq 1} \frac{\omega + x^2 - y^2}{\omega^2 + 4x^2y^2} = \sum_{m\geq 1} \frac{\left(j_{\nu,m}^{(n)}\right)^2}{\omega^2 + 4x^2y^2} = 0,$$

which is a contradiction. Thus, for $\nu > n$ the zeros of $J_\nu^{(n+1)}(x)$ indeed are all real.

Now, we prove that these zeros are all simple, except the origin. If we suppose that $\rho \neq 0$ is a double zero of $J_\nu^{(n)}(z)$ it follows that the derivative of the quotient $J_\nu^{(n)}(z)/J_\nu^{(n-1)}(z)$ also vanishes at ρ. It is a contradiction since by (2.2) for $\nu > n-1$ we have that

$$\frac{d}{dz}\left(\frac{J_\nu^{(n)}(z)}{J_\nu^{(n-1)}(z)}\right) = -\frac{\nu - n + 1}{z^2} - 2\sum_{m\geq 1} \frac{\left(j_{\nu,m}^{(n-1)}\right)^2 + z^2}{\left(\left(j_{\nu,m}^{(n-1)}\right)^2 - z^2\right)^2} \neq 0.$$

(b) Since the zeros of $J_\nu^{(n)}(x)$ are all real, it follows that the function $\mathbb{J}_{\nu,n}$ is in the Laguerre-Pólya class of real entire functions as the exponential factors in the infinite product are canceled because of the symmetry of the zeros $\pm j_{\nu,m}^{(n)}$, $m \in \mathbb{N}$, with respect to the origin. Now, since $\mathbb{J}_{\nu,n} \in \mathcal{LP}$, it follows that it satisfies the Laguerre inequality [Sko, p. 67]

$$\left(\mathbb{J}_{\nu,n}^{(k)}(x)\right)^2 - \mathbb{J}_{\nu,n}^{(k-1)}(x)\mathbb{J}_{\nu,n}^{(k+1)}(x) > 0,$$

where $n \in \mathbb{N}_0$, $k \in \mathbb{N}$, $\nu > n-1$ and $x \in \mathbb{R}$. Choosing $k = 1$ in the above Laguerre inequality we get

$$\left(xJ_\nu^{(n+1)}(x)\right)^2 - x^2 J_\nu^{(n+2)}(x)J_\nu^{(n)}(x) + (n-\nu)\left(J_\nu^{(n)}(x)\right)^2 > 0,$$

which implies that

$$\left(J_\nu^{(n+1)}(x)\right)^2 - J_\nu^{(n+2)}(x)J_\nu^{(n)}(x) > (\nu - n)\left(J_\nu^{(n)}(x)\right)^2/x^2 > 0,$$

where $\nu > n \geq 0$ and $x \neq 0$. Consequently, the function $J_\nu^{(n+1)}(x)/J_\nu^{(n)}(x)$ is strictly decreasing on each interval $\left(j_{\nu,m-1}^{(n)}, j_{\nu,m}^{(n)}\right)$, $m \in \mathbb{N}$ (note that $j_{\nu,0}^{(n)} = 0$). On the other hand, for fixed $m \in \mathbb{N}$ the function $J_\nu^{(n+1)}(x)/J_\nu^{(n)}(x)$ approaches ∞ when $x \searrow j_{\nu,m-1}^{(n)}$ and $-\infty$ when $x \nearrow j_{\nu,m}^{(n)}$. Summarizing, for arbitrary $m \in \mathbb{N}$ the graph of the restriction of the function $J_\nu^{(n+1)}(x)/J_\nu^{(n)}(x)$ to each interval $\left(j_{\nu,m-1}^{(n)}, j_{\nu,m}^{(n)}\right)$ intersects the horizontal axis only once, and the abscissa of this intersection point is exactly $j_{\nu,m}^{(n+1)}$. Thus we have proved that when $\nu > n$ the positive zeros of $J_\nu^{(n+1)}(x)$ and $J_\nu^{(n)}(x)$ are interlacing.

The monotonicity of $J_\nu^{(n+1)}(x)/J_\nu^{(n)}(x)$ can also be verified by using the Mittag-Leffler expansion (2.2). Namely, we have

$$\frac{d}{dx}\left(\frac{J_\nu^{(n+1)}(x)}{J_\nu^{(n)}(x)}\right) = -\frac{\nu-n}{x^2} - 2\sum_{m\geq 1}\frac{\left(j_{\nu,m}^{(n)}\right)^2 + x^2}{\left(\left(j_{\nu,m}^{(n)}\right)^2 - x^2\right)^2} < 0$$

for all $\nu > n$, $n \in \mathbb{N}_0$ and $x \neq j_{\nu,m}^{(n)}$, $m \in \mathbb{N}_0$.

(c) Laguerre's theorem on separation of zeros [Bo, p. 23] states that if $f(z)$ is a nonconstant entire function of genus 0 or 1, which is real for real z and has only real zeros, then the zeros of f' are also real and are separated by the zeros of f. The proof of part (a) shows that $\mathbb{J}_{\nu,n}$ is a real entire function of genus zero. Thus, by using part (a) of this theorem and Laguerre's separation theorem it follows that the zeros of $(n-\nu)J_\nu^{(n)}(x) + xJ_\nu^{(n+1)}(x)$ are real when $\nu > n-1$ and are interlacing with the zeros of $J_\nu^{(n)}(x)$. $\square$

Proof of Theorem 2. By (2.1) and Theorem 1 the function

$$\begin{aligned}\mathbb{J}_{\nu,n}(2\sqrt{x}) &= 2^n\Gamma(\nu+1-n)x^{\frac{n-\nu}{2}}J_\nu^{(n)}(2\sqrt{x})\\ &= \sum_{m\geq 0}\frac{(-1)^m\Gamma(\nu+2m+1)\Gamma(\nu+1-n)x^m}{m!\Gamma(\nu+2m-n+1)\Gamma(\nu+m+1)}\end{aligned}$$

belongs to the Laguerre-Pólya class $\mathcal{LP}$. Consequently by using a well-known theorem of Jensen (see [Je] or [DC, Theorem A]) it follows that the Jensen polynomial of $\mathbb{J}_{\nu,n}(2\sqrt{x})$ has only real zeros. Now, the Jensen polynomial in question is

$$\sum_{m=0}^{s}(-1)^m\binom{s}{m}\frac{\Gamma(\nu+2m+1)\Gamma(\nu+1-n)}{\Gamma(\nu+2m-n+1)\Gamma(\nu+m+1)}x^m,$$

which after some transformations and in view of the Legendre duplication formula

$$\Gamma(2x)\sqrt{\pi} = 2^{2x-1}\Gamma(x)\Gamma\left(x+\tfrac{1}{2}\right) \tag{2.3}$$

can be rewritten as

$${}_3F_3\left(-s, \frac{\nu+1}{2}, \frac{\nu}{2}+1; \nu+1, \frac{\nu-n+1}{2}, \frac{\nu-n}{2}+1; x\right).$$

Moreover, according to Csordas and Williamson [CW] the zeros of the Jensen polynomials are simple. This completes the proof of the theorem. $\square$

Proof of Theorem 3. (a) and (b) We know that for $\nu \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ all zeros of $\mathbf{H}_\nu'(x)$ are real and simple; see [BY]. For convenience we denote the mth positive zero of

$\mathbf{H}_\nu^{(n)}(x)$ by $h_{\nu,m}^{(n)}$, where $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$. Since

$$\lim_{m\to\infty} \frac{m \log m}{\log 2^{2m} + \log \Gamma\left(m+\frac{3}{2}\right) + \log \Gamma\left(m+\nu+\frac{3}{2}\right) - Q_{n,m}(\nu)} = \frac{1}{2},$$

where $Q_{n,m}(\nu) = \log |(2m+\nu+1)(2m+\nu)\ldots(2m+\nu-n+2)|$, the real entire function

$$\begin{aligned} \mathbb{H}_{\nu,n}(x) &= 2^{\nu+1}x^{n-\nu-1}\mathbf{H}_\nu^{(n)}(x) \\ &= \sum_{m\geq 0} \frac{(-1)^m\Gamma(\nu+2m+1)x^{2m}}{2^{2m}\Gamma\left(\nu+\frac{3}{2}\right)\Gamma(\nu+2m-n+2)\Gamma\left(\nu+m+\frac{3}{2}\right)} \end{aligned} \tag{2.4}$$

is of order $\frac{1}{2}$. By the Hadamard factorization theorem [Le, p. 26] it follows that the following Weierstrassian decomposition is valid for appropiate values of ν (for example $\nu \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, $\nu \neq 0$) and $n \in \mathbb{N}$:

$$\mathbf{H}_\nu^{(n)}(x) = \frac{(\nu+1)\nu\cdot\ldots\cdot(\nu-n+2)x^{\nu+1-n}}{\sqrt{\pi}2^\nu\Gamma\left(\nu+\frac{3}{2}\right)} \prod_{m\geq 1}\left(1-\frac{x^2}{(h_{\nu,m}^{(n)})^2}\right). \tag{2.5}$$

Thus, $\mathbf{H}_\nu^{(n)}(x)$ has infinitely many zeros. By using the fact that all zeros of $\mathbf{H}_\nu^{(n)}(x)$ are real (and simple) when $n=1$, it follows that for $n=1$ the function $\mathbb{H}_{\nu,n}$ is in the Laguerre-Pólya class of real entire functions (the exponential factors in the infinite product are canceled because of the symmetry of the zeros $\pm h_{\nu,m}^{(n)}$, $m \in \mathbb{N}$, with respect to the origin). Now, since for $n=1$ we have $\mathbb{H}_{\nu,n} \in \mathcal{LP}$, it satisfies the Laguerre inequality [Sko, p. 67]

$$\left(\mathbb{H}_{\nu,n}^{(k)}(x)\right)^2 - \mathbb{H}_{\nu,n}^{(k-1)}(x)\mathbb{H}_{\nu,n}^{(k+1)}(x) > 0,$$

where $n=1$, $k \in \mathbb{N}$, $\nu \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ and $x \in \mathbb{R}$. Choosing $k=1$ in the above Laguerre inequality we get

$$\left(x\mathbf{H}_\nu^{(n+1)}(x)\right)^2 - x^2\mathbf{H}_\nu^{(n+2)}(x)\mathbf{H}_\nu^{(n)}(x) + (n-\nu-1)\left(\mathbf{H}_\nu^{(n)}(x)\right)^2 > 0,$$

which implies that

$$\left(\mathbf{H}_\nu^{(n+1)}(x)\right)^2 - \mathbf{H}_\nu^{(n+2)}(x)\mathbf{H}_\nu^{(n)}(x) > (\nu+1-n)\left(\mathbf{H}_\nu^{(n)}(x)\right)^2/x^2 > 0,$$

where $\nu \in \left(0, \frac{1}{2}\right]$ and $x \neq 0$. Consequently, if $n=1$ the function $\mathbf{H}_\nu^{(n+1)}(x)/\mathbf{H}_\nu^{(n)}(x)$ is strictly decreasing on each interval $\left(h_{\nu,m-1}^{(n)}, h_{\nu,m}^{(n)}\right)$, $m \in \mathbb{N}$. Here we used $h_{\nu,0}^{(n)} = 0$. Since the zeros of $\mathbf{H}'_\nu$ are simple, the function $\mathbf{H}''_\nu$ does not vanish in $h'_{\nu,m}$. On the other hand, for $n=1$ and for fixed $m \in \mathbb{N}$ the function $\mathbf{H}_\nu^{(n+1)}(x)/\mathbf{H}_\nu^{(n)}(x)$ approaches ∞ when $x \searrow h_{\nu,m-1}^{(n)}$ and $-\infty$ when $x \nearrow h_{\nu,m}^{(n)}$. Summarizing, for $n=1$ and arbitrary $m \in \mathbb{N}$ the graph of the restriction of the function $\mathbf{H}_\nu^{(n+1)}(x)/\mathbf{H}_\nu^{(n)}(x)$ to each interval $\left(h_{\nu,m-1}^{(n)}, h_{\nu,m}^{(n)}\right)$ intersects the horizontal axis only once. The abscissa of this intersection point is exactly $h_{\nu,m}^{(n+1)}$. Moreover, it is clear that the zeros $h_{\nu,m}^{(n+1)}$ are simple because of the above monotonicity and limit properties. Concerning the distribution of the zeros, an analogous procedure shows that on the semi-axis $(-\infty, 0)$ we have a similar situation to the semi-axis $(0, \infty)$. Thus we have proved that for $\nu \in \left(0, \frac{1}{2}\right]$ all zeros of $\mathbf{H}''_\nu(x)$ are real and simple. Therefore the proof of part (a) becomes complete, moreover, we have also proved the statement of part (b).

(c) According to the above proof we know that $\mathbb{H}_{\nu,n}$ is a real entire function of genus zero. Thus, for $n \in \{0,1\}$ and $\nu \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ by using the fact that the zeros of $\mathbf{H}_\nu^{(n)}$ are all real and the Laguerre separation theorem, one obtains that the zeros of $(n-\nu-1)\mathbf{H}_\nu^{(n)}(x) + x\mathbf{H}_\nu^{(n+1)}(x)$ are real when $\nu \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ and $n \in \{0,1\}$ and are interlacing with the zeros of $\mathbf{H}_\nu^{(n)}(x)$. □

Proof of Theorem 4. The nth derivative with respect to the argument of the Struve function of order ν, denoted by $\mathbf{H}_\nu^{(n)}(x)$, is the power series

$$\begin{aligned}&\frac{x^{\nu-n+1}}{2^{\nu+1}} \sum_{m\geq 0} \frac{(-1)^m \Gamma(2m+\nu+2)\, x^{2m}}{2^{2m}\Gamma(m+\frac{3}{2})\Gamma(m+\nu+\frac{3}{2})\Gamma(2m+\nu-n+2)}\\ &= \frac{x^{\nu-n+1}}{2^{\nu-n+1}} \sum_{m\geq 0} \frac{(-1)^m \Gamma(m+\frac{\nu}{2}+1)\Gamma(m+\frac{\nu}{2}+\frac{3}{2})\, x^{2m}}{2^{2m}\Gamma(m+\frac{3}{2})\Gamma(m+\nu+\frac{3}{2})\Gamma(m+\frac{\nu-n}{2}+1)\Gamma(m+\frac{\nu-n}{2}+\frac{3}{2})},\end{aligned}$$

where the duplication formula (2.3) is employed in both numerator and denominator. This means that $\mathbf{H}_\nu^{(n)}(2\sqrt{x})$ can be written as

$$\frac{2^{1-n}\,\Gamma(\nu+2)\, x^{\frac{\nu-n+1}{2}}}{\sqrt{\pi}\,\Gamma(\nu+\frac{3}{2})\Gamma(\nu-n+2)} \sum_{m\geq 0} \frac{(-1)^m (1)_m (\frac{\nu}{2}+1)_m (\frac{\nu}{2}+\frac{3}{2})_m}{(\frac{3}{2})_m (\nu+\frac{3}{2})_m (\frac{\nu-n}{2}+1)_m (\frac{\nu-n}{2}+\frac{3}{2})_m} \frac{x^m}{m!},$$

where $\nu-n+2 \neq \{0,-1,\dots\}$, and we have the auxiliary function

$$\begin{aligned}\mathcal{H}_{\nu,n}(2\sqrt{x}) &= \frac{\sqrt{\pi}\,\Gamma(\nu+\frac{3}{2})}{2^{1-n}\,\Gamma(\nu+2)}\,\Gamma(\nu-n+2)\, x^{\frac{n-\nu-1}{2}}\, \mathbf{H}_\nu^{(n)}(2\sqrt{x})\\ &= {}_3F_4\left(\begin{matrix} 1, & \frac{\nu}{2}+1, & \frac{\nu}{2}+\frac{3}{2} \\ \frac{3}{2}, & \nu+\frac{3}{2}, & \frac{\nu-n}{2}+1, & \frac{\nu-n}{2}+\frac{3}{2}\end{matrix}; x\right).\end{aligned}$$

We recognize the coefficients in the associated Jensen polynomial [CVV, p. 113]

$$\gamma_m = \frac{(-1)^m (1)_m (\frac{\nu}{2}+1)_m (\frac{\nu}{2}+\frac{3}{2})_m}{(\frac{3}{2})_m (\nu+\frac{3}{2})_m (\frac{\nu-n}{2}+1)_m (\frac{\nu-n}{2}+\frac{3}{2})_m}.$$

The related Jensen polynomial becomes the Laguerre-type hypergeometric polynomial

$$\begin{aligned}\mathcal{P}_s^{\mathbf{H}}(x;n) &= \sum_{m=0}^{s} (-1)^m \binom{s}{m} \gamma_m\, x^m\\ &= {}_4F_4\left(\begin{matrix} -s, & 1, & \frac{\nu}{2}+1, & \frac{\nu}{2}+\frac{3}{2} \\ \frac{3}{2}, & \nu+\frac{3}{2}, & \frac{\nu-n}{2}+1, & \frac{\nu-n}{2}+\frac{3}{2}\end{matrix}; x\right). \qquad (2.6)\end{aligned}$$

The special case $n=0$ simplifies to

$$\mathcal{P}_s^{\mathbf{H}}(x;0) = {}_2F_2\left(-s, 1; \frac{3}{2}, \nu+\frac{3}{2}; x\right),$$

while, for $n=1$ we have

$$\mathcal{P}_s^{\mathbf{H}}(x;1) = {}_3F_3\left(-s, 1, \frac{\nu}{2}+\frac{3}{2}; \frac{3}{2}, \nu+\frac{3}{2}, \frac{\nu}{2}+\frac{1}{2}; x\right).$$

Now, by using the fact that for $\nu \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ the zeros of $\mathbf{H}_\nu$ and $\mathbf{H}'_\nu$ are real and also part (a) of Theorem 3 it follows that the function $\mathcal{H}_{\nu,n}(2\sqrt{x})$ belongs to the Laguerre-Pólya class $\mathcal{LP}$ under the assumption that $\nu \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ and $n \in \{0, 1\}$, or $\nu \in \left(0, \frac{1}{2}\right]$ and $n = 2$. Consequently by using the theorem of Jensen (see [Je] or [DC, Theorem A]) it follows that the Jensen polynomial $\mathcal{P}_s^{\mathbf{H}}(x; n)$ has only real zeros. Now, the hypergeometric nature of the Jensen polynomial $\mathcal{P}_s^{\mathbf{H}}(x; n)$ in question is shown in (2.6). Moreover, according to Csordas and Williamson [CW] the zeros of the Jensen polynomials are simple. This completes the proof of the theorem. □

Proof of Theorem 5. By equating the infinite series and the infinite product representation for $\mathbf{H}_\nu(z)$ given by [BPS]

$$\sqrt{\pi}2^\nu x^{-\nu-1}\Gamma\left(\nu + \frac{3}{2}\right)\mathbf{H}_\nu(x) = \prod_{n\geq 1}\left(1 - \frac{x^2}{h_{\nu,n}^2}\right)$$

and

$$\mathbf{H}_\nu(x) = \left(\frac{x}{2}\right)^{\nu+1}\sum_{n\geq 0}\frac{(-1)^n\left(\frac{x}{2}\right)^{2n}}{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(n+\nu+\frac{3}{2}\right)},$$

we obtain

$$\begin{aligned}\frac{1}{\Gamma(\frac{3}{2})\Gamma(\nu+\frac{3}{2})} &- \frac{(\frac{x}{2})^2}{\frac{3}{2}\Gamma(\frac{3}{2})\Gamma(\nu+\frac{3}{2})(\nu+\frac{3}{2})} + \dots \\ &= \frac{2}{\sqrt{\pi}\Gamma(\nu+\frac{3}{2})}\left(1 - \frac{x^2}{h_{\nu,1}^2}\right)\left(1 - \frac{x^2}{h_{\nu,2}^2}\right)\dots\end{aligned}$$

or equivalently

$$\begin{aligned}1-&\left(\frac{x}{2}\right)^2\frac{2^2}{3(2\nu+3)} + \left(\frac{x}{2}\right)^4\frac{2^4}{3\cdot 5(2\nu+3)(2\nu+5)} + \dots \\ &= 1 - x^2\sum_{n\geq 1}\frac{1}{h_{\nu,n}^2} + x^4\sum_{n\geq 1}\frac{1}{h_{\nu,n}^2}\sum_{k\geq 1,\ k\neq n}\frac{1}{h_{\nu,k}^2} + \dots.\end{aligned}$$

The coefficients of the same powers of x must be equal, so the equality of the coefficients of x^2 proves the first relation of the theorem, and the equality of the coefficients of x^4 gives

$$\sum_{n\geq 1}\frac{1}{h_{\nu,n}^2}\sum_{k\geq 1,\ k\neq n}\frac{1}{h_{\nu,k}^2} = \frac{1}{15(2\nu+3)(2\nu+5)}.$$

Since the zeros $h_{\nu,n}$ are symmetric around the origin and because of the first Rayleigh sum of $h_{\nu,n}$-s, the previous equation becomes

$$\frac{1}{2}\sum_{n\geq 1}\frac{1}{h_{\nu,n}^2}\left(\sum_{k\geq 1}\frac{1}{h_{\nu,k}^2} - \frac{1}{h_{\nu,n}^2}\right) = \frac{1}{15(2\nu+3)(2\nu+5)}.$$

Using the first Rayleigh sum again, it finally becomes the second Rayleigh sum as in the statement of the theorem.

To prove the relations on the Rayleigh sums of the zeros of $\mathbf{H}'_\nu$ we first equate the infinite sum and the infinite product representations of $\mathbf{H}'_\nu$ given by (2.4) and

(2.5), that is,

$$\sum_{n\geq 0}\frac{(-1)^n(2n+\nu+1)(\frac{x}{2})^{2n}}{(\nu+1)(\frac{3}{2})_n(\nu+\frac{3}{2})_n}=\prod_{n\geq 1}\left(1-\frac{x^2}{(h'_{\nu,n})^2}\right)$$

or equivalently

$$1-\left(\frac{x}{2}\right)^2\frac{(\nu+3)}{(\nu+1)(\frac{3}{2})_1(\frac{\nu+3}{2})_1}+\left(\frac{x}{2}\right)^4\frac{(\nu+5)}{(\nu+1)(\frac{3}{2})_2(\frac{\nu+3}{2})_2}+\dots$$
$$=1-x^2\sum_{n\geq 1}\frac{1}{(h'_{\nu,n})^2}+x^4\sum_{n\geq 1}\frac{1}{(h'_{\nu,n})^2}\sum_{k\geq 1,\ k\neq n}\frac{1}{(h'_{\nu,k})^2}+\dots.$$

The equality of the coefficients of the same power of x on both sides gives the desired relations for the zeros of $\mathbf{H}'_\nu$.

Finally, to deduce the first two Rayleigh sums for the zeros of $\mathbf{H}''_\nu$ we proceed similarly to what was done above. □

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References

[AM] R. Álvarez-Nodarse and F. Marcellán, *A generalization of the classical Laguerre polynomials*, Rend. Circ. Mat. Palermo (2) **44** (1995), no. 2, 315–329, DOI 10.1007/BF02850839. MR1355489

[BCD] Árpád Baricz, Murat Çağlar, and Erhan Deniz, *Starlikeness of Bessel functions and their derivations*, Math. Inequal. Appl. **19** (2016), no. 2, 439–449, DOI 10.7153/mia-19-35. MR3458758

[BPS] Árpád Baricz, Saminathan Ponnusamy, and Sanjeev Singh, *Turán type inequalities for Struve functions*, J. Math. Anal. Appl. **445** (2017), no. 1, 971–984, DOI 10.1016/j.jmaa.2016.08.026. MR3543805

[BS] Árpád Baricz and Róbert Szász, *Close-to-convexity of some special functions and their derivatives*, Bull. Malays. Math. Sci. Soc. **39** (2016), no. 1, 427–437, DOI 10.1007/s40840-015-0180-7. MR3439869

[BY] Árpád Baricz and Nihat Yağmur, *Geometric properties of some Lommel and Struve functions*, Ramanujan J. **42** (2017), no. 2, 325–346, DOI 10.1007/s11139-015-9724-6. MR3596935

[Bo] Ralph Philip Boas Jr., *Entire functions*, Academic Press Inc., New York, 1954. MR0068627

[CVV] George Csordas, Richard S. Varga, and István Vincze, *Jensen polynomials with applications to the Riemann ζ-function*, J. Math. Anal. Appl. **153** (1990), no. 1, 112–135, DOI 10.1016/0022-247X(90)90269-L. MR1080122

[CW] George Csordas and Jack Williamson, *The zeros of Jensen polynomials are simple*, Proc. Amer. Math. Soc. **49** (1975), 263–264, DOI 10.2307/2039828. MR0361017

[DC] Dimitar K. Dimitrov and Youssèf Ben Cheikh, *Laguerre polynomials as Jensen polynomials of Laguerre-Pólya entire functions*, J. Comput. Appl. Math. **233** (2009), no. 3, 703–707, DOI 10.1016/j.cam.2009.02.039. MR2583006

[Hu] A. Hurwitz, *Ueber die Nullstellen der Bessel'schen Function* (German), Math. Ann. **33** (1888), no. 2, 246–266, DOI 10.1007/BF01443855. MR1510541

[IM] Mourad E. H. Ismail and Martin E. Muldoon, *Bounds for the small real and purely imaginary zeros of Bessel and related functions*, Methods Appl. Anal. **2** (1995), no. 1, 1–21, DOI 10.4310/MAA.1995.v2.n1.a1. MR1337450

[Je] J. L. W. V. Jensen, *Recherches sur la théorie des équations* (French), Acta Math. **36** (1913), no. 1, 181–195, DOI 10.1007/BF02422380. MR1555086

[Ke] M. K. Kerimov, *Studies on the zeros of Bessel functions and methods for their computation*, Comput. Math. Math. Phys. **54** (2014), no. 9, 1337–1388, DOI 10.1134/S0965542514090073. MR3258608

[KK] Haseo Ki and Young-One Kim, *On the number of nonreal zeros of real entire functions and the Fourier-Pólya conjecture*, Duke Math. J. **104** (2000), no. 1, 45–73, DOI 10.1215/S0012-7094-00-10413-9. MR1769725

[Ko] Roelof Koekoek, *Generalizations of the classical Laguerre polynomials and some q-analogues*, ProQuest LLC, Ann Arbor, MI. Thesis (Dr.)–Technische Universiteit Delft (The Netherlands) (1990). MR2714461

[KL] Chrysi G. Kokologiannaki and Andrea Laforgia, *Simple proofs of classical results on zeros of $J_\nu(x)$ and $J'_\nu(x)$*, Tbilisi Math. J. **7** (2014), no. 2, 35–39, DOI 10.2478/tmj-2014-0014. MR3313053

[KP] Chrysi G. Kokologiannaki and Eugenia N. Petropoulou, *On the zeros of $J'''_\nu(x)$*, Integral Transforms Spec. Funct. **24** (2013), no. 7, 540–547, DOI 10.1080/10652469.2012.720253. MR3171971

[Le] B. Ya. Levin, *Lectures on entire functions* (Russian), translated by V. Tkachenko, written in collaboration with and with a preface by Yu. Lyubarskii, M. Sodin and V. Tkachenko, Translations of Mathematical Monographs, vol. 150, American Mathematical Society, Providence, RI, 1996. MR1400006

[LM] Lee Lorch and Martin E. Muldoon, *The real zeros of the derivatives of cylinder functions of negative order*, dedicated to Richard A. Askey on the occasion of his 65th birthday, Part III, Methods Appl. Anal. **6** (1999), no. 3, 317–326, DOI 10.4310/MAA.1999.v6.n3.a3. MR1803312

[LP] Lee Lorch and Peter Szego, *Monotonicity of the zeros of the third derivative of Bessel functions*, Methods Appl. Anal. **2** (1995), no. 1, 103–111, DOI 10.4310/MAA.1995.v2.n1.a7. MR1337456

[Me] A. McD. Mercer, *The zeros of $az^2J''_\nu(z)+bzJ'_\nu(z)+cJ_\nu(z)$ as functions of order*, Internat. J. Math. Math. Sci. **15** (1992), no. 2, 319–322, DOI 10.1155/S0161171292000395. MR1155524

[OLBC] Frank W. J. Olver, Daniel W. Lozier, Ronald F. Boisvert, and Charles W. Clark (eds.), *NIST handbook of mathematical functions*, with 1 CD-ROM (Windows, Macintosh and UNIX), U.S. Department of Commerce, National Institute of Standards and Technology, Washington, DC; Cambridge University Press, Cambridge, 2010. MR2723248

[Ske] E. A. Skelton, *A new identity for the infinite product of zeros of Bessel functions of the first kind or their derivatives*, J. Math. Anal. Appl. **267** (2002), no. 1, 338–344, DOI 10.1006/jmaa.2001.7779. MR1886832

[Sko] H. Skovgaard, *On inequalities of the Turán type*, Math. Scand. **2** (1954), 65–73, DOI 10.7146/math.scand.a-10396. MR0063415

[St] J. Steinig, *The real zeros of Struve's function*, SIAM J. Math. Anal. **1** (1970), 365–375, DOI 10.1137/0501034. MR0267162

[Wa] G. N. Watson, *A treatise on the theory of Bessel functions*, Cambridge University Press, Cambridge, England; The Macmillan Company, New York, 1944. MR0010746

[WL] R. Wong and T. Lang, *On the points of inflection of Bessel functions of positive order. II*, Canad. J. Math. **43** (1991), no. 3, 628–651, DOI 10.4153/CJM-1991-037-4. MR1118013

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